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Spaces of Urelements.

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Dedicated to Professor PRACHAR on his 60-th birthday.

1. Introduction.

We will prove a topological characterization of a class of spaces which can be constructed from a space of urelemente. Space of this kindoccur, when independence results on the axiom of choice AC are derived by applying standard topological procedures to sets whose existence contradicts the AC. We will consider the problem of their characterization in the ordered Mostowski model only. There the space U of urelemente in its order topology is a source of many independence theorems. Our main result asserts:

In the Mostowski model a Hausdorff space X is a continuous one-to-one image of a Dedekind-finite subset of U^ω, if and only if every infinite set $Y \subseteq X$ has an infinite compact subset.

Our notation will follow [6] and [7]. When viewed from outside the model, the set U of urelemente is $\mathbb{Q}$. But in the model most subsets of $\mathbb{Q}$ are deleted so that U becomes a connected, locally compact dense and Dedekind-complete linearly ordered space. As is easily seen, every infinite subset of U contains a closed, nontrivial interval which is compact. So the above topological condition is satisfied. It was first introduced by Bankston [1] under the name antianticompact. It is a hereditary property. We observe that in the presence of AC there are no antianticompact T_2 spaces.

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1.1. Lemma. If $P(\omega)$ is well orderable, then every antianticompact T_2 space is Dedekind-finite.

Proof. Let X be antianticompact, T_2 and countable. Then the topology $\mathbf{X}$ is well orderable, too. Therefore we may perform the usual argument of constructing an infinite discrete subset of X, thereby obtaining a contradiction to antianticompactness. Q.E.D.

A similar proof shows, that there are no antidiscrete T_2 spaces, either. A related result is due to J. Tong [9]: AC^ω implies, that there are no antianticompact R_0 spaces with an ascending chain of open sets.

On the other hand, in the Mostowski model (where $P(\omega)$ is well orderable) there are many antianticompact T_2 spaces (Dedekind-finite, of course).

1.2 Proposition. If a T_2 space X is a continuous one to one image of a Dedekind-finite subset D of U^ω then Y is antianticompact.

Proof. Since it is easily verified that a continuous one to one image of an antianticompact space is antianticompact, it suffices to show that D is antianticompact. Be $T \subseteq D$ infinite and in $\Delta(e)$ for some finite $e \subseteq U$ ($\Delta(e)$ is the class of all sets which are supported by e). It was observed in [3], that there is a one to one mapping $f: I \to T$ in $\Delta(e)$, where $I \subseteq U$ is an open interval between points in $\Delta(e)$. A permutation argument shows, that f is of the form $f(u) = (f_i(u))_{i\in\omega}$, where f_i is the identity map or f_i is a constant $a \in e$. Hence f induces on I the order topology which is antianticompact. So T contains an infinite antianticompact subset. Q.E.D.

2. Main result.

It was observed in [3] that the coarsest T_2 topology on U which is supported by φ is the order topology $\mathbf{U}_0$. We extend this result to sets of the form $X = \mathrm{orb}_e\, x = \{px: p \in \mathrm{fix}\, e\}$. If supp (x) denotes the least support of X and $\mathrm{supp}\, x \setminus e = \{a_i: i \in n\}$, $a_0 < a_1 < \ldots < a_{n-1}$, then there is a mapping $f: \mathrm{orb}_e\, x \to U^n$ which is defined through $f = \{p(x, \boldsymbol{a}): p \in \mathrm{fix}\, e\}$ $(\boldsymbol{a} = (a_i)_{i\in n})$; $f \in \Delta(e)$ (i.e.: e supports f). It is one to one. This canonical mapping induces a natural topology $\mathbf{X}_0$ on X which is generated by the product topology $\mathbf{U}_0^n$ on U^n.

2.1 Lemma. Be X, $\mathbf{X}_0$ and e as above. If $\mathbf{X} \in \Delta(e)$ is a T_2 topology on X then $\mathbf{X}_0 \subseteq \mathbf{X}$.

