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Existence and Uniqueness Theorems for Third Order Boundary Value Problems.

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The second order boundary value problems have been studied extensively. The existence and uniqueness of certain higher order boundary value problems also deserve a good deal of attention since they occur in a wide variety of applications. The purpose of this paper is to show existence and uniqueness results for third order differential equations of the type

$$y''' = f(x, y, y', y''), \tag{1}$$

or

$$y''' = f(x, y, y'), \tag{2}$$

where $f \in C[[0, 1]\times R\times R\times R, R]$ under the following types of boundary conditions:

$$y(0) = y_0, \quad y'(0) = \bar{y}_0, \quad y'(1) = y_1; \tag{3}$$

or

$$y(0) = y_0, \quad y''(0) - hy'(0) = 0, \quad y''(1) + ky'(1) = 0, \quad h,\ k \geqslant 0,\ h + k > 0. \tag{4}$$

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Our method is similar to the one used in [1] for fourth order boundary value problems in the following way:

We transform equation (1) or (2) into a second order integro-differential equation. Then we apply known results for second order boundary value problems and Schauder's fixed point theorem, to obtain exsistence and uuiqueness results for third order boundary value problem (1) or (2).

We present some results, which help to simplify the proofs of our main results.

LEMMA 1. The homogeneous boundary value problem

$$u'' = 0 ,$$

together with

$$u(a) = u(b) = 0 ,$$

has the Green's function $G(x, t)$, defined by

$$G(x, t) = -(b-a)^{-1}\begin{cases} (b-x)(t-a), & a \leqslant t \leqslant x \leqslant b \\ (b-t)(x-a), & a \leqslant x \leqslant t \leqslant b, \end{cases}$$

with the following properties:

$$\int_a^b |G(x, t)|\, dt \leqslant \frac{(b-a)^2}{8} ,$$

and

$$\int_a^b |G_x(x, t)|\, dt \leqslant \frac{b-a}{2} .$$

LEMMA 2 [4]. If $y(x) \in C^1[0, 1]$ and $y(0) = 0$, then

$$\int_0^1 y^2(x)\, dx \leqslant \frac{4}{\pi^2} \int_0^1 [y'(x)]^2 dx .$$

LEMMA 3. If $y(x) \in C^1[0,1]$ and $y(0) = y(1) = 0$, then

$$\int_1^1 y^2(x)\,dx \leqslant \frac{1}{\pi^2}\int_0^1 [y'(x)]^2 dx .$$

LEMMA 4 [2]. If $y(x) \in C^1[0,1]$ and $y(a) = y(b) = 0$, then

$$\sup_{0\leqslant x\leqslant 1} |y(x)| \leqslant \frac{(b-a)^{\frac{1}{2}}}{2}\left[\int_a^b [y'(x)]^2 dx\right]^{\frac{1}{2}} .$$

LEMMA 5 [2]. The second order linear boundary value problem

$$y'' = a(x)y + b(x) ,$$

$$y(a) = y(b) = 0 ,$$

where $a(x)$, $b(x) \in C[a, b]$ and $a(x) \geqslant -m > -\pi^2/(b-a)^2$ has a unique solution $y(x)$ and

$$\sup|y(x)| \leqslant \frac{\pi(b-a)^2}{2[\pi^2 - m(b-a)^2]}\sup|b(x)| , \qquad a \leqslant x \leqslant b .$$

LEMMA 6 [3]. The second order boundary value problem

$$y'' = a(x)y + b(x)$$

$$y'(0) - hy(0) = 0 , \qquad y'(1) + ky(1) = 0 , \qquad h, k \geqslant 0,\ h + k > 0 ,$$

where $a(x)$, $b(x) \in C[0, 1]$ and $a(x) \geqslant m > 0$, has a unique solution $y(x)$ and

$$\sup|y(x)| \leqslant \frac{1}{m}\sup|b(x)|, \qquad 0 \leqslant x \leqslant 1 .$$

LEMMA 7. Consider the third order linear differential equation

(5) $$y''' = a(x)y'' + b(x)y' + c(x)y + d(x) ,$$

with

$$y(0) = y'(0) = y'(1) = 0\,, \tag{6}$$

where $a(x) \in C^1[0,1]$, $b(x)$, $c(x)$, $d(x) \in C[0,1]$, $a'(x) \leqslant a_0$, $b(x) \geqslant -b_0$, and $|c(x)| \leqslant c_0$. Suppose

$$4c_0 + \pi(a_0 + 2b_0) < 2\pi^3,$$

then any solution $y(x)$ of (5), (6) satisfies the following:

$$\sup|y(x)| \leqslant \frac{\pi^2}{2\pi^3 - \pi(a_0 + 2b_0) - 4c_0} \sup|d(x)|\,, \qquad 0 \leqslant x \leqslant 1\,.$$

and

$$\sup|y'(x)| \leqslant \frac{\pi^2}{2\pi^3 - \pi(a_0 + 2b_0) - 4c_0} \sup|d(x)|\,, \qquad 0 \leqslant x \leqslant 1\,.$$

