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Remarks on the Yamabe problem and the Palais-Smale condition

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0. Introduction.

Let M be a C^∞ compact Riemannian manifold of dimension $n \geqslant 3$. Let $g(x)$ be its metric and $R(x)$ its scalar curvature. An important problem concerning the scalar curvature is the Yamabe problem: does there exist a metric g' conformal to g, such that the scalar curvature $\bar{R}$ of the metric is constant?

Since the pionering paper of Yamabe [11] appeared, several authors have studied this problem (cp. [1, 9, 10] and the references contained in [3]).

If we consider the conformal deformation in the form $g' = u^{4/(n-2)} g$ (with $u \in C^\infty$, $u > 0$), the Yamabe problem is reduced (cp. [1, 9, 11]) to the following eigenvalue problem:

$$
(0.1) \qquad \begin{cases} \text{find } u \in C^\infty, \ u > 0 \text{ and } \bar{R} \in \mathbb{R} \text{ s.t.}, \\ \gamma \Delta u + R(x) u = \bar{R} u^{2^*-1}, \end{cases}
$$

where $\gamma = 4(n-1)/(n-2)$, Δ denotes the Laplace-Beltrami operator corresponding to g and $2^* = 2n/(n-2)$.

We denote by H^1 the Sobolev space on M, i.e. H^1 is the completion

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of the space $C^\infty(M)$ with respect to the norm

$$\|\varphi\| = \Big(\int_M (|\varphi|^2 + |\nabla\varphi|^2)\, dM\Big)^{\frac{1}{2}}$$

where $|\nabla\varphi|^2 = g^{ij}\varphi_{x_i}\varphi_{x_j}$.

$L^\alpha (\alpha > 1)$ denotes the space of the functions on M which are α-integrable. Moreover we set

$$|\varphi|_\alpha = \Big(\int_M |\varphi|^\alpha dM\Big)^{1/\alpha}.$$

Clearly if u minimizes the functional

$$\psi(u) = \int_M (\gamma|\nabla u|^2 + R(x)u^2)\, dM \tag{0.2}$$

on the manifold in H^1

$$V = \{u \in H^1 \,|\, |u|_{2^*} = 1\} \tag{0.3}$$

u is a weak solution of (0.1) and $\bar{R}$ is the minimum of ψ on V. Then by a regularity result of Trudinger (cp. [9, Th. 3]) u is C^∞. Moreover, since $|\nabla|u|| = |\nabla u|$ (cp. [3, prop. 3.49]), we can assume that $u \geqslant 0$. Then (cp. [3, prop. 3.75]) we deduce that $u > 0$.

Therefore the Yamabe problem is solved if the functional ψ can be minimized on V. Since H^1 is not compactly embedded into L^{2^*} it is not easy to find a minimum directly. In [1, 9, 11] the following approximation-procedure has been used. Since the embedding $H^1 \hookrightarrow L^q$ $(q < 2^*)$ is compact the problem

$$-\gamma\Delta u + R(x)u = \omega_q u^{q-1} \qquad |u|_q = 1$$

can be easily solved; then, by taking the limit for $q \to 2^*$, it is possible, in some cases, to solve (0.1).

If Ω is a bounded domain in $\mathbb{R}^n$, set

$$S = \inf\Big\{\int_\Omega |\nabla\varphi|^2 dx \colon |\varphi|_{2^*} = 1, \quad \varphi \in C_0^\infty(\mathbb{R}^n)\Big\} \quad (n \geqslant 3).$$

It is known (cp. [2, 8]) that S is independent of Ω and

$$S = \frac{\omega_n^{2/n}}{4} n(n-2)$$

where ω_n denotes the area of the unit n-dimensional sphere S_n.

Using the « approximation-method » Aubin [1, Th. 7] proved the following Theorem.

THEOREM 0.1. *Suppose that*

$$\inf\{\psi(u) : u \in V\} < S \cdot \gamma .$$

Then there exists $u \in V$ *which minimized* $\psi|_V$.