PROOF. By the foregoing remarks we may assume that $X = \mathrm{orb}_e\, \boldsymbol{a}$, where $\boldsymbol{a}: n \to U\setminus e$ is increasing (i.e.: $\boldsymbol{a}(i) < \boldsymbol{a}(i+1)$). Hence X is the set of all increasing functions $x \in \prod_{i\in n} I_i$, where I_i is an interval between two consecutive elements of e. $\boldsymbol{X}_0$ is the subspace topology which is inherited from U_0^n. It is generated by the subbase sets $O(i, a) = \{x \in X: x(i-1) < a < x(i)\}$, where $a \in U$ and $0 \leqslant i \leqslant n$ ($x(-1)$ and $x(n)$ define void clauses). If $x \in O(i, a)$, then $O(i, a) = \mathrm{orb}_{e\cup\{a\}}(x)$. From this it follows with a permutation argument, that if $O(i, a) \notin \boldsymbol{X}$ for some i and some $a \in I_i$, then $O(i, b)^0 = \emptyset$ for all $b \in \mathrm{orb}_e\, a = I_i$ (for the other values of b it follows from the definition, that $O(i, b) = \emptyset$ or $O(i, b) = X$). In order to obtain a contradiction, we assume the latter and observe that $0^- \cap X^\varrho \neq \emptyset$ whenever $0 \in \boldsymbol{X}$ is nonempty and $^-$ and ϱ (boundary operator) are formed with respect to $(oU)^n$ (oU is the order compactification of $(U, \boldsymbol{U}_0)$). For if 0 is in $\Delta(f)$, then there is a $x \in 0\setminus \bigcup\{O(i, a): a \in f \cap I_i\}$ ($O(i, a)^0 = \emptyset$) and $\mathrm{orb}_f\, x$ (which is an intersection of at most n sets $O(j, a)$, $a \in f \cap I_j$ and $j \neq i$) has boundary points in X^ϱ. It follows from compactness that

$$C = \bigcap \{O^- \cap X^\varrho: x \in O \in \boldsymbol{X}\}$$

is nonempty and closed. Since subsets of $(oU)^n$ are definable from a finite subset of U and the ordering relation on U, every nonempty closed subset of $(oU)^n$ has a maximal element in the lexicographic order. Applied to C this yields a mapping $f: X \to X^\varrho$ in $\Delta(e)$ such that $f(x) \in O^-$ if $x \in O \in \boldsymbol{X}$. Since $|\mathrm{supp}\,(fx)\setminus e| < n = |\mathrm{supp}\,(x)\setminus e|$, a standard permutation argument assures that there is a $y \in X^\varrho$ such that the set $f^{-1}(y)$ is infinite. We choose $3^n + 1$ elements x_i of this set and get by T_2 pairwise disjoint sets O_i, $x_i \in O_i \in \boldsymbol{X}$. Then $y \in \bigcap_i O_i^-$. This gives a contradiction (hence all sets $O(i, a)$ are in $\boldsymbol{X}$). For if $y \in A^-$, $A \subseteq X^- \subseteq (oU)^n$, then for some $R_i \in \{<, =, >\}$ and some a_i, $y(i) R_i a_i$, A^- contains the set $\{x \in X^-: \forall i \in n: y(i) R_i X(i) R_i a_i\}$, whence at most 3^n pairwise disjoint subsets of X^- can have a common element y in their closures (a similar estimate holds for $(oU)^n$). Q.E.D.

We next improve this lemma in the case of an antianticompact topology on X.

2.2. LEMMA. Let X and e be as above and assume that $\boldsymbol{X} \in \Delta(e)$ is an antianticompact T_2 topology on X. Then $\boldsymbol{X} = \boldsymbol{X}_0$.

PROOF. According to 1.2, $\boldsymbol{X}_0$ is antianticompact. In view of lemma 2.1 we prove that $\boldsymbol{X} \subseteq \boldsymbol{X}_0$. Be $x \in 0 \in \boldsymbol{X}$ let $f \supseteq e$ be a support of x and 0 and fix c_i, d_i, $i \in n$, such that $c_i < d_i < c_{i+1}$, $x \in P = \prod_{i\in n}]c_i, d_i[\subseteq X$ and $]c_i, d_i[\cap g = \{x(i)\}$, where $g = f \cup g_0$, $g_0 = \{c_i, d_i : i \in n\}$. This is possible, since X is open in $(U^n, \boldsymbol{U}_0^n)$. We will prove that $Q = O \cap P = P \in \boldsymbol{X}_0$. We set for $E \subseteq n$ and $y \in P$, $L(E, y) = \{z \in P : z|n\setminus E = y|n\setminus E\}$ and prove by induction on $|E|$ that $L(E, x) \subseteq Q$. $|E| = 0$ says $x \in Q$ and $|E| = n$ gives $L(n, x) = P \subseteq Q$. Assume that $L(i, x) = L(\{0, \ldots, i-1\}, x) \subseteq Q$. We show that for each $y \in L(i, x) L(\{i\}, y) \subseteq Q$, whence $L(i+1, x) = L(i \cup \{i\}, x) \subseteq Q$. To this end we observe, that $\boldsymbol{X}/L(\{i\}, y)$ is a T_2 topology on $]c_i, d_i[$ in $\Delta(e \cup g_0 \cup y'n\setminus\{i\})$ and since $(e \cup g_0 \cup y'n\setminus\{i\}) \cap]c_i d_i[= \emptyset$, we may conclude from [3] that $\boldsymbol{X}|L$ is one of the following topologies: discrete, half open interval (these 3 topologies are anticompact by [8]) or the order topology which is the only antianticompact one (and therefore it is $X|L$). We next consider the interval $]a_i, b_i[$ around $y(i) = x(i)$ which corresponds to the connectedness component of $L(\{i\}, y) \cap Q$ around y: $a_i < x(i) < b_i$ and a_i, b_i are in $[c_i, d_i] \cap \Delta(g \cup y'n)$. Since $]c_i, d_i[\cap \Delta(g \cup y'n) = \{x(i)\}$, $a_i = c_i$, $b_i = d_i$ and $L(\{i\}, y) \cap Q = L(\{i\}, y)$. Q.E.D.

