Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 75 (1986), p. 247-251

<http://www.numdam.org/item?id=RSMUP_1986__75__247_0>

Rend. Sem. Mat. Univ. Padova, Vol. 75 (1986)

Extended Strings and Admissible Words.

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Let m be a natural number $\geqslant 2$, let Γ be the quiver

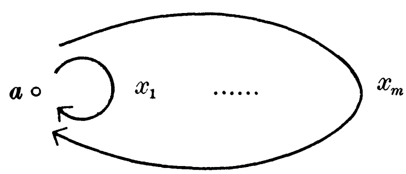

Fig. 1.

and let A be the path algebra of Γ over an algebraically closed field k (see [1] for the definitions). Then, even if $m = 2$, the injective representation $I(a)$ with socle given by the simple representation $S(a)$ of the form

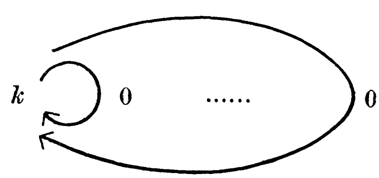

Fig. 2.

seems to be quite complicated. As a first step to investigate its structure, we will examine some special representations of Γ similar to infinite « strings ».

With a terminology suggested by [2], we call any infinite sequence $W = (l_n)_{n \in \mathbf{N}}$, where $l_n \in \{x_1, \ldots, x_m\}$ for any $n \in \mathbf{N}$, a *word* in the letters

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$x_1, \ldots, x_m$. For any word $W = (l_n)_{n \in \mathbf{N}}$, we denote by $M(W)$ the representation of Γ illustrated, in an obvious way, by the following picture:

$$\underset{v_0}{\circ} \xleftarrow{l_0} \underset{v_1}{\circ} \xleftarrow{l_1} \underset{v_2}{\circ} \xleftarrow{l_2} \underset{v_3}{\circ} \longleftarrow \circ \longleftarrow \circ \ldots$$

Finally, we denote by $\overline{M}(W)$ the representation of Γ of the form

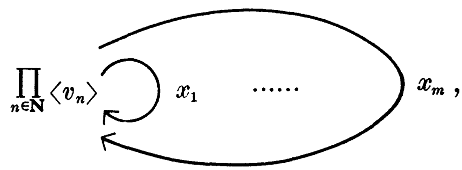

Fig. 3.

with the x_i's componentwise defined on the vector space $V = \prod_{n \in \mathbf{N}} \langle v_n \rangle$ by means of their action on the underlying vector space $\bigoplus_{n \in \mathbf{N}} \langle v_n \rangle$ of $M(W)$. More precisely, if $i = 1, \ldots, m$ and $v = (t_n v_n)_{n \in \mathbf{N}}$ with $t_n \in k$ for any n, then $x_i(v) = (t_{i,n} v_n)_{n \in N}$, where

$$t_{i,n} = \begin{cases} t_{n+1} & \text{if } l_n = x_i \\ 0 & \text{otherwise} \end{cases}$$

If the simple module generated by v_0 is the socle of $\overline{M}(W)$, then regarding $\overline{M}(W)$ as a submodule of $I(a)$, we shall say that $\overline{M}(W)$ is an *extended string contained in* $I(a)$, or, more briefly, that the word W is an *admissible word.*

As we shall see, there exist completely different admissible words. Roughly speaking, we can say that a word is admissible if and only if it is not « locally periodic ». The idea of this characterization arises from the direct check that, if x and y are two distinct letters, then the following facts hold:

(*) The words $(x, y, y, y, y, y, y, y, y, y, \ldots)$ and $(x, y, x, x, y, x, x, x, y, x, x, x, x, y, \ldots)$ are admissible.

(**) The word $(x, x, x, x, x, x, x, x, x, x, \ldots)$ is not admissible.

With all notation as above, we prove

Theorem 1. *Let $W = (l_n)_{n\in\mathbf{N}}$ be a word. The following conditions are equivalent*:

(i) *W is an admissible word.*

(ii) *There exists $d \in \mathbf{N}$ such that $l_0 \dots l_d \neq l_{n-d} \dots l_n$ for any $n > d$.*