In [1] Aubin has also pointed out that

$$\mu = \inf\{\psi(u) : u \in V\}$$

is a conformal invariant and that the inequality

$$\mu \leqslant \gamma S$$

holds for any compact n-dimensional Riemmanian manifold.

In this paper we prove that $\psi|_V$ satisfies the Palais-Smale (P-S) condition in the range $]-\infty, \gamma S[$ (cp. lemma 1.1). Then Theorem 0.1 can be proved directly using standard variational arguments.

Moreover we show that $\psi|_V$ does not satisfy the (P-S) condition at $c = \gamma S$ (cp. Th. 2.1, 2.2).

It can be proved also that slight perturbations of (0.1) have always a solution. More precisely we consider the following perturbed eigenvalue problem:

$$\text{(0.4)} \qquad \begin{cases} \text{find } u \in C^\infty ,\ u > 0 \text{ and } \bar{R} \in \mathbb{R} \text{ s.t.} , \\ -\gamma \Delta u + (R(x) - \varepsilon) u = \bar{R} u^{2^*-1} , \qquad \varepsilon > 0 , \end{cases}$$

and we prove the following

THEOREM 0.2. *Let* $n \geqslant 4$, *than for any* $\varepsilon > 0$, (0.4) *possesses a solution.*

Observe that Theorem 0.2 is similar to a result obtained by Brezis and Nirenberg (cp. [4, Th. 11]) in the context of elliptic boundary value problems on a bounded domain $\Omega \subset \mathbb{R}^n$.

Let us finally recall that another direct proof of Theorem 0.1 is contained in [10].

1. Proof of Theorem 0.1, 0.2.

The following lemma plays a fundamental role in proving Theorems 0.1, 0.2.

LEMMA 1.1. *The functional* $\psi|_V$ *defined by* (0.2) *and* (0.3) *satisfies the Palais-Smale condition in* $]-\infty, \gamma S[$ *in the following sense*:

(P-S) *If* $\{u_j\} \subset V$ *s.t. as* $j \to \infty$

$$d\psi|_V(u_j) \to 0 \text{ strongly in } H^{-1} \tag{1.1}$$

and

$$\psi(u_j) \to c, \qquad c < \gamma S \tag{1.2}$$

Then $\{u_j\}$ *contains a subsequence converging strongly in* H^1.

PROOF. Let $\{u_j\} \subset V$ be a sequence which satisfies (1.1) and (1.2). Then, by (1.2), we easily deduce that we can select a subsequence, which we continue to denote by $\{u_j\}$, such that

$$\begin{cases} u_j \to u \text{ weakly in } H^1, \\ u_j \to u \text{ strongly in } L^p, \qquad 1 \leqslant p < 2^*. \end{cases} \tag{1.3}$$

Set $\varphi(u) = |u|_{2^*}^{2^*}$ $u \in H^1$.

From (1.1) we deduce that there exists a sequence $\{\lambda_j\} \subset \mathbb{R}$ s.t.

$$d\psi(u_j) - \lambda_j d\varphi(u_j) \to 0 \quad \text{in } H^{-1} = (H^1)'. \tag{1.4}$$

Then, since $|u_j|_{2^*} = 1$ and by using (1.2), we obtain

$$\lambda_j = \frac{\langle d\psi(u_j), u_j \rangle}{2^*} + o(1) = \frac{2}{2^*}\psi(u_j) + o(1) = \frac{2}{2^*}c + o(1). \tag{1.5}$$

We show now that u solves the equation

$$(1.6) \qquad -\gamma\Delta u + R(x)u - cu|u|^{2^*-2} = 0 .$$

Let $\zeta \in C^\infty(M)$, then by (1.4), (1.3), (1.5) we deduce that

$$o(1) = \langle d\psi(u_j) - \lambda_j\, d\varphi(u_j), \zeta\rangle = \langle d\psi(u_j), \zeta\rangle - \frac{2}{2^*} c\langle d\varphi(u_j), \zeta\rangle + o(1) =$$

$$= 2\langle -\gamma\, \Delta u + R(x)u, \zeta\rangle - 2c\int_M u|u|^{2^*-2}\cdot \zeta\, dM + o(1) .$$

Then u is a weak solution of (1.6). Then (cp. [9, Th. 3]) u is a C^∞ solution of (1.6).

