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A result on \( m \)-flats in \( \mathbb{A}^n_k \)

P. C. CRAIGHERO (*)

Introduction.

Let \( k \) be an algebraically closed field of characteristic zero. An automorphism \( \Phi: \mathbb{A}^n_k \to \mathbb{A}^n_k \) which is the product of linear and triangular automorphisms is called \textit{tame}. A variety \( \mathcal{F}^{(m)} \) which is isomorphic to \( \mathbb{A}^m_k \), will be called an \( m \)-flat. A 1-flat is called a line. Two varieties \( \mathcal{V}', \mathcal{V}'' \) such that there exists an automorphism \( \Phi: \mathbb{A}^n_k \to \mathbb{A}^n_k \) with \( \Phi(\mathcal{V}') = \mathcal{V}'' \), will be called \textit{equivalent}. An \( m \)-flat which is equivalent to a linear subspace \( \mathcal{S}^{(m)} \) of \( \mathbb{A}^n_k \) will be called shortly \textit{linearizable}. A linearizable 1-flat will be called \textit{rectifiable}. If an \( m \)-flat \( \mathcal{F}^{(m)} \) is transformed into a linear subspace by means of a tame automorphism of \( \mathbb{A}^n_k \), we say that \( \mathcal{F}^{(m)} \) is \textit{tamely linearizable}. In Chapter 11 of [1], p. 413, Prof. Abhyankar raises the following interesting

Question: is it true that in \( \mathbb{A}^n_k \), with \( n \geq 3 \), there are \( m \)-flats which are not equivalent?

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This question is exactly the same as to ask the following: is it true that in $\mathbb{A}_n^3$, with $n > 3$, there are $m$-flats which are not linearizable?

In this paper we give a partial answer to this last question by showing in Theorem 1) that, if $m < \frac{1}{3}(n - 1)$, any $m$-flat in $\mathbb{A}_n^3$ is tamely linearizable. This has the interesting consequence that any line in $\mathbb{A}_n^3$, with $n > 4$, is tamely rectifiable, which corners down the possibility for a line not to be rectifiable in $\mathbb{A}_n^3$, so that Conjecture 1), p. 413 in [1], can be true only for $n = 3$ (in $\mathbb{A}_k^2$ every line is tamely rectifiable, as a consequence of the well known Theorem of Abhyankar and Moh [2]). On the other hand, in $\mathbb{A}_n^3$, there are examples of rigid lines which are very difficult to rectify, namely the

$$C_n: (t + t^n, t^{n-1}, t^{n-2}) \quad \text{for } n > 5$$

(see Conjecture 3), [1], p. 414). However in a previous work [3], we managed to rectify just $C_5$, by means of an automorphism which, according to a Conjecture of M. Nagata (see [4], p. 47), should not be tame (we recall that, for $n > 3$, it is not yet known whether a non tame automorphism of $\mathbb{A}_n^3$ exists).

Let us consider in $\mathbb{A}_n^3$, an arbitrary $m$-flat $\mathcal{F}^{(m)}$, with $m < \frac{1}{3}(n - 1)$ (of course it will be $n > 4$). $\mathcal{F}^{(m)}$ admits a biregular parametric representation by polynomials:

$$\begin{align*}
&x_1 = F_1(u_1, \ldots, u_m) \\
&\vdots \\
&x_n = F_n(u_1, \ldots, u_m)
\end{align*} \quad (**)

\begin{align*}
&u_1 = G_1(x_1, \ldots, x_n) \\
&\vdots \\
&u_m = G_m(x_1, \ldots, x_n)
\end{align*} \quad (***)

with $F_1, \ldots, F_n \in k[X_1, \ldots, X_n]$ and $G_1, \ldots, G_m \in k[X_1, \ldots, X_n]$. Let us call a straight line a chord of $\mathcal{F}^{(m)}$ if it meets $\mathcal{F}^{(m)}$ in at least two distinct points. The union of all the chords of $\mathcal{F}^{(m)}$ is contained in the (unirational) algebraic variety $V$, whose parametric representation is