PROOF. On multiplying (5) by $y'(x)$, and integrating the results from 0 to 1, and using $y'(0) = y'(1) = 0$, we have

$$-\int_0^1 [y''(x)]^2\,dx = \int_0^1 a(x)y'y''\,dx + \int_0^1 b(x)(y')^2\,dx + \int_0^1 c(x)yy'\,dx + \int_0^1 d(x)y'\,dx\,,$$

or

$$\int_0^1 [y''(x)]^2\,dx = \tfrac{1}{2}\int_0^1 a'(x)(y')^2\,dx - \int_0^1 b(x)(y')^2\,dx - \int_0^1 c(x)yy'\,dx - \int_0^1 d(x)y'\,dx$$

Now applying Lemmas 2, 3 and the Cauchy-Schwartz inequality we obtain

$$\int_0^1 [y''(x)]^2 dx \leqslant \frac{1}{2}\,a_0\int_0^1 (y')^2 dx + b_0\int_0^1 (y')^2 dx +$$

$$+ c_0\left[\int_0^1 y^2 dx\right]^{\frac{1}{2}}\left[\int_0^1 (y')^2 dx\right]^{\frac{1}{2}} + \sup|dx|\left[\int_0^1 (y')^2 dx\right]^{\frac{1}{2}}$$

or

$$\int_0^1 [y''(x)]^2 dx \leqslant \frac{\pi(a_0 + 2b_0) + 4c_0}{2\pi^3} \int_0^1 [y''(x)]^2 dx + \frac{1}{\pi} \sup|d(x)| \left[\int_0^1 [y''(x)]^2 dx\right]^{\frac{1}{2}}$$

or

$$\left[\int_0^1 [y''(x)]^2 dx\right]^{\frac{1}{2}} \leqslant \frac{2\pi^2}{2\pi^3 - \pi(a_0 + 2b_0) - 4c_0} \sup|d(x)|, \qquad 0 \leqslant x \leqslant 1 .$$

Then from Lemma 4

$$\sup|y'(x)| \leqslant \frac{\pi^2}{2\pi^3 - \pi(a_0 + 2b_0) - 4c_0} \sup|d(x)|, \qquad 0 \leqslant x \leqslant 1 .$$

Also from $y(0) = 0$, we have

$$y(x) = \int_0^x y'(t)\, dt$$

or

$$\sup |y(x)| \leqslant sup\ |y'(x)|, \qquad 0 \leqslant x \leqslant 1 .$$

Proof is complete.

LEMMA 8. Suppose all conditions of Lemma 7 are satisfied. Then equation (5) with boundary condition (3) has a unique solution.

PROOF. First we show that if (5), (3) has a solution, it is unique. Suppose $u(x)$ and $v(x)$ are solutions of (5), (3). Let $w(x) = u(x) - v(x)$, then

$$w'''(x) = a(x) w''(x) + b(x) w'(x) + c(x) w(x) \tag{7}$$

and

$$w(0) = w'(0) = w'(1) = 0 . \tag{8}$$

Problem (7), (8) is a form of (5), (6), Then by Lemma 7

$$\sup_{0 \leqslant x \leqslant 1} |w(x)| \leqslant 0 .$$

This implies that $u(x) = v(x)$. Thus if (5), (3) has a solution it must be unique.

