RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

L. BICAN
L. SALCE
J. ŠTĚPÁN

A characterization of countable Butler groups

Rendiconti del Seminario Matematico della Università di Padova, tome 74 (1985), p. 51-58
<http://www.numdam.org/item?id=RSMUP_1985__74__51_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 74 (1985)

A Characterization of Countable Butler Groups.

L. BICAN - L. SALCE - J. ŠTĚPÁN (*)

In this paper by the word « group » we shall always mean an additively written abelian group.

The class of pure subgroups of completely decomposable torsion free groups of finite rank was introduced and investigated by Butler in [6]. The first author characterized in [3] the Butler groups as those torsion-free groups of finite rank H such that $r_p(H) = r(p^\omega H)$ for each prime p and H/B has finitely many non-zero primary components for all full generalized subgroups B of H.

Recently, Arnold collected the known results on Butler groups in [1] and the first two authors showed in [5] that a torsion free group of finite rank H is a Butler group if and only if Bext $(H, T) = 0$ for all torsion groups T (Bext is the subfunctor of Ext consisting of the equivalence classes of the balanced exact sequences). This characterization is used in [5] to define Butler groups of arbitrary rank.

In this paper we introduce some kind of ascending chain conditions on torsion free groups, which enable us to obtain new characterizations of Butler groups of finite and countable rank, and to show that countable pure subgroups of Butler groups with linearly ordered type set are Butler groups. Notice that the class of arbitrary Butler groups is not closed under pure subgroups (see [2, Lemma 12] and [4]).

If p is a prime and g is an element of a torsion free group H, then

(*) Indirizzo degli AA.: L. BICAN and J. ŠTĚPÁN: MFF UK, Sokolovská 83, Praha 8-Karlín, Czechoslovakia; L. SALCE: Seminario Matematico dell'Università, via Belzoni 7, 35100 Padova.

Lavoro eseguito con il contributo del Ministero della Pubblica Istruzione.

$h_p^H(g)$ denotes the p-height of g in H. The set of all elements g of H with $h_p^H(g) = \infty$ is a subgroup of H which will be denoted by $p^\omega H$. It is well known (see [8]) that if H is a torsion free group of finite rank and K is a maximal free subgroup, then the number $r_p(H)$ of summands isomorphic to $Z(p^\infty)$ in H/K does not depend on the particular choice of K, and this number is called the p-rank of H.

In T is a torsion group, then $r_p^*(T)$ denotes the rank of the divisible part of the p-primary component of T. If K is a subgroup of a torsion free group H of finite rank and H/K is torsion, then $r_p(H) = r_p(K) + r_p^*(H/K)$ (see [8, Theorem 5]).

A subgroup K of a torsion free group H is called generalized regular if, for every $g \in K$, the p-heights $h_p^H(g)$ and $h_p^K(g)$ differ for a finite number of primes p, only.

If S is a subset of the set of all primes P and M is a subset of a torsion free group H, then $\langle M\rangle_S$ (or more precisely $\langle M\rangle_S^H$) denotes the S-pure closure of M in H. We shall also write $\langle M\rangle_*$ instead of $\langle M\rangle_P$.

For all other unexplained notation and terminology we refer to [7].

DEFINITION 1. A chain $H_1 \subseteq H_2 \subseteq \ldots$ of subgroups of a torsion free group H such that for every $h \in H$, $\langle h\rangle_* \subseteq H_m$ for some $m \in \mathbf{N}$, will be called a $*$-chain of H. A $*$-chain of H is said full if H/H_m is torsion for each $m \in \mathbf{N}$.

REMARK 2. If H is a torsion free group of finite rank, then obviously in every $*$-chain $H_1 \subseteq H_2 \subseteq \ldots$ of H there is a member H_m such that H/H_m is torsion and, consequently, $H_m \subseteq H_{m+1} \subseteq \ldots$ is a full $*$-chain of H.

We start with two lemmas on $*$-chains.

