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A Property of Multiplication in Sobolev Spaces. Some Applications.

TULLIO VALENT (*)

SUMMARY - Let Ω be an open subset of $\mathbb{R}^n$ having the cone property. In Sect. 1, Theorem 1 concerns the conditions on the numbers p, q, r and m for (the pointwise) multiplication is a continuous function of $W^{m,p}(\Omega) \times W^{m,q}(\Omega)$ into $W^{m,r}(\Omega)$. As a consequence of Theorem 1, multiplication is a continuous function of $W^{m,p}(\Omega) \times W^{m,q}(\Omega)$ into $W^{m,q}(\Omega)$ if the following conditions are satisfied: $q \leqslant p$, $mp > n$ and, if $p \neq q$ and the volume of Ω is infinite, $mq \leqslant n$. In particular one deduces the well known fact that $W^{m,p}(\Omega)$ is a Banach algebra if $mp > n$. In Sect. 2 we apply Theorem 1 in showing a property of the Nemytsky operator: see Theorem 2. The proof of such a property given in [2] (see Lemma 1) is not completely correct.

1. A property of multiplication in Sobolev spaces.

Let Ω be an open subset of $\mathbb{R}^n (n \geqslant 1)$, let m be an integer $\geqslant 1$ and let p, q, r be real numbers $\geqslant 1$. $W^{m,p}(\Omega)$ will denotes the vector space $\{v \in L^p(\Omega): D^\alpha v \in L^p(\Omega),\ o \leqslant |\alpha| \leqslant m\}$ with the norm $\|\cdot\|_{m,p}$ defined by

$$\|v\|_{m,p} = \left(\sum_{|\alpha| \leqslant m} \|D^\alpha v\|_{0,p}^p\right)^{1,p},$$

where $\|\cdot\|_{0,p}$ is the usual norm of $L^p(\Omega)$. We will put $D_i = \partial/\partial x_i$, $(i = 1, \dots, n)$.

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THEOREM 1. *Assume that Ω has the cone property, and that $p \geqslant r$, $q \geqslant r$ and*

$$\frac{m}{n} > \frac{1}{p} + \frac{1}{q} - \frac{1}{r}.$$

If the volume of Ω is infinite, assume further that $mp \leqslant n$ when $q \neq r$, that $mq \leqslant n$ when $p \neq r$ and that

$$\frac{m-1}{n} \leqslant \frac{1}{p} + \frac{1}{q} - \frac{1}{r} \quad \text{when } p \neq r,\ q \neq r.$$

Then, if $u \in W^{m,p}(\Omega)$ and $v \in W^{m,q}(\Omega)$, we have $uv \in W^{m,r}(\Omega)$ and there exists a positive number c independent of u and v such that $\|uv\|_{m,r} \leqslant \leqslant c\|u\|_{m,p}\, \|v\|_{m,q}$.

PROOF (by induction on m). As a first step we prove that the statement is true for $m = 1$. Then we suppose that $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ with $p \geqslant r$, $q \geqslant r$, $p \leqslant n$ if $q \neq r$, $q \leqslant n$ if $p \neq r$,

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{r} < \frac{1}{n} \tag{1.1}$$

and

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{r} \geqslant 0 \qquad (\text{if } p \neq r \neq q), \tag{1.2}$$

and we show that $uv \in W^{1,r}(\Omega)$ and that $\|uv\|_{1,r} \leqslant c\|u\|_{1,p}\, \|v\|_{1,q}$, where c is a positive number independent of u and v. Moreover we show that, if Ω has finite volume, the conclusion holds without assumption (1.2) and without the conditions $q \neq r \Rightarrow p \leqslant n$ and $p \neq r \Rightarrow q \leqslant n$.