To show existence, let $y_1(x)$, $y_2(x)$, and $y_3(x)$ be solutions of the following initial value problems, respcetively:

$$\text{(i)} \quad y_1''' = a(x)y_1'' + b(x)y_1' + c(x)y_1 + d(x)\,,$$
$$y_1(0) = y_0,\ y_1'(0) = y_1''(0) = 0;$$

$$\text{(ii)} \quad y_2''' = a(x)y_2'' + b(x)y_2' + c(x)y_2\,,$$
$$y_2'(0) = \bar{y}_0,\ y_2(0) = y_2''(0) = 0\,;$$

$$\text{(iii)} \quad y_3''' = a(x)y_3'' + b(x)y_3' + c(x)y_3\,,$$
$$y_3''(0) = 1,\ y_3(0) = y_3'(0) = 0\,.$$

We notice that $y_1(x)$, $y_2(x)$, and $y_3(x)$ exist and are unique. Moreover, $y_3'(1) \neq 0$, because if $y_3'(1) = 0$, then equation (iii) and

$$y_3(0) = y_3'(0) = y_3'(1) = 0$$

imply that $y_3(x) \equiv 0$, which contradicts the fact that $y_3''(0) = 1$. Therefore by linearity

$$y(x) = y_1(x) + y_2(x) + \frac{y_1 - y_1'(1) - y_2'(1)}{y_3'(1)} \cdot y_3(x)$$

is the solution of (5), (3).

THEOREM 1. Suppose f is bounded on $[0, 1] \times \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$. Then problem (1), (3) has a solution.

PROOF. Let $y' = u$. Then

$$y(x) = y_0 + \int_0^x u(t)\,dt\,. \tag{9}$$

Using $u = y'$ and (9), problem (1), (3) becomes

$$u'' = f\Big(x, y_0 + \int_0^x u(t)dt, u, u'\Big) \tag{10}$$

with

$$u(0) = \bar{y}_0, \; u(1) = y_1, \tag{11}$$

or

$$u(x) = \bar{y}_0 + x(y_1 - \bar{y}_0) + \int_0^1 G(x, t) f\left[t, y_0 + \int_0^t u(s)\, ds, u(t), u'(t)\right] dt . \tag{12}$$

Define an operator T on $E = C[[0, 1], R]$ by

$$Tu(x) = \bar{y}_0 + x(y_1 - \bar{y}_0) + \int_0^1 G(x, t) f\left[t, y_0 + \int_0^t u(s)\, ds, u(t), u'(t)\right] dt . \tag{13}$$

If $U \in E$, the norm is defined by

$$|u|_E = \max |u(x)| + \max |u'(x)|, \qquad 0 \leqslant x \leqslant 1 .$$

Let M be the bound of f on $[0, 1] \times R \times R \times R$. Then from (13) and the estimates on $G(x, t)$ and $G_x(x, t)$, it follows that

$$|Tu(x)| \leqslant |\bar{y}_0| + |y_1 - \bar{y}_0| + \tfrac{1}{8} M , \tag{14}$$

and

$$|T' u(x)| \leqslant |y_1 - y_0| + \tfrac{1}{2} M .$$

Hence, T maps the closed, bounded, and convex set

$$B = \{u \in E: |u(x)| \leqslant |\bar{y}_0| + |y_1 - \bar{y}_0| + \tfrac{1}{8} M, \; |u'(x)| \leqslant |y_1 - \bar{y}_0| + \tfrac{1}{2} M\}$$

into itself. Moreover, since $|(Tu)''| \leqslant M$, T is completely continuous by Ascoli's theorem. The Schauder's fixed point theorem then yields a fixed point of T which is a solution of (10), (11). Since $u = y'$, then (9) is a solution of (1), (3).

THEOREM 2. Suppose there exist positive numbers r and $m < \pi^2$ such that

(i) $\sup |f(x, y, 0)| \leqslant \dfrac{2(\pi^2 - m)}{\pi} r, \quad$ for $0 \leqslant x \leqslant 1, \quad |y| \leqslant r;$

(ii) $f(x, y, z)$ has a continuous partial derivative with respect to z on $[0, 1]\times R\times R$ and

$$f_z(x, y, z)\geqslant -m > -\pi^2, \qquad \text{for } 0\leqslant x\leqslant 1, \quad |y|\leqslant r, \quad z\in R.$$

Then equation (2) with the boundary condition

$$y(0) = y'(0) = y'(1) = 0 \tag{16}$$

has a solution

PROOF. Assuming $u = y'$, we have, from $y(0) = 0$,

$$y(x) = \int_0^x u(t)\,dt. \tag{17}$$

Problem (2), (16) is equivalent to

$$u'' = f\left(x, \int_0^x u\,dt, u\right), \tag{18}$$

and

$$u(0) = u(1) = 0. \tag{19}$$

Let $B_r = \{u \in C[0, 1]\colon |u|\leqslant r\}$. For $u\in B_r$ and $0\leqslant x\leqslant 1$, define a mapping $T\colon C[0, 1] \to C[0, 1]$ by $(Tu)(x) = v(x)$, where

$$v'' = f\left(x, \int_0^x u\,dt, v\right), \tag{20}$$

and

$$v(0) = v(1) = 0. \tag{21}$$

Equation (20) is equivalent to

$$v'' = f\left(x, \int_0^x u\,dt, v\right) - f\left(x, \int_0^x u\,dt, 0\right) + f\left(x, \int_0^x u\,dt, 0\right),$$

or

$$v'' = \left(\int_0^1 f_v\left[x, \int_0^x u(t)\,dt,\ \tau v\right] d\tau \right) v + f\left(x, \int_0^x u(t)\,dt,\ 0\right). \tag{22}$$

