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A result on m-flats in $\mathbb{A}_k^n$

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RIASSUNTO - In questa nota si dimostra che ogni varietà $V^{(m)}$ di $\mathbb{A}_k^n$, che sia isomorfa ad $\mathbb{A}_k^m$, con $m \leqslant 1/3(n-1)$, è la trasformata di un sottospazio lineare $S^{(m)} \subset \mathbb{A}_k^n$ mediante un automorfismo globale $\Phi: \mathbb{A}_k^n \to \mathbb{A}_k^n$ il quale risulta prodotto di automorfismi lineari e triangolari. Come conseguenza di ciò si ha il fatto che ogni linea di $\mathbb{A}_k^n$, con $n \geqslant 4$, risulta elementarmente rettificabile.

Introduction.

Let k be an algebraically closed field of characteristic zero. An automorphism $\Phi: \mathbb{A}_k^n \to \mathbb{A}_k^n$ which is the product of linear and triangular automorphisms is called *tame*. A variety $\mathcal{F}^{(m)}$ which is isomorphic to $\mathbb{A}_k^m$, will be called an m-flat. A 1-flat is called a line. Two varieties V', V'' such that there exists an automorphism $\Phi: \mathbb{A}_k^n \to \mathbb{A}_k^n$ with $\Phi(V') = V''$, will be called *equivalent*. An m-flat which is equivalent to a linear subspace $S^{(m)}$ of $\mathbb{A}_k^n$ will be called shortly *linearizable*. A linearizable 1-flat will be called *rectifiable*. If an m-flat $\mathcal{F}^{(m)}$ is transformed into a linear subspace by means of a tame automorphism of $\mathbb{A}_k^n$, we say that $\mathcal{F}^{(m)}$ is *tamely linearizable*. In Chapter 11 of [1], p. 413, Prof. Abhyankar raises the following interesting

Question: is it true that in $\mathbb{A}_k^n$, with $n \geqslant 3$, there are m-flats which are not equivalent?

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This question is exactly the same as to ask the following: is it true that in $\mathbb{A}_k^n$, with $n \geqslant 3$, there are m-flats which are not linearizable?

In this paper we give a partial answer to this last question by showing in Theorem 1) that, if $m \leqslant \frac{1}{3}(n-1)$, any m-flat in $\mathbb{A}_k^n$ is tamely linearizable. This has the interesting consequence that any line in $\mathbb{A}_k^n$, with $n \geqslant 4$, is tamely rectifiable, which corners down the possibility for a line not to be rectifiable in $\mathbb{A}_k^3$, so that Conjecture 1), p. 413 in [1], can be true only for $n = 3$ (in $\mathbb{A}_k^2$ every line is tamely rectifiable, as a consequence of the well known Theorem of Abhyankar and Moh [2]). On the other hand, in $\mathbb{A}_k^3$, there are examples of rigid lines which are very difficult to rectify, namely the

$$\mathcal{C}_n\colon (t + t^n, t^{n-1}, t^{n-2}) \qquad \text{for } n \geqslant 5$$

(see Conjecture 3), [1], p. 414). However in a previous work [3], we managed to rectify just $\mathcal{C}_5$, by means of an automorphism which, according to a Conjecture of M. Nagata (see [4], p. 47), should not be tame (we recall that, for $n \geqslant 3$, it is not yet known whether a non tame automorphism of $\mathbb{A}_k^n$ exists).