To show that $u_j \to u$ strongly in $H^1(M)$, let

$$v_j = u_j - u .$$

Testing (1.4) with v_j, we obtain

$$(1.7) \quad o(1) = \langle d\psi(u_j), v_j\rangle - \lambda_j\langle d\varphi(u_j), v_j\rangle =$$

$$= 2\int_M [\gamma\big((\nabla u|\nabla v_j) + |\nabla v_j|^2\big) + R(x)\,(uv_j + v_j^2)]\,dM -$$

$$- \lambda_j < d\varphi(u_j),\ v_j\rangle .$$

By (1.3) we have

$$(1.8) \qquad \int_M [\gamma(\nabla u|\nabla v_j) + R(x)\,(uv_j + v_j^2)]\,dM = o(1) .$$

Whence from (1.7), (1.8) we deduce that

$$(1.9) \quad 2\gamma|\nabla v_j|_2^2 = \lambda_j < d\varphi(u_j), v_j > + o(1) =$$

$$= 2^*\lambda_j \int_M |u + v_j|^{2^*-2}\,(u + v_j)\,v_j dM + o(1)$$

Now we claim that

$$(1.10) \qquad \gamma\|v_j\|^2 = \gamma|\nabla v_j|_2^2 + o(1) = c|v_j|_{2^*}^{2^*} + o(1) .$$

In fact, following an analogous argument as in [5, lemma 2.1], we obtain:

$$\left|\int_M [(u+v_j)|u+v_j|^{2^*-2}v_j - |v_j|^{2^*}]\, dM\right| =$$

$$= \left|\int_M \int_0^{u(x)} \frac{\partial}{\partial\zeta}[(v_j+\xi)|v_j+\xi|^{2^*-2}]v_j\, d\xi\, dM\right| =$$

$$= (2^*-1)\left|\int_M \int_0^1 |v_j+tu|^{2^*-2} v_j u\, dt\, dM\right| \leqslant$$

$$\leqslant \text{const}\left(\int_M (|u|\,|v_j|^{2^*-1} + |v_j|\,|u|^{2^*-1})\, dM\right).$$

Then, by (1.3) and since $u \in L^{2^*}(M)$, we have

$$\int_M (u+v_j)\,|u+v_j|^{2^*-2} v_j dM = |v_j|_{2^*}^{2^*} + o(1)\,. \tag{1.11}$$

So (1.10) easily follows from (1.9), (1.11) and (1.5). If $c \leqslant 0$ from (1.10) we deduce that

$$\|v_j\| = o(1)\,.$$

Whence

$$u_j \to u \text{ strongly in } H^1\,.$$

Now suppose that

$$0 < c < \gamma S\,. \tag{1.12}$$

Using Theorem 2.21 in [3] (cp. also [5, lemma 2.5]) we have

$$\|v_j\|^2 \geqslant S|v_j|_{2^*}^2 + o(1)\,. \tag{1.13}$$

By (1.10), (1.13)

$$\|v_j\|^{2^*} \geqslant S^{2^*/2}\frac{\gamma}{c}\|v_j\|^2 + o(1)\,,$$

$$\|v_j\|^2\left(\frac{\gamma}{c}S^{2^*/2} - \|v_j\|^{2^*-2}\right) \leqslant o(1)\,. \tag{1.14}$$

Let us now prove that

$$\|v_j\|^2 \leq \frac{c}{\gamma}, \tag{1.15}$$

Easy calculations show that

$$\psi(u_j) = \psi(u + v_j) = \gamma\int_M (|\nabla u|^2 + |\nabla v_j|^2)\, dM + \\ + \int_M R(x) u^2 dM + o(1). \tag{1.16}$$

By (1.16) and (1.2) we deduce that

$$\gamma\|v_j\|^2 = c - \int_M (\gamma|\nabla u|^2 + R(x)u^2)\, dM + o(1). \tag{1.17}$$

Moreover, since u solves (1.6), we have

$$\int_M (\gamma|\nabla u|^2 + R(x)u^2)\, dM = c|u|_{2^*}^{2^*}. \tag{1.18}$$

From (1.17) and (1.18) we easily deduce (1.15).