Now, using Lemma 5, and condition (ii) we have

$$\sup|v(x)| \leqslant \frac{\pi}{2[\pi^2 - m]} \sup \left| f\left(x, \int_0^x u(t)\,dt, 0\right)\right|, \qquad 0 \leqslant x \leqslant 1\,,$$

or from (i)

$$\sup_{0 \leqslant x \leqslant 1} |v(x)| \leqslant r\,. \tag{23}$$

This shows that T maps the closed, bounded, and convex set B_r into itself. Also, from (20), (21)

$$v(x) = \int_0^1 G_x(x, t) f\left(t, \int_0^t u(s)\,ds,\ v(t)\right) dt\,,$$

and

$$v'(x) = \int_0^1 G_x(x, t) f\left(t, \int_0^t u(s)\,ds,\ v(t)\right) dt\,.$$

Since for $0 \leqslant x \leqslant 1$, $|y| \leqslant r$, $|z| \leqslant r$,

$$|f(x, y, z)| \leqslant k\,.$$

Then

$$|v'(x)| \leqslant \tfrac{1}{2} k\,.$$

All of these considerations imply that T is completely continuous. Schauder's fixed point theorem then yields a fixed point of T which is a solution of (18), (19). Thus $u = y'$ implies that (17) is a solution of (2), (16).

THEOREM 3. Assume that there exist positive numbers m and r such that

(i) $\sup |f(x, y, 0)| \leqslant \mathrm{mr}\,, \qquad 0 \leqslant x \leqslant 1\,, \qquad |y| \leqslant |y_0| + r\,;$

(ii) $f(x, y, z)$ has a continuous partial derivative with respect to z on $[0, 1]\times R\times R$ and

$$f_z(x, y, z)\geqslant m > 0 \text{ for } 0\leqslant x\leqslant 1\,, \quad |y|\leqslant|y_0| + r\,, \quad z\in R\,.$$

Then the boundary value problem (2), (4) has a solution.

PROOF. Let $y' = u$, then

$$y(x) = y_0 + \int_0^x u(t)\,dt\,. \tag{24}$$

Hence (2), (4) becomes

$$u'' = f\Big(x, y_0 + \int_0^x (ut)\,dt, u\Big) \tag{25}$$

and

$$u'(0) - hu(0) = 0\,, \quad u'(1) + ku(1) = 0\,. \tag{26}$$

Let

$$B_r = \{u\in C[0, 1]\colon |u|\leqslant r\}\,.$$

For $u\in B_r$, define a mapping $T\colon C[0, 1]\to C[0, 1]$ by $(Tu)(x) = v(x)$, where

$$v'' = f\Big(x, y_0 + \int_0^x u(t)\,dt, v\Big) \tag{27}$$

and

$$v'(0) - hv(0) = 0\,, \quad v'(1) + kv(1) = 0\,. \tag{28}$$

Equation (27) is equivalent to

$$v'' = \Big(\int_0^1 f_z\Big[x, y_0 + \int_0^x u(t)\,dt,\ \tau v\Big]\,d\tau\Big)v + f\Big(x, y_0 + \int_0^x u(t)\,dt, 0\Big)\,. \tag{29}$$

Using the fact that $|y_0 + \int_0^x u(t)\,dt|\leqslant|y_0| + r$, for $u\in B_r$ and applying

Lemma 6, we obtain, by virtue of conditions (i) and (ii),

$$\sup_{0\leqslant x\leqslant 1}|v(x)|\leqslant\frac{1}{m}\sup_{0\leqslant x\leqslant 1}\left|f\left(x, y_0+\int_0^x u(t)\,dt, 0\right)\right|\leqslant\frac{mr}{m}\leqslant r\,. \tag{30}$$