LEMMA 3. Let L be a pure subgroup of a torsion free group H. Then the following facts are equivalent:

(i) H contains a full generalized regular subgroup B such that $L/(L \cap B)$ has infinitely many non-zero components;

(ii) there is a full $*$-chain $H_1 \subseteq H_2 \subseteq \ldots$ of H such that $L/(L \cap H_m)$ has infinitely many non-zero primary components for each $m \in \mathbf{N}$.

PROOF. (i) $\rightarrow$ (ii). There is a natural monomorphism

$$L/(L \cap B) \rightarrow H/B$$

and so the set $S = \{p_1, p_2, \ldots\}$ of all primes p such that the p-primary component of H/B is non-zero, is infinite. Denoting $P_m = \{p_1, p_2, \ldots, p_m\}$ and $H_m = \langle B \rangle_{P_m}$, $(m \in \mathbf{N})$, we have $H_1 \subset H_2 \subset \ldots$. If $0 \neq h \in H$ is arbitrary, then $0 \neq nh \in B$ for some $n \in \mathbf{N}$ and the p-heights $h_p^H(nh)$ and $h_p^B(nh)$ differ for a finite set of primes from S, say $p_{i_1}, \ldots, p_{i_k}$. If $m \geqslant \max\{i_1, \ldots, i_k\}$, then clearly $\langle nh \rangle_* = \langle h \rangle_* \subseteq H_m$ and $H_1 \subset H_2 \subset \ldots$ is a $*$-chain of H. Since obviously $L \cap H_m = \langle L \cap B \rangle_{P_m}$, the factor group $(L \cap H_m)/(L \cap B)$ has only a finite number of non-zero primary components and consequently the hypothesis together with the exact sequence

$$0 \to (L \cap H_m)/(L \cap B) \to L/(L \cap B) \to L/(L \cap H_m) \to 0$$

shows that $L/(L \cap H_m)$ has infinitely many non-zero components.

(ii) $\to$ (i). Choose an infinite set $S = \{p_1, p_2, \ldots\}$ of primes such that for each $m \in \mathbf{N}$ the p_m-primary component of $L/(L \cap H_m)$ is non-zero. Take $x_m \in L \setminus H_m$ such that $p_m x_m \in H_m$ and let B_m be a subgroup of H maximal with respect to $H_m \subseteq B_m$ and $x_m \notin B_m$. Then H/B_m is cocyclic with socle $\langle x_m + B_m \rangle$ and so $0 \neq L/(L \cap B_m)$ is cocyclic p_m-primary owing to the natural embedding $L/(L \cap B_m) \to H/B_m$. Moreover, setting $B = \bigcap_{i \in \mathbf{N}} B_i$, $L/(L \cap B)$ is torsion since $H_1 \subseteq B$ and so the natural embedding $L/(L \cap B) \to \prod_{i \in \mathbf{N}} L/(L \cap B_i)$ yields the isomorphism $L/(L \cap B) \cong \bigoplus_{i \in N} L/(L \cap B_i)$. Finally, if $b \in B$ is arbitrary, then there exists $m \in \mathbf{N}$ such that $\langle b \rangle_* \subseteq H_m \subseteq \bigcap_{i \geqslant m} B_i$. Thus $h_p^H(b) = b_p^B(b)$ for each prime $p \notin \{p_1, p_2, \ldots, p_m\}$ and B is generalized regular in H. //

Lemma 4. *Let H be a torsion free group of finite rank and p be a prime. Then the following facts are equivalent:*

(i) $r_p(H) > r(p^\omega H)$;

(ii) *there is a strictly increasing $*$-chain $H_1 \subset H_2 \subset \ldots$ of H such that $H/H_i \cong Z(p^\infty)$ for each $i \in \mathbf{N}$;*

(iii) *there is a strictly increasing $*$-chain $H_1 \subset H_2 \subset \ldots$ of H such that H/H_i is divisible p-primary for each $i \in \mathbf{N}$.*