If $q > r$ [resp. $p > r$] let α [resp. β] be the real number such that

$$\frac{1}{\alpha} + \frac{1}{q} = \frac{1}{r}, \quad \left[\text{resp. } \frac{1}{p} + \frac{1}{\beta} = \frac{1}{r}\right],$$

(i.e., $\alpha = qr/(q-r)$ and $\beta = pr/(p-r)$). Holder's inequality yields the following implications

$$(1.3)\quad \begin{cases} w_1 \in L^p(\Omega)\,, \quad p > r\,, \quad w_2 \in L^\beta(\Omega) \Rightarrow w_1 w_2 \in L^r(\Omega)\,, \\ \qquad\qquad \|w_1 w_2\|_{0,r} \leqslant c_1 \|w_1\|_{0,p} \|w_2\|_{0,\beta}\,, \\ w_1 \in L^q(\Omega)\,, \quad q > r\,, \quad w_2 \in L^\alpha(\Omega) \Rightarrow w_1 w_2 \in L^r(\Omega)\,, \\ \qquad\qquad \|w_1 w_2\|_{0,r} \leqslant c_1 \|w_1\|_{0,q} \|w_2\|_{0,\alpha}\,, \end{cases}$$

where c_1 is a positive number independent of w_1 and w_2.

Note that, by virtue of (1.1), $p \leqslant n$ implies $q > r$ and $q \geqslant n$ implies $p > r$.

A basic remark is that, if $p < n$ [resp. $q < n$], condition (1.1) is equivalent to the condition

$$\alpha < \frac{np}{n-p} \qquad \left[\text{resp. } \beta < \frac{nq}{n-q}\right],$$

while condition (1.2) is equivalent to the condition $p \leqslant \alpha$ [resp. $q \leqslant \beta$].

Hence, by the Sobolev imbedding theorem (see e.g. Adams [1], Theorem 5.4) the following continuous imbedding holds if $p \leqslant n$ [resp. $q \leqslant n$]:

$$(1.4)\qquad W^{1,p}(\Omega) \subseteq L^\alpha(\Omega)\,, \qquad [\text{resp. } W^{1,q}(\Omega) \subseteq L^\beta(\Omega)]\,.$$

We are now in a position to easily recognize that

$$(1.5)\quad \begin{cases} w_1 \in L^p(\Omega)\,, \quad w_2 \in W^{1,q}(\Omega) \Rightarrow w_1 w_2 \in L^r(\Omega)\,, \\ \qquad\qquad \|w_1 w_2\|_{0,r} \leqslant c_2 \|w_1\|_{0,p} \|w_2\|_{1,q}\,, \\ w_1 \in L^q(\Omega)\,, \quad w_2 \in W^{1,p}(\Omega) \Rightarrow w_1 w_2 \in L^r(\Omega)\,, \\ \qquad\qquad \|w_1 w_2\|_{0,r} \leqslant c_2 \|w_1\|_{0,q} \|w_2\|_{1,p}\,, \end{cases}$$

where c_2 is a positive number independent of w_1 and w_2.

Indeed, if $p \leqslant n$ and $q \leqslant n$, then by (1.1) we have $p > r$ and $q > r$; thus (1.5) is an immediate consequence of (1.3) and (1.4). If $q > n$ and $p \leqslant n$ [resp. $p > n$ and $q \leqslant n$] then $p = r$ and $W^{1,q}(\Omega) \subseteq L^\beta(\Omega)$ [resp. $q = r$ and $W^{1,p}(\Omega) \subseteq L^\alpha(\Omega)$], and therefore the first [resp. second] of the implications (1.5) follows from Hölder's inequality because $W^{1,q}(\Omega)$ [resp. $W^{1,p}(\Omega)$], by the Sobolev imbedding theorem, can be continuously imbedded into $L^\infty(\Omega)$, while the second [resp. first] of the implications (1.5) is a consequence of the second [resp. first] of the implications (1.3). Finally, if $p > n$ and $q > n$, then $p = r = q$ and $W^{1,p}(\Omega)$ can be continuously imbedded into $L^\infty(\Omega)$; thus (1.5) follows once more from Hölder's inequality.