Combining these results we may conclude:

2.3 THEOREM. In the Mostowski model a Hausdorff space is antianticompact, if and only if it is a continuous one-to-one image of a Dedekind-finite subset of U^ω, U with the order topology.

PROOF. We consider an antianticompact T_2 space $(X, \boldsymbol{X})$ in $\Delta(e)$. By 2.2 to each orbit $o = \mathrm{orb}_e\, x$ there corresponds naturally an embedding (topologically) $f_0 : o \to U^{n(0)}$ where $f_0' o$ is homeomorphic to some orbit $\mathrm{orb}_e\, \boldsymbol{a}$, $\boldsymbol{a} \in U^{n(0)}$. Since the set of all orbits $\mathrm{orb}_e\, \boldsymbol{a}$, $\boldsymbol{a} \in U^n$, $n \in \omega$ is countable, also the set O of all e-orbits of X is countable, for otherwise there are uncountably many orbits $o(\alpha)$, $\alpha \in \omega_1$, with the same image $f'_{o(\alpha)} o(\alpha) = \mathrm{orb}_e\, \boldsymbol{a}$, whence $\{f^{-1}_{o(\alpha)}(\boldsymbol{a}) : \alpha \in \omega_1\}$ would be an uncountable subset of X, contradicting 1.1. Consequently the topological sum D of O can be embedded in U^ω and the functions f_0^{-1} induce a continuous bijective mapping $f : D \to X$. Since X is Dedek indfinite, so is D. This proves « only if ». The converse implication is 1.2. Q.E.D.

It follows, than in the Mostowski model finite products of antianticompact T_2 spaces are antianticompact.

3. Additional remarks.

Using lemma 2.1, we can answer a question from [4] concerning the following properties of a topological space $(X, \mathbf{X})$. X is $A1$, if for every open covering $\mathbf{0}$ there is a neighborhood choice function $f: X \to \mathbf{0}$ such that $x \in f(x)$. X is $A2$, if there is a $f: X \to \mathbf{X}$ such that $x \in f(x)$ and $f'X$ refines $\mathbf{0}$. AC implies that every space is $A1$, and conversely, the assertion « every T_2 space is $A1$ » implies AC and « every T_2 space is $A2$ » implies MC (every set is a union of a well orderable family of finite sets). In ZF^0 $AC \Rightarrow MC \Rightarrow PW$, where ZF^0 is set theory minus foundation and PW asserts that the power set of an ordinal is well orderable, in ZF (ZF^0 + foundation) $PW \Rightarrow AC$, but in ZF^0 $PW \not\Rightarrow MC$, $MC \not\Rightarrow AC$. In [4] it was shown that the assertion « every hereditarily $A2$ T_2-space is a union of a well orderable family of discrete sets (property $D2$) » is in strength between MC and PW. The problem was left open, if it implies MC (in ZF^0, of course). The following partial answer was provided: In the ordered Mostowski model every hereditarily $A1 + T_2$ space is well orderable.

3.1 THEOREM. In the ordered Mostowski model every hereditarily $A2$ T_2-space is $D2$. Hence this assertion does not imply MC in ZF_0.

PROOF. Be $(X, \mathbf{X}) \in \Delta(e)$. Since the family of all orbits $\mathrm{orb}_e(x)$, $x \in X$, is well orderable, it suffices to show that $\mathrm{orb}_e\, x$ is discrete. As was observed in 2.2, $\mathrm{orb}_e\, x$ is covered by a family of open sets $P = \prod_{i \in n}]c_i, d_i[$, where $P \subseteq \mathrm{orb}_e\, x$. We show that P is discrete. Since by lemma 2.1 $\mathbf{X}_0|P \subseteq \mathbf{X}|P$, $\mathbf{0} = \{0 \in \mathbf{X}|P\colon \forall i \in n\colon \sup O|i < d_i\}$ is an open cover of $P(O|i = \{x(i)\colon x \in 0\} \subseteq [c_i, d_i])$. Let f be an $A2$ mapping for $\mathbf{0}$ in $\Delta(h)$ and consider $f_i(y) = \sup f(y)|i \in (h \cup y'n) \cap]c_i, d_i[$. For some y and all i $h \cap]c_i, d_i[< y(i) < d_i$. Hence $f_i(y) = y(i)$ for all i and therefore $V(y) = \{z \in P\colon \forall i\colon z(i) \leqslant y(i)\}$ is a neighborhood of these points y. Since $P = \mathrm{orb}.\, y$ and $\mathbf{X}|P \in \Delta(g)$, where $g = e \cup \cup \{c_i, d_i\colon i \in n\}$, $V(y)$ is a neighborhood of y for every point $y \in P$. Similarly $W(y) = \{z \in P\colon z(i) \geqslant y(i)$ for all $i\}$ is a neighborhood of y, whence $\{y\} = V(y) \cap W(y)$ is isolated. Q.E.D.