$$\begin{align*}
y'_1 &= F'_1(u_1, \ldots, u_m) + \lambda [F_1(v_1, \ldots, v_m) - F'_1(u_1, \ldots, u_m)] \\
&\vdots \\
y'_n &= F'_n(u_1, \ldots, u_m) + \lambda [F_n(v_1, \ldots, v_m) - F'_n(u_1, \ldots, u_m)]
\end{align*}

where $u_1, \ldots, u_m$, $v_1, \ldots, v_m$, $\lambda$ are algebraically independent over $k$. Of course $\dim V \leq 2m + 1$. It can be shown that $V$ contains also the union of all the tangent straight lines to $\mathcal{F}^{(m)}$, which is contained
in the variety $W$ whose parametric representation is

$$\begin{align*}
y_1'' &= F_1(u_1, \ldots, u_m) + \lambda_1 \left[ \frac{\partial F_1}{\partial X_1} \right] + \cdots + \lambda_m \left[ \frac{\partial F_1}{\partial X_m} \right] \\
& \vdots \\
y_n'' &= F_n(u_1, \ldots, u_m) + \lambda_1 \left[ \frac{\partial F_n}{\partial X_1} \right] + \cdots + \lambda_m \left[ \frac{\partial F_n}{\partial X_m} \right]
\end{align*}$$

where $u_1, \ldots, u_m$, $\lambda_1, \ldots, \lambda_m$ are algebraically independent over $k$, and $[\partial F_i/\partial X_j]$ means $\partial F_i/\partial X_j$ calculated in $(u_1, \ldots, u_m)$.

Anyway, even without proving that $W \subset V$, we have $\dim W \leq 2m$. Let us embed canonically $A^n_k$ in $P^n_k$. Let be $\bar{V}$, $\bar{W}$ the projective closures of $V$, $W$ and $V_\infty$, $W_\infty$ respectively the intersections of $\bar{V}$ and $\bar{W}$ with the hyperplane at infinity $\pi_\infty$. We have

$$\dim V_\infty \leq 2m, \quad \dim W_\infty \leq 2m - 1.$$ 

Let us identify $\pi_\infty$ with $\mathbb{P}_k^{n-1}$. Since by assumption it is $m \leq \frac{1}{3}(n - 1)$, we have

$$\dim (V_\infty \cup W_\infty) + (m - 1) \leq 2m + m - 1 = 3m - 1 \leq n - 2;$$

this means that in $\pi_\infty$ we can surely find a linear subspace $S^{(m-1)}_\infty$, of dimension $m - 1$, which does not intersect $V_\infty \cup W_\infty$. Now we can state the following

**Lemma 1.** With the previous notations, any linear subspace $S^{(m)}$ of dimension $m$ in $A^n_k$, such that $S^{(m)} \cap \pi_\infty = S^{(m-1)}_\infty$, cannot meet $S^{(m)}$ in more than one point; moreover, if it meets $S^{(m)}$ in one point $P$, it cannot be tangent in $P$ to $S^{(m)}$.

**Proof.** Suppose $P'$, $P''$ two distinct points of $S^{(m)}$ and that there exists an $S^{(m)}$ such that $P', P'' \in S^{(m)} \cap S^{(m)}$, with $S^{(m)} \cap \pi_\infty = S^{(m-1)}_\infty$; then the chord $l$ of $S^{(m)}$ through $P'$, $P''$ is contained in $S^{(m)}$, so that $l$ meets $S^{(m-1)}_\infty$ and cannot meet by consequence $V_\infty$ which is disjoint from $S^{(m-1)}_\infty$: this is absurd because $l \subset \bar{V}$. Next suppose $P \in S^{(m)} \cap S^{(m)}$, and that $S^{(m)}$ is tangent in $P$ to $S^{(m)}$; then every straight line $l$ of $S^{(m)}$ through $P$ is tangent in $P$ to $S^{(m)}$: again this leads to an absurd, because