Observe that, if Ω has finite volume, then $s_1 \leqslant s_2 \Rightarrow L^{s_2}(\Omega) \subseteq L^{s_1}(\Omega)$; therefore, in this case, the continuous imbedding (1.4) does not need condition (1.2), and the deduction of (1.5) does not need the implications $q \neq r \Rightarrow p \leqslant n$ and $p \neq r \Rightarrow q \leqslant n$.

In view of (1.5) we have

$$(1.6) \qquad uv \in L^r(\Omega), \qquad vD_i u \in L^r(\Omega), \qquad uD_i v \in L^r(\Omega), \qquad (i = 1, \ldots, n)\,.$$

Let now $(u_k)_{k\in\mathbf{N}}$ and $(v_k)_{k\in\mathbf{N}}$ be sequences in $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ and in $C^\infty(\Omega) \cap W^{1,q}(\Omega)$ respectively such that

$$(1.7) \qquad \lim_{k\to\infty} \|u_k - u\|_{1,p} = 0\,, \qquad \lim_{k\to\infty} \|v_k - v\|_{1,q} = 0\,.$$

Since by (1.5) we have

$$\begin{aligned}\|v_k D_i u_k - vD_i u\|_{0,r} &\leqslant \|v_k(D_i u_k - D_i u)\|_{0,r} + \\ &+ \|(v_k - v)\, D_i u\|_{0,r} \leqslant \|D_i u_k - D_i u\|_{0,p}\, \|v_k\|_{1,q} + \\ &+ \|v_k - v\|_{1,q}\, \|D_i u\|_{0,p} \leqslant \|u_k - u\|_{1,p}\, \|v_k\|_{1,q} + \|v_k - v\|_{1,q}\, \|u\|_{1,p}\,,\end{aligned}$$

and

$$\begin{aligned}\|u_k D_i v_k - uD_i v\|_{0,r} &\leqslant \|u_k(D_i v_k - D_i v)\|_{0,r} + \\ &+ \|(u_k - u)\, D_i v\|_{0,r} \leqslant \|D_i v_k - D_i v\|_{0,p}\, \|u_k\|_{1,p} + \\ &+ \|u_k - u\|_{1,p}\, \|D_i v\|_{0,q}\, \|v_k - v\|_{1,q}\, \|u_k\|_{1,p} + \|u_k - u\|_{1,p}\, \|v\|_{1,q}\,,\end{aligned}$$

then from (1.7) it follows that

$$(1.8) \qquad \lim_{k\to\infty} \|(v_k D_i u_k + u_k D_i v_k) - (vD_i u + uD_i v)\|_{0,r} = 0\,.$$

Using Holder's inequality we can immediately deduce from (1.8) that

$$\int_\Omega (vD_iu + uD_iv)\varphi\,dx + \int_\Omega uvD_i\varphi\,dx =$$

$$= \lim_{k\to\infty}\left[\int_\Omega (v_kD_iu_k + u_kD_iv_k)\varphi\,dx + \int_\Omega u_kv_kD_i\varphi\,dx\right] \quad \forall\varphi\in\mathfrak{D}(\Omega),$$

whence

$$\int_\Omega (vD_iu + uD_iv)\varphi\,dx + \int_\Omega uvD_i\varphi\,dx = 0 \qquad \forall\varphi\in\mathfrak{D}(\Omega), \tag{1.9}$$

because, being $D_i(u_kv_k) = v_kD_iu_k + u_kD_iv_k$, we have

$$\int_\Omega (v_kD_iu + u_kD_iv_k)\varphi\,dx + \int_\Omega u_kv_kD_i\varphi\,dx = 0$$

Note that (1.9) means that

$$vD_iu + uD_iv = D_i(uv);$$

hence $D_i(uv)\in L^r(\Omega)$ because of (1.6). Moreover by (1.6) uv belongs to $L^r(\Omega)$. Thus we conclude that $uv\in W^{1,r}(\Omega)$. Finally, from (1.5) we obtain

$$\|uv\|_{1,r}\leqslant c_3\left(\|uv\|_{0,r} + \sum_{i=1}^n \|vD_iu + uD_iv\|_{0,r}\right)\leqslant$$

$$\leqslant c_4(\|u\|_{1,p}\|v\|_{1,q}) + \sum_{i=1}^n (\|D_iu\|_{0,p}\|v\|_{1,q} + \|u\|_{1,p}\|D_iv\|_{0,q}) \leqslant c_5\|u\|_{1,p}\|v\|_{1,q},$$

where c_3, c_4 and c_5 are positive numbers independent of u and v.