As was observed in [4], there are compact (hence $A1$) T_2 spaces in the Mostowski model which are not $D2$.

While antianticompact T_2 spaces do not exist in the presence of AC, the large class of anticompact spaces does not conflict with AC. A space is anticompact, if compact subsets are finite (example: discrete spaces or D-finite subsets of $\mathbf{R}$). We next investigate, if nondiscrete first countable anticompact T_2 spaces can exist. We shall relate this question to the countable multiple choice axiom MC^ω (if $(E_n)_{n\in\omega}$ is a countable sequence of nonempty sets, there is a sequence $(F_n)_{n\in\omega}$ of finite sets such that $\emptyset \neq F_n \subseteq E_n$. In ZF^0, $MC^\omega \not\Rightarrow AC^\omega$ (unknown for ZF) and $AC^\omega \Rightarrow MC^\omega$ (AC^ω: countable AC).

3.2 LEMMA. (1) In $ZF_0 + MC^\omega$ a T_2 space with a countable local base is a Kelley k-space (A is closed, if and only if $A \cap K$ is closed, K all compact sets).

(2) In ZF^0 anticompact $T_2 + k$-spaces are discrete.

PROOF. For (2) see [1]. (1) is a modification of standard arguments. Be $p \in A^- \setminus A$ and consider a neighborhood base $(U_n)_{n\in\omega}$ at p, $U_n \supseteq U_{n+1}$. By MC^ω there is a sequence $(F_n)_{n\in\omega}$ of finite sets such that $\emptyset \neq F_n \subseteq U_n \cap A$. $K = \{p\} \cup \bigcup_{n\in\omega} F_n$ is compact, because the open sets containing p are cofinite in K, and $p \in (K \cap A)^-$. So $K \cap A$ is not closed. Q.E.D.

3.3 THEOREM. MC^ω is equivalent to the proposition that anticompact metrizable topological groups are discrete.

PROOF. If MC^ω holds, we get « discrete » by an application of the previous lemma. For the proof of the converse, we will start with a counterexample $(E_n)_{n\in\omega}$ of PMC^ω, $E_n \cap E_m = \emptyset$ for $n \neq m$, and construct an anticompact metric group with no isolated points. PMC^ω is the axiom that there is an infinite set $A \subseteq \omega$ and a sequence $(F_n)_{n\in A}$ of finite sets such that for $n \in A$, $\emptyset \neq F_n \subseteq E_n$ (« P » stands for « partial »). As was shown in [5], $MC^\omega \Leftrightarrow PMC^\omega$. We set $E = \bigcup_{n\in\omega} E_n$, $E(n) = \bigcup_{m\in n} E_m$ and $X = [E]^{<\omega}$, the system of all finite subsets of E, $X_n = [E(n)]^{<\omega}$. On X we consider the Baire-metric: $d(x,x) = 0$ and $d(x,y) = 1/(n+1)$, if $x \cap E(n) = y \cap E(n)$ and $x \cap E_n \neq y \cap E_n$. The group-multiplication is the symmetric difference $(A\setminus B) \cup (B \setminus A)$. As is easily verified, X is a metric topological group without isolated points. We show that X is anticompact. Let K be compact. First we observe, that X_n is closed and discrete, since $d(x,y) > 1/(m+1)$, whenever $x \in X_n$, $y \in X_m$, $n \leqslant m$, and because $X = \bigcup_{n\in\omega} X_n$. Hence

$K \cap X_n$ is finite. This implies that $A = \{n \in \omega : K \cap (X_{n+1} \setminus X_n) \neq \emptyset\}$ is finite, whence $K = \bigcup_{n \in A} (K \cap X_{n+1})$ is finite, too. For if $n \in A$, then $F_n = E_n \cap (\cup K)$ is nonempty and as $F_n \subseteq \bigcup (K \cap X_{n+1})$, F_n is finite. So $(F_n)_{n \in A}$ would define a PMC-function of $(E_n)_{n \in \omega}$, a contradiction.
Q.E.D.

In [2] the same construction with finite sets E_n was used to obtain a σ-compact group which is not Lindelöf. 3.3 shows, that the finiteness of the sets E_n was essential there.

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