Let us consider in $\mathbb{A}_k^n$, an arbitrary m-flat $\mathcal{F}^{(m)}$, with $m \leqslant \frac{1}{3}(n-1)$ (of course it will be $n \geqslant 4$). $\mathcal{F}^{(m)}$ admits a biregular parametric representation by polynomials:

$$(*) \quad \begin{cases} x_1 = F_1(u_1, \ldots, u_m) \\ \vdots \\ x_n = F_n(u_1, \ldots, u_m) \end{cases}, \qquad (**) \quad \begin{cases} u_1 = G_1(x_1, \ldots, x_n) \\ \vdots \\ u_m = G_m(x_1, \ldots, x_n) \end{cases}$$

with $F_1, \ldots, F_n \in k[X_1, \ldots, X_m]$ and $G_1, \ldots, G_m \in k[X_1, \ldots, X_n]$. Let us call a straight line a *chord* of $\mathcal{F}^{(m)}$ if it meets $\mathcal{F}^{(m)}$ in at least two distinct points. The union of all the chords of $\mathcal{F}^{(m)}$ is contained in the (unirational) algebraic variety V, whose parametric representation is

$$\begin{cases} y_1' = F_1(u_1, \ldots, u_m) + \lambda[F_1(v_1, \ldots, v_m) - F_1(u_1, \ldots, u_m)] \\ \vdots \\ y_n' = F_n(u_1, \ldots, u_m) + \lambda[F_n(v_1, \ldots, v_m) - F_n(u_1, \ldots, u_m)] \end{cases}$$

where $u_1, \ldots, u_m, v_1, \ldots, v_m, \lambda$ are algebraically independent over k. Of course $\dim V \leqslant 2m + 1$. It can be shown that V contains also the union of all the tangent straight lines to $\mathcal{F}^{(m)}$, which is contained in the variety W whose parametric representation is

$$\begin{cases} y_1'' = F_1(u_1, \ldots, u_m) + \lambda_1 \left[\dfrac{\partial F_1}{\partial X_1}\right] + \ldots + \lambda_m \left[\dfrac{\partial F_1}{\partial X_m}\right] \\ \vdots \\ y_n'' = F_n(u_1, \ldots, u_m) + \lambda_1 \left[\dfrac{\partial F_n}{\partial X_1}\right] + \ldots + \lambda_m \left[\dfrac{\partial F_n}{\partial X_m}\right] \end{cases}$$

where $u_1, \ldots, u_m$, $\lambda_1, \ldots, \lambda_m$ are algebraically independent over k, and $[\partial F_i/\partial X_j]$ means $\partial F_i/\partial X_j$ calculated in $(u_1, \ldots, u_m)$.

Anyway, even without proving that $W \subset V$, we have dim $W \leqslant 2m$. Let us embed canonically $\mathbb{A}_k^n$ in $\mathbb{P}_k^n$. Let be $\tilde{V}$, $\tilde{W}$ the projective closures of V, W and V_∞, W_∞ respectively the intersections of $\tilde{V}$ and $\tilde{W}$ with the hyperplane at infinity π_∞. We have

$$\dim V_\infty \leqslant 2m\,, \qquad \dim W_\infty \leqslant 2m - 1\,.$$

Let us identify π_∞ with $\mathbb{P}_k^{n-1}$. Since by assumption it is $m \leqslant \frac{1}{3}(n-1)$, we have

$$\dim (V_\infty \cup W_\infty) + (m-1) \leqslant 2m + m - 1 = 3m - 1 \leqslant n - 2\,;$$

this means that in π_∞ we can surely find a linear subspace $S_\infty^{(m-1)}$, of dimension $m-1$, which does not interect $V_\infty \cup W_\infty$. Now we can state the following

LEMMA 1. *With the previous notations, any linear subspace $S^{(m)}$ of dimension m in $\mathbb{A}_k^n$, such that $S^{(m)} \cap \pi_\infty = S_\infty^{(m-1)}$, cannot meet $\mathcal{F}^{(m)}$ in more than one point; moreover, if it meets $\mathcal{F}^{(m)}$ in one point P, it cannot be tangent in P to $\mathcal{F}^{(m)}$.*