Proof. (i) $\to$ (ii). It is easy to see that H contains a subgroup K such that $p^\omega H \subseteq K$ and $H/K \cong Z(p^\infty)$. So H contains a chain $H_1 \subset H_2 \subset \ldots$ such that $H_i/K \cong Z(p^i)$ and $H/H_i \cong Z(p^\infty)$ for each $i \in \mathbf{N}$. If $0 \neq h \in H$ is an arbitrary element, then for $h \in p^\omega H$ it is $\langle h \rangle_* \subseteq K \subseteq H_1$. If $h \notin p^\omega H$, then $h_p^H(h) = r < \infty$ and $p^r x = h$ for some $x \in H$. Moreover,

$p^s h \in K$ for some $s \in N$, so that $p^{r+s}x \in K$ and $x \in H_{r+s}$. Thus $\langle h \rangle_* \subseteq \subseteq H_{r+s}$ and $H_1 \subset H_2 \subset \ldots$ is a $*$-chain of H with the desired property.

(ii) $\to$ (iii). Obvious.

(iii) $\to$ (i). Let $r_p(H) = r(p^\omega H)$ and $H_1 \subseteq H_2 \subseteq \ldots$ be an arbitrary $*$-chain of H such that H/H_i is a divisible p-primary group for each $i \in \mathbf{N}$. With respect to Remark 2, we can assume that this chain is full and it is easy to see that $p^\omega H \subseteq H_m$ for some $m \in \mathbf{N}$. Now [8, Theorem 5] gives $r_p(H) = r_p(H_m) + r_p^*(H/H_m) = r(p^\omega H) + r_p(H_m/p^\omega H) + r_p^*(H/H_m)$ and so the equality $r_p(H) = r(p^\omega H)$ yields $r_p^*(H/H_m) = 0$. Thus H/H_m is a finite group and consequently $H_k = H$ for some $k \geqslant m$, which contradicts the hypothesis. //

Definition 5. A torsion free group H is said to be $*$-noetherian if it has no strictly increasing infinite $*$-chains.

We can now give a new characterization of Butler groups of finite rank.

Theorem 6. A torsion free group H is $*$-noetherian if and only if it is a Butler group of finite rank.

Proof. Suppose that H is a Butler group of finite rank and let $H_1 \subset H_2 \subset \ldots$ be any strictly increasing $*$-chain of H. With respect to Remark 2 we can assume that this chain is full. By [3] and by Lemma 3 we can suppose that H/H_1 has only a finite number of non-zero primary components. So it is easy to see that there is a prime p such that the p-primary component of H/H_1 is non-zero for each $i \in \mathbf{N}$ and the sequence $\langle H_1 \rangle_S \subset \langle H_2 \rangle_S \subset \ldots$, where $S = P \setminus \{p\}$, is strictly increasing. Since $H/\langle H_1 \rangle_S$ is a finite direct sum of groups $Z(p^k)$, $k \in \mathbf{N} \cup \{\infty\}$, there is an $m \in \mathbf{N}$ such that $H/\langle H_m \rangle_*$ is a divisible p-primary group. Thus $r_p(H) > r(p^\omega H)$ by Lemma 4, which is absurd by [3].

Conversely, if H is $*$-noetherian then it is obviously of finite rank. Moreover, $r_p(H) = r(p^\omega H)$ by Lemma 4 and for every generalized regular subgroup B of H the factor group H/B has only a finite number of non-zero primary components by Lemma 3, and consequently H is a Butler group by [3]. //

We introduce now two new concepts, which are useful in the investigation of Butler groups of arbitrary rank.

Definition 7. A full $*$-chain $H_1 \subseteq H_2 \subseteq \ldots$ of a torsion free group H is said to be locally finite if, for every pure subgroup of finite rank L of H, the chain $(H_1 \cap L) \subseteq (H_2 \cap L) \subseteq \ldots$ is finite.

DEFINITION 8. A torsion free group H is called locally $*$-noetherian if every full $*$-chain of H is locally finite.