As a second step of our induction argument, we now suppose that the statement of Theorem 1 is true for an $m(\geqslant 1)$ and we will prove that, consequently, it is true even when m is replaced by $m+1$. Accordingly, let p_1, q_1, r_1 be real numbers $\geqslant 1$ such that $p_1\geqslant r_1$, $q_1\geqslant r_1$ and

$$\frac{m+1}{n} > \frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1} \tag{1.10}$$

and let $u_1 \in W^{m+1,p}(\Omega)$ and $v \in W^{m+1,q}(\Omega)$. If the volume of Ω is infinite we also suppose that

$$(1.11)\qquad \frac{m}{n} \leqslant \frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1} \quad \text{in the case } p_1 \neq r_1 \neq q_1 ,$$

that $(m+1)p_1 \leqslant n$ in the case $q_1 \neq r_1$ and that $(m+1)q_1 \leqslant n$ in the case $p_1 \neq r_1$.

We begin by considering the case when $mp_1 \leqslant n$ and $mq_1 \leqslant n$, with $m > 1$. We set

$$\tilde{p}_1 = \frac{np_1}{n - p_1} \quad \text{and} \quad \tilde{q}_1 = \frac{nq_1}{n - q_1} .$$

By the Sobolev imbedding theorem, under our hypotheses, we have

$$u \in W^{m,\tilde{p}_1}(\Omega), \qquad v \in W^{m,\tilde{q}_1}(\Omega) ,$$

besides

$$D_i u \in W^{m,p_1}(\Omega), \qquad D_i v \in W^{m,q_1}(\Omega), \qquad (i = 1, \dots, n) .$$

Remark that, since

$$(1.12)\qquad \frac{1}{\tilde{p}_1} = \frac{1}{p_1} - \frac{1}{n} \quad \text{and} \quad \frac{1}{\tilde{q}_1} = \frac{1}{q_1} - \frac{1}{n} ,$$

(1.10) implies

$$(1.13)\qquad \frac{m}{n} > \frac{1}{\tilde{p}_1} + \frac{1}{q_1} - \frac{1}{r_1} \quad \text{and} \quad \frac{m}{n} > \frac{1}{p_1} + \frac{1}{\tilde{q}_1} - \frac{1}{r_1} .$$

If the volume of Ω is finite, this suffices to deduces (via the induction hypothesis) that

$$vD_i u \in W^{m,r_1}(\Omega), \qquad uD_i v \in W^{m,r_1}(\Omega)$$

and that

$$(1.14)\quad \|vD_i u\|_{m,r_1} \leqslant c_6 \|v\|_{m,\tilde{q}_1} \, \|D_i u\|_{m,p_1} , \quad \|uD_i v\|_{m,r_1} \leqslant c_6 \|u\|_{m,\tilde{p}_1} \, \|D_i v\|_{m,q_1} ,$$

where c_6 is a positive number independent of u, v and i; then, in view of the Sobolev imbedding theorem, there exists a positive number c_7 independent of u, v and i such that

$$(1.15)\quad \|vD_iu\|_{m,r_1}\leqslant c_7\|u\|_{m+1,p_1}\|v\|_{m+1,q_1},\quad \|uD_iv\|_{m,r_1}\leqslant c_7\|u\|_{m+1,p_1}\|v\|_{m+1,q_1}.$$

If the volume of Ω is infinite, it is not difficult to realize that our assumptions imply that

$$(1.16)\quad \frac{m-1}{n}\leqslant\frac{1}{\tilde{p}_1}+\frac{1}{q_1}-\frac{1}{r_1}\quad\text{and}\quad\frac{m-1}{n}\leqslant\frac{1}{p_1}+\frac{1}{\tilde{q}_1}-\frac{1}{r_1}.$$

Indeed, by (1.12) each of the conditions (1.16) is equivalent to (1.11) and therefore (1.16) holds if $p_1\neq r_1\neq q_1$; moreover (1.16) also holds if $p_1=r_1$ and if $q_1=r_1$, because (1.11) becomes $mq_1\leqslant n$ if $p_1=r_1$ and becomes $mq_1\leqslant n$ if $q_1=r_1$.