$$l \subset S^{(m)} \cap W \Rightarrow \emptyset = S^{(m-1)}_\infty \cap W_\infty \cap l \cap \pi_\infty \neq \emptyset.$$
Now choose \( n - m \) hyperplanes \( \pi_1, \ldots, \pi_{n-m} \) in \( \mathbb{A}^n_k \), so that
\[
\pi_1 \cap \cdots \cap \pi_{n-m} \cap \pi_\infty = S^{(m-1)}_\infty
\]
and, calling \( A \) a linear automorphism of \( \mathbb{A}^n_k \) such that
\[
A(\pi_i) = \{x_i = 0\} \quad (i = 1, \ldots, n - m)
\]
let
\[
\begin{cases}
  x'_1 = F'_1(u_1, \ldots, u_m) \\
  \vdots \\
  x'_n = F'_n(u_1, \ldots, u_m)
\end{cases}
\]
and
\[
\begin{cases}
  u_1 = G'_1(x'_1, \ldots, x'_n) \\
  \vdots \\
  u_m = G'_m(x'_1, \ldots, x'_n)
\end{cases}
\]
be the biregular parametric representation of the \( m \)-flat \( \tilde{A}(\mathbb{F}^{(m)}) \) that we obtain from \((\ast)\) and \((\ast\ast)\) above by applying \( \Lambda \). Calling \( \tilde{A} \) the extension of \( A \) to \( \mathbb{P}^n_k \), we have of course that:

1. \( \Lambda(V), \Lambda(W) \) are the varieties containing the chords and the tangents of \( \Lambda(\mathbb{F}^{(m)}) \);
2. \( \Lambda(V)_{\infty} = \tilde{\Lambda}(V) \cap \pi_\infty = \tilde{A}(V)_{\infty}, \) and \( \Lambda(W)_{\infty} = \tilde{A}(W)_{\infty} \);
3. \( \tilde{A}(S^{(m-1)}_\infty) = \tilde{S}^{(m)}_0 \cap \pi_\infty = (S^{(m)}_0)_{\infty}, \) where
   \[
   S^{(m)}_0 = \{x_1 = \ldots = x_{n-m} = 0\};
   \]
4. \( (\Lambda(V)_{\infty} \cup \Lambda(W)_{\infty}) \cap (S^{(m)}_0)_{\infty} = \emptyset; \)
5. The above Lemma 1 holds substituting respectively \( S^{(m)}, \tilde{S}^{(m)}, \)
   \( S^{(m-1)}_\infty, \mathbb{F}^{(m)} \) with \( \Lambda(S^{(m)}), \tilde{\Lambda}(S^{(m)}), (S^{(m)}_0)_{\infty}, \Lambda(\mathbb{F}^{(m)}) \).

Now let us consider the linear subspace
\[
S^{(n-m)} = \{x_{n-m+1} = \ldots = x_n = 0\}
\]
and let
\[
\Psi: \mathbb{A}^n_k \rightarrow S^{(n-m)}
\]
be the projection of \( \mathbb{A}^n_k \) on to \( S^{(n-m)} \) from \( (S^{(m)}_0)_{\infty} \). We call
\[
\psi: \Lambda(\mathbb{F}^{(m)}) \rightarrow S^{(n-m)}
\]
the restriction of \( \Psi \) to \( \Lambda(\mathbb{F}^{(m)}) \).
We can state the following