PROOF. Suppose P', P'' two distinct points of $\mathcal{F}^{(m)}$ and that there exists an $S^{(m)}$ such that $P', P'' \in S^{(m)} \cap \mathcal{F}^{(m)}$, with $S^{(m)} \cap \pi_\infty = S_\infty^{(m-1)}$; then the chord l of $\mathcal{F}^{(m)}$ through P', P'' is contained in $S^{(m)}$, so that $\tilde{l}$ meets $S_\infty^{(m-1)}$ and cannot meet by consequence V_∞ which is disjoint from $S_\infty^{(m-1)}$: this is absurd because $\tilde{l} \subset \tilde{V}$. Next suppose $P \in S^{(m)} \cap \mathcal{F}^{(m)}$, and that $S^{(m)}$ is tangent in P to $\mathcal{F}^{(m)}$; then every straight line l of $S^{(m)}$ throught P is tangent in P to $\mathcal{F}^{(m)}$: again this leeds to an absurd, because

$$l \subset S^{(m)} \cap W \;\Rightarrow\; \emptyset = S_\infty^{(m-1)} \cap W_\infty \supset \tilde{l} \cap \pi_\infty \neq \emptyset\,.$$

Now choose $n-m$ hyperplanes $\pi_1, \ldots, \pi_{n-m}$ in $\mathbb{A}_k^n$, so that

$$\tilde{\pi}_1 \cap \ldots \cap \tilde{\pi}_{n-m} \cap \pi_\infty = S_\infty^{(m-1)}$$

and, calling Λ a linear automorphism of $\mathbb{A}_k^n$ such that

$$\Lambda(\pi_i) = \{X_i = 0\} \qquad (i = 1, \ldots, n-m)$$

let

$$(\circ) \quad \begin{cases} x_1' = F_1'(u_1, \ldots, u_m) \\ \vdots \\ x_n' = F_n'(u_1, \ldots, u_m) \end{cases}, \qquad (\circ\circ) \quad \begin{cases} u_1 = G_1'(x_1', \ldots, x_n') \\ \vdots \\ u_m = G_m'(x_1', \ldots, x_n') \end{cases}$$

be the biregular parametric representation of the m-flat $\Lambda(\mathcal{F}^{(m)})$ that we obtain from (*) and (**) above by applying Λ. Calling $\tilde{\Lambda}$ the extension of Λ to $\mathbf{P}_k^n$, we have of course that:

(1) $\Lambda(V)$, $\Lambda(W)$ are the varieties containing the chords and the tangents of $\Lambda(\mathcal{F}^{(m)})$;

(2) $\Lambda(V)_\infty = \widetilde{\Lambda(V)} \cap \pi_\infty = \tilde{\Lambda}(V_\infty)$, and $\Lambda(W)_\infty = \tilde{\Lambda}(W_\infty)$;

(3) $\tilde{\Lambda}(S_\infty^{(m-1)}) = \widetilde{S_0^{(m)}} \cap \pi_\infty = (S_0^{(m)})_\infty$, where

$$S_0^{(m)} = \{X_1 = \ldots = X_{n-m} = 0\};$$

(4) $\big(\Lambda(V)_\infty \cup \Lambda(W)_\infty\big) \cap (S_0^{(m)})_\infty = \emptyset$;

(5) The above Lemma 1 holds substituting respectively $S^{(m)}$, $\widetilde{S^{(m)}}$, $S_\infty^{(m-1)}$, $\mathcal{F}^{(m)}$ with $\Lambda(S^{(m)})$, $\tilde{\Lambda}(\widetilde{S^{(m)}})$, $(S_0^{(m)})_\infty$, $\Lambda(\mathcal{F}^{(m)})$.

Now let us consider the linear subspace

$$S^{(n-m)} = \{X_{n-m+1} = \ldots = X_n = 0\}$$

and let

$$\Psi\colon \mathbb{A}_k^n \to S^{(n-m)}$$

be the projection of $\mathbb{A}_k^n$ on to $S^{(n-m)}$ from $(S_0^{(m)})_\infty$. We call

$$\psi\colon \Lambda(\mathcal{F}^{(m)}) \to S^{(n-m)}$$

the restriction of Ψ to $\Lambda(\mathcal{F}^{(m)})$.