Furthermore, since $m\tilde{p}_1=(m+1)p_1$ and $m\tilde{q}_1=(m+1)q_1$, if the volume of Ω is infinite we have $m\tilde{p}_1\leqslant n$ in the case $q_1\neq r_1$ and $m\tilde{q}_1\leqslant n$ in the case $p_1\neq r_1$.

Therefore, by the induction hypothesis, estimates (1.15), and consequently (1.16), are true even when the volume of Ω is infinite.

By an analogous way as we obtained (1.15) we can show that $uv\in W^{m,r_1}(\Omega)$ and that a positive number c_8 independent of u and v exists such that

$$(1.17)\quad \|uv\|_{m,r_1}\leqslant c_8\|u\|_{m+1,p_1}\,\|v\|_{m+1,q_1}.$$

We now prove that estimates (1.15) and (1.17) hold also in the four cases: $mp_1>n$, $mq_1>n$, $p_1=n$ with $m=1$ and $q_1=n$ with $m=1$.

If $mp_1>n$ or $mq_1>n$ it is easily seen that all hypotheses of the statement of Theorem 1 are satisfied, so that (by the induction assumption) multiplication is a continuous operator from $W^{m,p_1}(\Omega)\times W^{m,q_1}(\Omega)$ to $W^{m,r_1}(\Omega)$. This is obvious if the volume of Ω is finite; if the volume of Ω is infinite we need only remark that, if $mp_1>n$ [resp. $mq_1>n$], then $q_1=r_1$ [resp. $p_1=r_1$].

Let now $p_1=n$ [resp. $q_1=n$] and $m=1$. If the volume of Ω is finite it may occurs that $q_1>r_1$ [resp. $p_1>r_1$]: in this case all hypotheses of the statement of Theorem 1 are again satisfied. If the volume of Ω is infinite we have $q_1=r_1$ [resp. $p_1=r_1$]. Note that, in the case when $p_1=n$, $q_1=r_1$ [resp. $q_1=n$, $p_1=r_1$] and $m=1$, the hypotheses of the statement of Theorem 1 are satisfied provided p_1 [resp. q_1] is replaced by $\tilde{p}_1$ [resp. $\tilde{q}_1$], where $\tilde{p}_1$ [resp. $\tilde{q}_1$] is any number $> p_1$ [resp. $> q_1$]. Thus, recalling that (by the Sobolev imbedding theorem) $W^{2,p_1}(\Omega)$ [resp. $W^{2,q_1}(\Omega)$] can be continuously imbedded into $W^{1,\tilde{p}_1}(\Omega)$ [resp. $W^{1,\tilde{q}_1}(\Omega)$], from the induction hypothesis we get that, if $p_1 = n$ [resp. $q_1 = n$], then multiplication is a continuous operator from $W^{2,p_1}(\Omega) \times \times W^{1,q_1}(\Omega)$ [resp. $W^{1,p_1}(\Omega) \times W^{2,q_1}(\Omega)$] to $W^{1,r_1}(\Omega)$.

This evidently shows what we wanted: that (1.15) and (1.17) are true also in the four cases $mp_1 > n$, $mq_1 > n$, $p_1 = n$ with $m = 1$, and $q_1 = n$ with $m = 1$.