Recall that a torsion free group of arbitrary rank is said a Butler group if $\mathrm{Bext}(H, T) = 0$ for all torsion groups T (see [5]).

PROPOSITION 9. Every Butler group is locally $*$-noetherian.

PROOF. Let H be a torsion free group which is not locally $*$-noetherian. Then there is a full $*$-chain $H_1 \subset H_2 \subset \ldots$ of H and a pure subgroup of finite rank L of H such that the chain $(H_1 \cap L) \subset (H_2 \cap L) \subset \ldots$ is strictly increasing. Suppose that $L/(H_1 \cap L)$ has only a finite number of non-zero p-primary components. Then (see the first part of the proof of Theorem 6) $r_p(L) > r(p^\omega L)$. On the other hand, $p^\omega L = L \cap p^\omega H$ gives $L/p^\omega L \simeq (L + p^\omega H)/p^\omega H \subseteq H/p^\omega H$. By [5, Theorem 2.1], we have $r_p(H/p^\omega H) = 0$, so that, by [9, Corollary 2], $r_p(\langle (L + p^\omega H)/p^\omega H\rangle_*) = 0$ and consequently $r_p(L/p^\omega L) = 0$ by [8, Theorem 5]. Thus we can suppose that $L/(H_m \cap L)$ has infinitely many nonzero primary components for each $m \in \mathbf{N}$. So, by Lemma 3, H contains a full generalized regular subgroup B such that $L/(L \cap B)$ has infinitely many nonzero primary components. Let $\{p_1, p_2, \ldots\}$ be the set of all primes p for which $(L/(L \cap B))_p \neq 0$. Take $x_i \in L \setminus B$ such that $p_i x_i \notin B$ and let C_i be a subgroup of H maximal with respect to $B \subseteq C_i$ and $x_i \notin C_i$. Then $B \subseteq C = \bigcap_{i \in \mathbf{N}} C_i$ and C is generalized regular in H. Moreover the factor group H/C is naturally embedded into $\prod_{i \in N} (H/C_i)$, so, being a torsion group, it is in fact isomorphic to $\bigoplus_{i \in \mathbf{N}} (H/C_i)$. Applying the same method as in the proof of [5, Theorem 3.4], we get that H is not a Butler group. //

REMARK 10. If every pure subgroup of finite rank of a torsion free group H is a Butler group, then H is locally $*$-noetherian by Theorem 6. The example $H = Z^{\mathbf{N}}$ (see the remark at the end of [5]) shows that the converse of Proposition 9 does not hold in general. However, in the case of countable torsion free groups the Butler groups are just the locally $*$-noetherian groups.

THEOREM 11. A countable torsion free group H is a Butler group if and only if it is locally $*$-noetherian.

PROOF. With respect to Proposition 9, only the sufficiency requires a verification. Assume that H is not a Butler group. By [5, Theorem 3.4], H contains a pure subgroup of finite rank L which is not a Butler group. Using the technique of the proof of [5, Theorem 3.4] we can costruct a generalized regular subgroup B of H such that the factor group $L/(L \cap B)$ has infinitely many non-zero primary components. Thus H is not locally $*$-noetherian by Lemma 3. //

In order to prove that countable rank pure subgroups of Butler groups with linearly ordered type set are still Butler groups, we need the following result.

LEMMA 12. Let L be a pure subgroup of finite rank of a torsion free group H with linearly ordered type set $\tau(H)$. Then there is an L-high subgroup K of H such that for every $h \in H$ the factor group $(\langle h\rangle^* + L + K)/(L + K)$ is finite.