Proof (i) $\Rightarrow$ (ii) Assume that W is admissible. For any $i \in \mathbf{N}$, let π_i denote the projection of V onto $\langle v_i \rangle$ with $\operatorname{Ker} \pi_i = \prod_{n\neq i} \langle v_n \rangle$. Next, let v denote the vector $(v_n)_{n\in\mathbf{N}} \in V$, and choose some $f \in A$ such that $f(v) = v_0$. View f as a polynomial in the (non commutative) variables $x_1, \dots, x_m$, and let d be the degree of f. We claim that $l_0 \dots l_d \neq l_{n-d} \dots l_n$ for all $n > d$. To see this, fix any $n > d$, and let v' and v'' denote the following vectors:

$$v' = v_0 + \dots + v_d \quad \text{and} \quad v'' = v_{n-d} + \dots + v_n .$$

Evidently $\pi_0 f(v')$ and $\pi_{n-d} f(v'')$ depend on f and on the paths $l_0 \dots l_d$ and $l_{n-d} \dots l_n$. Hence, to show that these two paths are different, it suffices to check that $\pi_0 f(v') \neq 0$, while $\pi_{n-d} f(v'') = 0$. Since the endomorphisms $x_1, \dots, x_m$ of V are componentwise defined, the same holds for f. Combining this observation with the hypothesis that $\deg f = d$, we obtain $\pi_i f\left(\prod_{n>i+d} \langle v_n \rangle\right) = 0$ for any $i \in \mathbf{N}$. Since $f(v) = v_0$, it follows that $\pi_0 f(v') = \pi_0 f(v) = v_0$, while $\pi_{n-d} f(v'') = \pi_{n-d} f(v) = 0$, as we whished to show. Hence

$$l_0 \dots l_d \neq l_{n-d} \dots l_n ,$$

and so condition (ii) holds.

(ii) $\Rightarrow$ (i) Let d be as in condition (ii). To prove that W is admissible, take any $0 \neq v = (t_n v_n)_{n\in\mathbf{N}} \in V$. If $t_n = 0$ for any $n > d + 1$, then, clearly, $v_0 \in Av$. Otherwise, let $m = \min\{n \in \mathbf{N} : n > d + 1,\ t_n \neq 0\}$, and let $u = l_{d+1} \dots l_{m-1}(v)$. Then the choice of m guarantees that $\pi_{d+1}(u) \neq 0$, while the definition of d implies that

$$0 \neq l_0 \dots l_d(u) \in \langle v_0 \rangle .$$

Consequently, $v_0 \in Au \subseteq Av$; thus W is admissible, as claimed in (i).

If $W = (l_n)_{n \in \mathbf{N}}$ is a word and $i \in \mathbf{N}$, we shall denote by W_i the following word:

$$W_i = (l_i, l_{i+1}, l_{i+2}, \ldots) .$$

The relation between W and the W_i's is given by the following corollary.

COROLLARY 2. *Let* $W = (l_n)_{n \in \mathbf{N}}$ *be a word and let* $i > 0$. *The following facts hold*:

(1) *If* W_i *is admissible, then also* W *is admissible.*

(2) *If* $W_i = W$, *then* W *is not admissible.*

Proof (1) Assume that W_i is admissible. Then, by Theorem 1, we can find some $d \geqslant i$ such that $l_i \ldots l_d \neq l_{n-(d-i)} \ldots l_n$ for any $n > d$. This means that $l_0 \ldots l_d \neq l_{n-d} \ldots l_n$ for all $n > d$. Hence, applying again Theorem 1, we conclude that W is admissible.

(2) Suppose that $W_i = W$. Then, for any $d \in \mathbf{N}$, we obviously have $l_0 \ldots l_d = l_{ir} \ldots l_{ir+d}$ for all $r \in \mathbf{N}$. Thus W cannot satisfy condition (ii) of Theorem 1, and so W is not admissible.

Finally, we point out a «translated» version of Theorem 1.

COROLLARY 3. *Let* $W = (l_n)_{n \in \mathbf{N}}$ *be a word. The following conditions are equivalent*:

(*a*) *W is an admissible word.*

(*b*) *There exist* $r, s \in \mathbf{N}$ *with* $r \geqslant s$ *such that*

$$l_{r-s} \ldots l_r \neq l_{n-s} \ldots l_n \quad \textit{for all } n > r .$$

Proof $(a) \Rightarrow (b)$ If W is admissible and d satisfies condition (ii) of Theorem 1, then (b) holds for $r = d$ and $s = d$.

$(b) \Rightarrow (a)$ If r and s satisfy (b), then, by Theorem 1, the word W_{r-s} is admissible. Hence, by Corollary 2, also W is admissible.

REFERENCES

[1] C. M. Ringel, *Tame Algebras,* Springer LMN **831** (1980), pp. 137-287.
[2] C. M. Ringel, *The indecomposable representations of the dihedral 2-groups,* Math. Ann., **214** (1975), pp. 19-34.

Manoscritto pervenuto in redazione il 6 dicembre 1984