Now, using the density of $C^\infty(\Omega) \cap W^{m,s}(\Omega)$ in $W^{m,s}(\Omega)$, $1 \leqslant s \in \mathbb{R}$, we can deduce, by a procedure quite analogous to the one developed in the first step, that $D_i(uv) = v\ D_i u + u\ D_i v$. Then, in view of (1.15) and (1.17), we can conclude that $uv \in W^{m+1,r_1}(\Omega)$ and that $\|uv\|_{m+1,r_1} \leqslant c_9 \|u\|_{m+1,p_1} \|v\|_{m+1,q_1}$, where c_9 is a positive number independent of u and v. Thus the induction argument is complete. □

2. A property of the Nemytsky operator.

Let N be an integer $\geqslant 1$ and let $(x, y) \mapsto f(x, y)$ be a real function defined in $\Omega \times \mathbb{R}^N$. For any function $\sigma: \Omega \to \mathbb{R}^N$ let $F(\sigma): \Omega \to \mathbb{R}$ be the function defined by setting

$$F(\sigma)(x) = f\big(x, \sigma(x)\big)\,, \qquad x \in \Omega\,. \tag{2.1}$$

We will denote by $C^m(\bar{\Omega} \times \mathbb{R}^N)$ the set of real functions defined in $\Omega \times \mathbb{R}^N$ which are restrictions to $\Omega \times \mathbb{R}^N$ of some C^m-function of $\mathbb{R}^n \times \mathbb{R}^N$ into $\mathbb{R}$.

THEOREM 2. *Assume that Ω is bounded and has the cone property, that $f \in C^m(\bar{\Omega} \times \mathbb{R}^N)$ and that $mp > n$. Then $\sigma \mapsto F(\sigma)$ is a continuous operator of $\big(W^{m,p}(\Omega)\big)^N$ into $W^{m,p}(\Omega)$.*

PROOF (by induction on m). We denote by $F_{x_i}(\sigma)$ and $F_{y_j}(\sigma)$, $(i = 1, \ldots, n;\ j = 1, \ldots, N)$, the real functions defined in Ω by setting

$$F_{x_i}(\sigma)(x) = \frac{\partial f}{\partial x_i}\big(x, \sigma(x)\big)\,, \qquad F_{y_j}(\sigma)(x) = \frac{\partial f}{\partial y_j}\big(x, \sigma(x)\big)\,.$$

We begin with the case $m = 1$. Accordingly, let $f \in C^1(\bar{\Omega} \times \mathbb{R}^N)$ and

$p > n$. By the Sobolev imbedding theorem each $v \in W^{1,p}(\Omega)$ is an equivalence class of functions containing a continuous and bounded function, which we still denote by v, and there exists a positive number $c_{1,p}$ independent of v such that

$$\|v\|_{0,\infty} \leq c_{1,p}\|v\|_{1,p} \qquad \forall v \in W^{1,p}(\Omega)\,, \tag{2.2}$$

where $\|\cdot\|_{0,\infty}$ is the norm of $L^\infty(\Omega)$. Then, if $\sigma \in \big(W^{1,p}(\Omega)\big)^N$, the equivalence classes $F(\sigma)$, $F_{x_i}(\sigma)$ and $F_{v_j}(\sigma)$ can be identified with continuous and bounded functions. Let $\sigma = (\sigma_j)_{j=1,\dots,N} \in \big(W^{1,p}(\Omega)\big)^N$ and let $(\sigma^k)_{k\in\mathbf{N}}$ be a sequence in $\big(C^\infty(\Omega) \cap W^{1,p}(\Omega)\big)^N$ which converges to σ in $\big(W^{1,p}(\Omega)\big)^N$, and therefore by (2.2) in $\big(L^\infty(\Omega)\big)^N$. We have

$$D_i F(\sigma^k) = F_{x_i}(\sigma^k) + \sum_{j=1}^N F_{v_j}(\sigma^k) D_i \sigma_j^k\,. \tag{2.3}$$