We can state the following

LEMMA 2. *With the previous notations, ψ is an isomorphic embedding.*

PROOF. ψ is a finite mapping (see [5], Th. 7, p. 50). Of course

$\mathfrak{X} = \Lambda(\mathcal{F}^{(m)})$ is a smooth variety. ψ, by construction, is injective, because, if P_1, P_2 are two distinct points of $\Lambda(\mathcal{F}^{(m)})$ such that $\psi(P_1) = \psi(P_2) = Q \in S^{(n-m)}$, then the two subspaces $S_1^{(m)}$ and $S_2^{(m)}$ projecting P_1 and P_2 from $(S_0^{(m)})_\infty$ would coincide with $S_Q^{(m)}$, projecting Q from $(S_0^{(m)})_\infty$: this $S_Q^{(m)}$ would then contradict Lemma 1 (modified according to (5) above). By this same Lemma the differential mapping of ψ in P, $d_P\psi\colon \theta_{P,\mathfrak{X}} \to \theta_{\psi(P),S^{n-m}} = S^{(n-m)}$, where $\theta_{P,\mathfrak{X}}$ is the tangent space in P to the variety $\mathfrak{X}$, is an isomorphic embedding for every $P \in \mathfrak{X} = \Lambda(\mathcal{F}^{(m)})$. Indeed, in our case, $d_P\psi$ is exactly the restriction of Ψ to $\theta_{P,\mathfrak{X}}$, and since, by Lemma 1, we have, $\forall P \in \mathfrak{X}$, $\theta_{P,\mathfrak{X}} \cap (S_0^{(m)})_\infty = \emptyset$, then $d_P\psi$ is injective: suppose in fact P_1, P_2 two distinct points of $\theta_{P,\mathfrak{X}}$, and suppose $d_P\psi(P_1) = d_P\psi(P_2)$; this implies $S_{P_1}^{(m)} = S_{P_2}^{(m)} = S^{(m)}$, so that the straight line $l(P_1, P_2)$ is contained in $S^{(m)} \cap \theta_{P,\mathfrak{X}}$, which is absurd because we would find $(\tilde{\theta}_{P,\mathfrak{X}} \cap (S_0^{(m)})_\infty = \emptyset$, as above, by Lemma 1)

$$\emptyset = \tilde{\theta}_{P,\mathfrak{X}} \cap (S_0^{(m)})_\infty = \tilde{\theta}_{P,\mathfrak{X}} \cap \tilde{S}^{(m)} \supset \tilde{l}(P_1, P_2) \cap \pi_\infty \neq \emptyset\,.$$

Beeing a linear injective mapping between linear subspaces of $\mathbf{A}_k^n$, $d_P\psi$ is an isomorphic embedding. Now we can apply the Lemma in [5], Ch. 2, p. 124, and conclude that ψ is an isomorphic embedding.

COROLLARY 1. *With the notations of Lemma 2, the affine variety $\psi(\Lambda(\mathcal{F}^{(m)}))$ is an m-flat.*

PROOF. Obvious: $\psi(\Lambda(\mathcal{F}^{(m)}))$ is isomorphic to $\Lambda(\mathcal{F}^{(m)})$, which is an m-flat, via ψ.

REMARK 1. The isomorphism (which we call again ψ)

$$\psi\colon \Lambda(\mathcal{F}^{(m)}) \to \psi(\Lambda(\mathcal{F}^{(m)}))$$

has a regular inverse, that is, ψ^{-1} is given by polynomials.

REMARK 2. For every point $P(y_1, y_2, \ldots, y_n) \in \mathbb{A}_k^n$, we have, by the choice of $(S_0^{(m)})_\infty$ and $S^{(n-m)}$,

$$\Psi(y_1, \ldots, y_n) = (y_1, \ldots, y_{n-m}, 0, \ldots, 0)$$

so that ψ and ψ^{-1} will have equations of the following type

$$\psi(y_1, \ldots, y_n) = (y_1, \ldots, y_{n-m}, 0, \ldots, 0)$$

$$\forall (y_1, \ldots, y_n) \in \Lambda(\mathcal{F}^{(m)}) ,$$

$$\psi^{-1}(z_1, \ldots, z_{n-m}, 0, \ldots, 0) = \big(H_1(z_1, \ldots, z_{n-m}), \ldots, H_n(z_1, \ldots, z_{n-m})\big)$$