**Lemma 2.** With the previous notations, $\psi$ is an isomorphic embedding.

**Proof.** $\psi$ is a finite mapping (see [5], Th. 7, p. 50). Of course $\mathcal{X} = \Lambda(\mathcal{F}^{(m)})$ is a smooth variety. $\psi$, by construction, is injective, because, if $P_1$, $P_2$ are two distinct points of $\Lambda(\mathcal{F}^{(m)})$ such that $\psi(P_1) = \psi(P_2) = Q \in S^{(n-m)}$, then the two subspaces $S^{(m)}_1$ and $S^{(m)}_2$ projecting $P_1$ and $P_2$ from $(S^{(m)}_0)_{\infty}$ would coincide with $S^{(m)}_Q$, projecting $Q$ from $(S^{(m)}_0)_{\infty}$: this $S^{(m)}_Q$ would then contradict Lemma 1 (modified according to (5) above). By this same Lemma the differential mapping of $\psi$ in $P$, $d_p\psi: \theta_{P,\mathcal{X}} \rightarrow \theta_{\psi(P),S^{(n-m)}} = S^{(n-m)}$, where $\theta_{P,\mathcal{X}}$ is the tangent space in $P$ to the variety $\mathcal{X}$, is an isomorphic embedding for every $P \in \mathcal{X} = \Lambda(\mathcal{F}^{(m)})$. Indeed, in our case, $d_p\psi$ is exactly the restriction of $\Psi$ to $\theta_{P,\mathcal{X}}$, and since, by Lemma 1, we have, $\forall P \in \mathcal{X}$, $\theta_{P,\mathcal{X}} \cap (S^{(m)}_0)_{\infty} = \emptyset$, then $d_p\psi$ is injective: suppose in fact $P_1$, $P_2$ two distinct points of $\theta_{P,\mathcal{X}}$, and suppose $d_p\psi(P_1) = d_p\psi(P_2)$; this implies $S^{(m)}_{P_1} = S^{(m)}_{P_2} = S^{(m)}$, so that the straight line $l(P_1, P_2)$ is contained in $S^{(m)} \cap \theta_{P,\mathcal{X}}$, which is absurd because we would find $(\bar{\theta}_{P,\mathcal{X}} \cap (S^{(m)}_0)_{\infty} = \emptyset$, as above, by Lemma 1)

$$\emptyset = \bar{\theta}_{P,\mathcal{X}} \cap (S^{(m)}_0)_{\infty} = \bar{\theta}_{P,\mathcal{X}} \cap S^{(m)} \cap l(P_1, P_2) \cap \pi_{\infty} = \emptyset.$$

Being a linear injective mapping between linear subspaces of $A^n_x$, $d_p\psi$ is an isomorphic embedding. Now we can apply the Lemma in [5], Ch. 2, p. 124, and conclude that $\psi$ is an isomorphic embedding.

**Corollary 1.** With the notations of Lemma 2, the affine variety $\psi(\Lambda(\mathcal{F}^{(m)}))$ is an $m$-flat.

**Proof.** Obvious: $\psi(\Lambda(\mathcal{F}^{(m)}))$ is isomorphic to $\Lambda(\mathcal{F}^{(m)})$, which is an $m$-flat, via $\psi$.

**Remark 1.** The isomorphism (which we call again $\psi$)

$$\psi: \Lambda(\mathcal{F}^{(m)}) \rightarrow \psi(\Lambda(\mathcal{F}^{(m)}))$$

has a regular inverse, that is, $\psi^{-1}$ is given by polynomials.
Remark 2. For every point \( P(y_1, y_2, \ldots, y_n) \in \mathbb{A}^n_k \), we have, by the choice of \((S_0^{(m)})_0\) and \(S^{(n-m)}\),

\[
\Phi(y_1, \ldots, y_n) = (y_1, \ldots, y_{n-m}, 0, \ldots, 0)
\]

so that \( \psi \) and \( \psi^{-1} \) will have equations of the following type

\[
\psi(y_1, \ldots, y_n) = (y_1, \ldots, y_{n-m}, 0, \ldots, 0)
\]

\[
\forall (y_1, \ldots, y_n) \in \Lambda(\mathcal{F}^{(m)})
\]

\[
\psi^{-1}(z_1, \ldots, z_{n-m}, 0, \ldots, 0) = (H_1(z_1, \ldots, z_{n-m}), \ldots, H_n(z_1, \ldots, z_{n-m}))
\]