Since $(\sigma^k)_{k\in\mathbf{N}}$ converges to σ in $\big(L^\infty(\Omega)\big)^N$, then $\big(F(\sigma^k)\big)_{k\in\mathbf{N}}$, $\big(F_{x_i}(\sigma^k)\big)_{k\in\mathbf{N}}$ and $\big(F_{v_j}(\sigma^k)\big)_{k\in\mathbf{N}}$ converge in $L^\infty(\Omega)$ respectively to $F(\sigma)$, $F_{x_i}(\sigma)$ and $F_{v_j}(\sigma)$, and therefore $\Big(F_{x_i}(\sigma^k) + \sum_{j=1}^N F_{v_j}(\sigma^k) D_i \sigma_j^k\Big)_{k\in\mathbf{N}}$ converges in $L^p(\Omega)$ to $F_{x_i}(\sigma) + \sum_{j=1}^N F_{v_j}(\sigma) D_i \sigma_j$. Consequently, by Hölder's inequality we have, for any $\varphi \in \mathfrak{D}(\Omega)$,

$$\begin{aligned} &\int_\Omega \Big(F_{x_i}(\sigma) + \sum_{j=1}^N F_{v_j}(\sigma) D_i \sigma_j\Big)\varphi\, dx + \int_\Omega F(\sigma) D_i \varphi\, dx = \\ &= \lim_{k\to\infty}\Big[\int_\Omega \Big(F_{x_i}(\sigma^k) + \sum_{j=1}^N F_{v_j}(\sigma^k) D_i \sigma_j^k\Big)\varphi\, dx + \int_\Omega F(\sigma^k) D_i \varphi\, dx\Big]. \end{aligned} \tag{2.4}$$

Because of (2.3) we have for any $k \in \mathbf{N}$ and any $\varphi \in \mathfrak{D}(\Omega)$

$$\int_\Omega \Big(F_{x_i}(\sigma^k) + \sum_{j=1}^N F_{v_j}(\sigma^k) D_i \sigma_j^k\Big)\varphi\, dx + \int_\Omega F(\sigma^k) D_i \varphi\, dx = 0$$

and therefore, by (2.4), we obtain

$$\int_\Omega \Big(F_{x_i}(\sigma) + \sum_{j=1}^N F_{v_j}(\sigma) D_i \sigma_j\Big)\varphi\, dx + \int_\Omega F(\sigma) D_i \varphi\, dx = 0 \qquad \forall \varphi \in \mathfrak{D}(\Omega)\,,$$

which means

$$(2.5) \qquad D_i F(\sigma) = F_{x_i}(\sigma) + \sum_{j=1}^{N} F_{v_j}(\sigma) D_i \sigma_j .$$

Since the equivalence classes $F(\sigma)$, $F_{x_i}(\sigma)$ and $F_{v_j}(\sigma)$ contain a continuous and bounded function, from (2.5) if follows that $F(\sigma) \in W^{1,p}(\Omega)$.

To prove that $F\colon \big(W^{1,p}(\Omega)\big)^N \to W^{1,p}(\Omega)$ is continuous we need only remark that, if a sequence $(\sigma^k)_{k\in\mathbf{N}}$ converges to σ in $\big(W^{1,p}(\Omega)\big)^N$, then, by (2.2), $(\sigma^k)_{k\in\mathbf{N}}$ converges to σ in $\big(L^\infty(\Omega)\big)^N$, and therefore the sequences $\big(F(\sigma^k)\big)_{k\in\mathbf{N}}$, $\big(F_{x_i}(\sigma^k)\big)_{k\in\mathbf{N}}$ and $\big(F_{v_j}(\sigma^k)\big)_{k\in\mathbf{N}}$ converge in $L^\infty(\Omega)$ respectively to $F(\sigma)$, $F_{x_i}(\sigma)$ and $F_{v_j}(\sigma)$: then $\big(D_i F(\sigma^k)\big)_{k\in\mathbf{N}}$ converges to $D_i F(\sigma)$ in $L^p(\Omega)$ in view of (2.5), and thus $\big(F(\sigma^k)\big)_{k\in\mathbf{N}}$ converges to $F(\sigma)$ in $W^{1,p}(\Omega)$.