$$\forall (z_1, \ldots, z_{n-m}, 0, \ldots, 0) \in \psi\big(\Lambda(\mathcal{F}^{(m)})\big)$$

and where (see Remark 1)) $H_1, \ldots, H_n$ are suitable polynomials $\in k[X_1, \ldots, X_{n-m}]$: consequently we have

$$(6) \qquad F_i'(u_1, \ldots, u_m) = H_i\big(F_1'(u_1, \ldots, u_m), \ldots, F_{n-m}'(u_1, \ldots, u_m)\big) \qquad (i = 1, \ldots, n)$$

and, by (∘∘) above, we also have

$$(7) \qquad u_i = G_i'\big(H_1(F_1', \ldots, F_{n-m}'), \ldots, H_n(F_1', \ldots, F_{n-m}')\big) \qquad (i = 1, \ldots, m)$$

where we write shortly F_i' for $F_i'(u_1, \ldots, u_m)$.

Now we can prove the following

THEOREM 1. *Every m-flat $\mathcal{F}^{(m)} \subset \mathbb{A}_k^n$, with $m \leqslant \frac{1}{3}(n-1)$, is tamely linearizable.*

PROOF. Let Λ be a linear automorphism such that the conditions for validity of Lemma 2, and Remark 1 and 2 are fulfilled, with the same notations, and consider the automorphism

$$\chi \circ \Phi \circ \Lambda$$

where Φ and χ are following tame automorphisms

$$\Phi = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-m} \\ X_{n-m+1} - H_{n-m+1}(X_1, \ldots, X_{n-m}) + G_1'(H_1(X_1, \ldots, X_{n-m}), \ldots, H_n(X_1, \ldots, X_{n-m})) \\ \vdots \\ X_n - H_n(X_1, \ldots, X_{n-m}) + G_m'(H_1(X_1, \ldots, X_{n-m}), \ldots, H_n(X_1, \ldots, X_{n-m})) \end{pmatrix}$$

$$\chi = \begin{pmatrix} X_1 - F_1'(X_{n-m+1}, \ldots, X_n) \\ \vdots \\ X_{n-m} - F_{n-m}'(X_{n-m+1}, \ldots, X_n) \\ X_{n-m+1} \\ \vdots \\ X_n \end{pmatrix}$$

with obvious meaning of the notations $F_i(X_{n-m+1}, \ldots, X_n)$. We find, by (7) and (6) above

$$\chi \circ \Phi \circ \Lambda(\mathcal{F}^{(m)}) = \chi \circ \Phi \begin{pmatrix} F_1'(u_1, \ldots, u_m) \\ \vdots \\ F_n'(u_1, \ldots, u_m) \end{pmatrix} =$$

$$= \chi \begin{pmatrix} F_1'(u_1, \ldots, u_m) \\ \vdots \\ F_{n-m}'(u_1, \ldots, u_m) \\ u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_1 \\ \vdots \\ u_m \end{pmatrix}$$

which means that $\chi \circ \Phi \circ \Lambda$ transforms our $\mathcal{F}^{(m)}$ into the linear subspace

$$S_0^{(m)} = \{X_1 = \ldots = X_{n-m} = 0\}$$

and our theorem is proved.

In particular we can state the following remarkable

COROLLARY 2. *Any line of $\mathbb{A}_k^n$, with $n \geqslant 4$, is tamely rectifiable.*

PROOF. Apply Theorem 1 to 1-flats in $\mathbf{A}_k^n$, with $n \geqslant 4$: we have $1 \leqslant \frac{1}{3}(n-1)$, so that any 1-flat of $\mathbf{A}_k^n$, with $n \geqslant 4$, is tamely linearizable, which is our statement according to the nomenclature in the introduction.

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