\[
\forall (z_1, \ldots, z_{n-m}, 0, \ldots, 0) \in \psi(\Lambda(\mathcal{F}^{(m)}))
\]

and where \((\text{see Remark 1})\) \( H_1, \ldots, H_n \) are suitable polynomials \( \in k[X_1, \ldots, X_{n-m}] \): consequently we have

\[(6)\quad F_i'(u_1, \ldots, u_m) = H_i(F_1'(u_1, \ldots, u_m), \ldots, F_{n-m}'(u_1, \ldots, u_m)) \quad (i = 1, \ldots, n)
\]

and, by \((\circ\circ)\) above, we also have

\[(7)\quad u_i = G_i'(H_1'(F_1', \ldots, F_{n-m}'), \ldots, H_n'(F_1', \ldots, F_{n-m}')) \quad (i = 1, \ldots, m)
\]

where we write shortly \( F_i' \) for \( F_i'(u_1, \ldots, u_m) \).

Now we can prove the following

**Theorem 1.** Every \( m \)-flat \( \mathcal{F}^{(m)} \subset \mathbb{A}^n_k \), with \( m \leq \frac{1}{3}(n - 1) \), is tamely linearizable.

**Proof.** Let \( \Lambda \) be a linear automorphism such that the conditions for validity of Lemma 2, and Remark 1 and 2 are fulfilled, with the same notations, and consider the automorphism

\[
\chi \circ \Phi \circ \Lambda
\]
where $\Phi$ and $\chi$ are following tame automorphisms

$$\Phi = \begin{pmatrix}
X_1 \\
\vdots \\
X_{n-m} \\
X_{n-m+1} - H_{n-m+2}(X_1, \ldots, X_{n-m}) + G'_1(H_1(X_1, \ldots, X_{n-m}), \ldots, H_n(X_1, \ldots, X_{n-m})) \\
X_n - H_n(X_1, \ldots, X_{n-m}) + G'_m(H_1(X_1, \ldots, X_{n-m}), \ldots, H_n(X_1, \ldots, X_{n-m}))
\end{pmatrix}$$

$$\chi = \begin{pmatrix}
X_1 - F'_1(X_{n-m+1}, \ldots, X_n) \\
\vdots \\
X_{n-m} - F'_{n-m}(X_{n-m+1}, \ldots, X_n) \\
X_{n-m+1} \\
X_n
\end{pmatrix}$$

with obvious meaning of the notations $F'_i(X_{n-m+1}, \ldots, X_n)$. We find, by (7) and (6) above

$$\chi \circ \Phi \circ A(F^{(m)}) = \chi \circ \Phi \begin{pmatrix} F'_1(u_1, \ldots, u_m) \\
\vdots \\
F'_n(u_1, \ldots, u_m) \end{pmatrix} = \chi \begin{pmatrix} F'_1(u_1, \ldots, u_m) \\
\vdots \\
F'_{n-m}(u_1, \ldots, u_m) \\
u_1 \\
\vdots \\
u_m \end{pmatrix} = \begin{pmatrix} 0 \\
\vdots \\
0 \\
u_1 \\
\vdots \\
u_m \end{pmatrix}$$

which means that $\chi \circ \Phi \circ A$ transforms our $F^{(m)}$ into the linear subspace

$$S^{(m)}_0 = \{X_1 = \ldots = X_{n-m} = 0\}$$

and our theorem is proved.

In particular we can state the following remarkable

**Corollary 2.** Any line of $A^n_k$, with $n > 4$, is tamely rectifiable.
PROOF. Apply Theorem 1 to 1-flats in $\mathbb{A}^n_k$, with $n \geq 4$: we have $1 \leq \frac{1}{3}(n - 1)$, so that any 1-flat of $\mathbb{A}^n_k$, with $n \geq 4$, is tamely linearizable, which is our statement according to the nomenclature in the introduction.

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