This shows that T maps B_r into itself. Also

$$v'(x)=v'(0)+\int_0^x f\left[t, y_0+\int_0^t u(s)\,ds, v(t)\right]dt\,.$$

From $v'(0)-hv(0)=0$ and (30)

$$|v'(0)|\leqslant hr\,. \tag{31}$$

Moreover, $|f(x, y, z)|\leqslant k$ for $0\leqslant x\leqslant 1$, $|y|\leqslant|y_0|+r$, and $|z|\leqslant r$, therefore

$$|v'(x)|\leqslant hr+k\,.$$

Thus, T is completely continuous by Ascoli's theorem. Schauder's fixed point theorem then yields a fixed point of T, which is a solution of (25), (26). Then $u=y'$ and (24) imply that (2), (4) has a solution

THEOREM 4. Suppose for all $(x, y_1, y_2, y_3), (x, \bar{y}_1, \bar{y}_2, \bar{y}_3)\in[0,1]\times R\times \times R\times R$ the function f satisfies the Lipschitz condition

$$|f(x, y_1, y_2, y_3)-f(x, \bar{y}_1, \bar{y}_2, \bar{y}_3)|\leqslant\sum_{i=1}^{3}L_i|y_i-\bar{y}_i|\,, \tag{32}$$

where $L_i>0$, $i=1,2,3$. If

$$L_1+L_2+4L_3<8\,, \tag{33}$$

then problem (1), (3) has a unique solution.

PROOF. Let B be the set of functions $u\in C'[[0,1], R]$ with the norm

$$|u|_B=(L_1+L_2)\max|u|+L_3\max|u'|,\qquad 0\leqslant x\leqslant 1\,.$$

Define the operators $T: B \to B$ by (13). For u and $v \in B$, we have

$$|Tu - Tv| \leqslant \int_0^1 |G(x,t)| \left[L_1 \int_0^t |u(s) - v(s)|\, ds + L_2 |u(t) - v(t)| + \right.$$
$$\left. + L_3 |u'(t) - v'(t)| \right] dt$$

or

$$|Tu - Tv| \leqslant \tfrac{1}{8}[(L_1 + L_2) \max |u(x) - v(x)| + L_3 \max |u'(x) - v'(x)|] ,$$
$$0 \leqslant x \leqslant 1$$

or

$$|Tu - Tv| \leqslant \tfrac{1}{8} |u - v|_B .$$

Also

$$|T'u - T'v| \leqslant \tfrac{1}{2} |u - v|_B .$$

It then follows that

$$|Tu - Tv|_B \leqslant \left(\frac{L_1 + L_2}{8} + \frac{L_3}{2} \right) |u - v|_B .$$

This in view of assumption (33), shows that T is a contraction mapping and thus has a unique fixed point which is the solution of (10), (11). Therefore, problem (1), (3) has a unique solution given by (9).

THEOREM 5. Suppose there exist positive numbers r, m, and c_0 such that

(i) $\sup |f(x, y, 0)| \leqslant \dfrac{2(\pi^2 - m)}{\pi} r$, for $0 \leqslant x \leqslant 1$, $|y| \leqslant r$;

(ii) $f(x, y, z)$ has a continuous partial derivative with respect to z on $[0,1] \times R \times R$ and

$$f_z(x, y, z) \geqslant -m > -\pi^2 , \quad \text{on } [0,1] \times R \times R ;$$

(iii) $f(x, y, z)$ has a continuous partial derivative with respect to y on $[0,1] \times R \times R$ and

$$|f_y(x, y, z)| \leqslant c_0 \quad \text{on } [0,1] \times R \times R .$$

If

(34) $$2c_0 + m\pi < \pi^3 ,$$

then the boundary value problem (2), (16) has a unique solution.

PROOF. Existence of a solution of (2), (16) follows from Theorem 2. Now, suppose that $u(x)$ and $v(x)$ are two solutions of (2), (16). Let $w(x) = u(x) - v(x)$, then

$$w'''(x) = f(x, u, u') - f(x, v, v')$$

or

$$w'''(x) = \left(\int_0^1 f_{u'}[x, \tau u + (1-\tau)v, \tau u' + (1-\tau)v']\,d\tau\right) w' + \\ + \left(\int_0^1 f_u[x, \tau u + (1-\tau)v, \tau u' + (1-\tau)v']\,d\tau\right) w(x)\,.$$

Since $w(0) = w'(0) = w'(1) = 0$, we have, by Lemma 7 and conditions (ii), (iii), and (34),

$$\sup_{0\leqslant x\leqslant 1} |w(x)| \leqslant 0\,.$$

This implies that $w(x) = 0$, or $u(x) = v(x)$. Hence problem (2), (16) has a unique solution.

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