As a next step, we suppose that the statement of the theorem is true for an $m \geqslant 1$ and we show that, consequently, it holds when m is replaced by $m+1$. In order to do this, we assume that $f \in C^{m+1}(\bar{\Omega} \times \mathbf{R}^N)$, that $(m+1)p > n$ and that $\sigma \in \big(W^{m+1,p}(\Omega)\big)^N$, and we prove that $F(\sigma) \in W^{m+1,p}(\Omega)$ and that $\sigma \mapsto F(\sigma)$ is a continuous operator from $\big(W^{m+1,p}(\Omega)\big)^N$ to $W^{m+1,p}(\Omega)$.

Let us recall that (by the Sobolev imbedding theorem) each $v \in W^{m+1,p}(\Omega)$ can be indentified with a continuous function and there is a positive number $c_{m+1,p}$ independent of v such that $\|v\|_{0,\infty} \leqslant c_{m+1,p}\|v\|_{m+1,p}$ $\forall v \in W^{m+1,p}(\Omega)$. Then, by arguments quite similar to the ones given in the case $m = 1$, we can show that F is a continuous operator from $\big(W^{m+1,p}(\Omega)\big)^N$ to $W^{1,p}(\Omega)$ and that (2.5) holds.

It is now convenient to distinguish the cases $p > n$, $p = n$ and $p < n$.

If $p > n$, from the (induction) assumption it follows that F_{x_i} and F_v are continuous operators of $\big(W^{m,p}(\Omega)\big)^N$ into $W^{m,p}(\Omega)$; therefore F is a continuous operator of $\big(W^{m+1,p}(\Omega)\big)^N$ into $W^{m+1,p}(\Omega)$, in view of (2.5), because $W^{m,p}(\Omega)$ is a Banach algebra.

Let now $p = n$, and let $q \in \mathbf{R}$ be such that $n < q$. Thus $mq > n$ $\forall m \geqslant 1$ and (by the Sobolev imbedding theorem) $W^{m+1,n}(\Omega)$ can be continuously imbedded into $W^{m,q}(\Omega)$; furthermore, by the (induction) assumption, F_{x_i} and F_{v_j} are continuous operators of $\big(W^{m,q}(\Omega)\big)^N$ into $W^{m,q}(\Omega)$.

Note that, since $mq > n$, from Theorem 1 it follows that the pointwise multiplication is a continuous operator of $W^{m,n}(\Omega) \times W^{m,q}(\Omega)$ into $W^{m,n}(\Omega)$. Then we can deduce by (2.5) that $\sigma \mapsto D_i F(\sigma)$ is a continuous operator of $\big(W^{m+1,n}(\Omega)\big)^N$ into $W^{m,n}(\Omega)$. Consequently

$\sigma \mapsto F(\sigma)$ is a continuous operator of $(W^{m+1,n}(\Omega))^N$ into $W^{m+1,n}(\Omega)$.

Finally, let us consider the case $p < n$. In this case the condition $(m+1)p > n$ is equivalent to the condition

$$m\frac{np}{n-p} > n .$$

Now: F_{x_i} and F_{y_j} are continuous operators of $(W^{m,np/(n-p)}(\Omega))^N$ into $W^{m,np/(n-p)}(\Omega)$ (because of the induction hypothesis), $W^{m+1,p}(\Omega)$ can be continuously imbedded into $W^{m,np/(n-p)}(\Omega)$ (by the Sobolev imbedding theorem), and the pointwise multiplication is a continuous operator of $W^{m,p}(\Omega)\times W^{m,np/(n-p)}(\Omega)$ into $W^{m,p}(\Omega)$ (by Theorem 1). This implies, by (2.5), that $\sigma \mapsto D_iF(\sigma)$ is a continuous operator of $(W^{m+1,p}(\Omega))^N$ into $W^{m,p}(\Omega)$. Therefore, also in this case $\sigma \mapsto F(\sigma)$ is a continuous operator of $(W^{m+1,p}(\Omega))^N$ into $W^{m+1,p}(\Omega)$. □

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