PROOF. The type set $\tau(L)$ of L is obviously finite and linearly ordered, $\tau(L) = \{\tau_1, \tau_2, \dots, \tau_r\}$, where $\tau_1 > \tau_2 > \dots > \tau_r$. Choose arbitrarily a basis M_1 of $H^*(\tau_1)$, add a basis N_1 of $L(\tau_1)$ and the linearly independent set $M_1 \cup N_1$ extends by M_2 to a basis $M_1 \cup M_2 \cup N_1$ of $H(\tau_1)$. Now there is a linearly independent set M_3 such that $M_1 \cup M_2 \cup M_3 \cup N_1$ is a basis of $H^*(\tau_2)$. Extending N_1 to a basis $N_1 \cup N_2$ of $L(\tau_2)$ we can extend the linearly independent set $M_1 \cup M_2 \cup M_3 \cup N_1 \cup N_2$ to a basis $M_1 \cup M_2 \cup M_3 \cup M_4 \cup N_1 \cup N_2$ of $H(\tau_2)$. Continuing in this way, we get a basis $(\cup (M_i : 1 \leqslant i \leqslant 2r)) \cup (\cup (N_i : 1 \leqslant i \leqslant r))$ of $H(\tau_r)$ and we can extend it by M_{2r+1} to a basis of H. Thus the set $N = \cup (N_i : 1 \leqslant i \leqslant r)$ is a basis of L and $N \cup M$, where $M = \cup (M_i : 1 \leqslant i \leqslant 2r + 1)$, is a basis of H. Setting $F_1 = \bigoplus_{x \in N} \langle x\rangle$, $F_2 = \bigoplus_{y \in M} \langle y\rangle$, $F = F_1 \oplus F_2$, $K = \langle F_2\rangle_*$, F is a maximal free subgroup of H and K is an L-high subgroup of H. Now it is easy to see that the claim of Lemma 12 is equivalent to the fact that, given $l + k \in F \setminus (F_1 \cup F_2)$, then $\tau^H(l + k) = \tau^H(l) \wedge \tau^H(k)$.

So suppose that the types $\tau^H(l)$ and $\tau^H(k)$ are different and that $\tau^H(l) > \tau^H(k)$. Then $h_p^H(l) > h_p^H(k)$ for an infinite set S_1 of primes and, obviously, $h_p^H(l + k) = h_p^H(k) < h_p^H(l)$ for every $p \in S_1$. Now the assumption $h_p^H(l + k) > h_p^H(l) = h_p^H(k)$ for an infinite set S_2 of primes yields the incomparability of $\tau^H(l + k)$ and $\tau^H(l + k)$ and $\tau^H(l)$, which contradicts the hypothesis. The case $\tau^H(l) < \tau^H(k)$ is similar.

Now we proceed to the case $\tau^H(l) = \tau^H(k) = \tau$. Then necessarily $\tau = \tau_i$ for some $i \in \{1, 2, \dots, r\}$ and so l linearly depends on $N_1 \cup N_2 \cup \dots \cup N_i$ and it has at least one non-zero component at N_i. Similarly, k linearly depends on $M_1 \cup M_2 \cup \dots \cup M_{2i}$ and it has at least one nonzero component at M_{2i}. Assuming that $\tau^H(l+k) > \tau_i$, we have $l+k \in H^*(\tau_i)$ and consequently, by the choice of $M_1, M_2, \ldots, M_{2i}$, $N_1, N_2, \ldots, N_i$, it linearly depends on the set $\big(\cup (M_j : 1 \leqslant j \leqslant 2i-1)\big) \cup \cup \big(\cup (N_j : 1 \leqslant j \leqslant i-1)\big)$, which gives an obvious contradiction with the above part, and we are through. //

We can now prove the following

THEOREM 13. Every countable pure subgroup of a Butler group H with linearly ordered type set is Butler.

PROOF. With respect to [5, Theorem 3.4], it suffices to show that any pure subgroup of finite rank of H is a Butler group. Assume, by way of contradiction, that H contains a pure subgroup of finite rank L which is not a Butler group. Since $r_p(L/p^\omega L) = 0$ (see the proof of Proposition 9), by Theorem 6 and Lemma 3, L contains a generalized regular subgroup B such that L/B is a torsion group with infinitely many non-zero primary components. If $P = \{p_1, p_2, \ldots\}$ is the set of all primes, denote $P_m = \{p_1, p_2, \ldots, p_m\}$, $m \in \mathbf{N}$. Now choose an L-high subgroup K of H by Lemma 12 and set

$$H_m = \langle B \oplus K \rangle_{P_m}, \qquad m \in \mathbf{N}\,.$$

It is easy to see that $\langle B \rangle_{P_m} = L \cap H_m$ and, from the exact sequence

$$0 \to (L \cap H_m)/B \to L/B \to L/(L \cap H_m) \to 0$$

it follows that, for each $m \in \mathbf{N}$, $L/(L \cap H_m)$ has infinitely many non-zero primary components, because $(L \cap H_m)/B$ has obviously only a finite number of ono-zero primary components.

Now it suffices to show that $H_1 \subseteq H_2 \subseteq \ldots$ is a $*$-chain of H, since in this case H is not locally $*$-noetherian, which contradicts Proposition 9. So, let $0 \neq h \in H$ be an arbitrary element. By Lemma 12 there is an $n \in \mathbf{N}$ such that $n\langle h \rangle_* \subseteq L \oplus K$. If $nh = l + k \in L \oplus K$, then the p-primary component of $\langle l \rangle_* / (\langle l \rangle_* \cap B)$ is non-zero for a finite set $S = \{p_{i_1}, \ldots, p_{i_r}\}$ of primes, B being generalized regular full subgroup of L. Let $m \in \mathbf{N}$ be such that P_m contains S and all prime divisors of n. If $p \notin P_m$ and the equation $p^j x = h$ is solvable in H, then $nx = l' + k' \in L \oplus K$, $p^j nx = nh = l + k$ and so $p^j l' = l$ and $l' \in \langle l \rangle_*$. Thus $sl' \in B$ for some $s \in \mathbf{N}$ divisible by the primes from S only. Consequently, $snx \in B \oplus K$ gives $x \in \langle B + K \rangle_{P_m} = H_m$, so that $\langle h \rangle_* \subseteq H_m$ and we are through. //

COROLLARY 14. (i) Every pure subgroup of finite rank of a Butler group with linearly ordered type set is completely decomposable.

(ii) Every countable pure subgroup of a homogeneous Butler group is completely decomposable.

PROOF. (i) Butler groups of finite rank with linearly ordered type set are completely decomposable (see [6]).

(ii) Pure subgroups of completely decomposable groups of finite rank are direct summands (see [7, § 66]). //

REFERENCES

[1] D. M. ARNOLD, *Pure subgroups of finite rank completely decomposable groups*, Proc. Oberwolfach, Lecture Notes n. 874, 1-31, Springer-Verlag, Berlin, 1981.

[2] L. BICAN, *Completely decomposable abelian groups any pure subgroup of which is completely decomposable*, Czech. Math. J., **24** (99) (1974), pp. 176-191.

[3] L. BICAN, *Splitting in abelian groups*, Czech. Math. J., **28** (103) (1978), pp. 356-364.

[4] L. BICAN, *Pure subgroups of Butler groups*, to appear on the Proc. Udine Conference on Abelian Groups and Modules.

[5] L. BICAN - L. SALCE, *Butler groups of infinite rank*, Proc. Honolulu, Lecture Notes n. 1006, 171-189, Springer-Verlag, Berlin, 1983.

[6] M. C. R. BUTLER, *A class of torsion free abelian groups of finite rank*, Proc. London Math. Soc., **15** (1965), pp. 680-698.

[7] L. FUCHS, *Infinite abelian groups*, Vol. I and II, Academic Press, London, New York, 1971 and 1973.

[8] L. PROCHÁZKA, *On p-rank of torsion free abelian groups of finite rank*, Czech. Math. J., **12** (87) (1962), pp. 3-43.

[9] L. PROCHÁZKA, *A note on completely decomposable torsion free abelian groups*, Comment. Math. Univ. Carolinae, **10** (1969), pp. 141-161

Manoscritto pervenuto in redazione il 22 maggio 1984.