By (1.14) and (1.15) we have

$$\|v_j\|^2 \left(\frac{\gamma}{c} S^{2^*/2} - \left(\frac{c}{\gamma}\right)^{(2^*-2)/2}\right) \leq o(1). \tag{1.19}$$

Since $c < \gamma S$, we have

$$\frac{\gamma}{c} S^{2^*/2} > \left(\frac{c}{\gamma}\right)^{(2^*-2)/2}. \tag{1.20}$$

Finally from (1.19) and (1.20) we deduce that

$$\|v_j\| \to 0 \qquad \text{Q.E.D.}.$$

PROOF OF THEOREM 0.1. The proof of Theorem 0.1 can be deduced from lemma 1.1 using standard variational arguments. Suppose

$$\mu = \inf \psi|_V < \gamma S$$

and arguing by contradiction assume that μ is not a critical value for $\psi|_V$. Since the (P-S) condition holds in $]-\infty, \gamma S[$ by well known results (cp. [6, 7]) there exists $\varepsilon > 0$ and an homeomorphism

$$\eta\colon V \to V \text{ s.t.}$$

$$(u \in V,\ \psi(u) \leqslant \mu + \varepsilon) \Rightarrow \bigl(\psi(\eta(u)) \leqslant \mu - \varepsilon\bigr)\,.$$

Obviously this contradicts $\mu = \inf \psi|_V$. Q.E.D.

REMARK 1.2. From the proofs given before it is easy to see that lemma 1.1 and Theorem 0.1 still hold if $R(x)$ is any smooth function which does not represent necessarily the scalar curvature of M.

Moreover lemma 1.1 and Theorem 0.1 hold also, with obvious changes, if we replace M with a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, and H^1 with the Sobolev space $H^1_0(\Omega)$. In this setting Theorem 0.1 becomes a variant of Lemma 1.2 in [4].

REMARK 1.3. In order to apply Theorem 0.1 it is important to find conditions which guarantee

$$\mu = \inf \psi|_V < \gamma S\,. \tag{1.21}$$

It is easy to see that if we assume $\int_M R(x)\,dM < (\text{meas } M)^{(n-2)/n}\,\gamma S$, then (1.21) is verified (cp. also [1, 3, 9]).

Other classes of manifolds which satisfy (1.21) can be found in [1, 3].

2. Remarks about the (P-S) condition and Proof of Theorem 0.2.

For any Riemannian manifold [1, lemma 4] we have

$$\mu = \inf \psi|_V \leqslant \gamma S\,.$$

Moreover there exist manifolds, which we call critical, for which

$$\mu = \gamma S\,.$$

An example of critical manifold is the n-dimensional sphere $S_n (n \geqslant 3)$ [1, cor. 4].

For this class of manifolds the (P-S) condition does not hold in $\mu = \gamma S$.

In fact the following Theorem holds

THEOREM 2.1. *Let M be a critical Riemannian manifold i.e.*

$$\mu = \inf \psi|_V = \gamma S .$$

Then $\psi|_V$ does not satisfy the (P-S) *condition in $\mu = \gamma S$, i.e. there exists a sequence $\{u_j\} \subset V$ s.t. $\psi(u_j) \to \gamma S$ and $d\psi|_V(u_j) \to 0$ and which is not precompact in H^1.*

PROOF. By a result of Aubin [3, Th. 2.21] there exists a sequence $\{\omega_j\} \subset V$ s.t.

$$|\omega_j|_2 \to 0 \quad \text{and} \quad |\nabla \omega_j|_2^2 \to S .$$

Then

$$\psi(\omega_j) \to \gamma S . \tag{2.1}$$

Since the manifold M is critical, by (2.1) we deduce that $\{\omega_j\}$ is a minimizing sequence for $\psi|_V$.

Now we argue by contradiction and suppose that $\psi|_V$ satisfies the (P-S) condition at $\mu = \gamma S$. Then by using standard variational arguments (cp. prop. A.1 in Appendix)

$$d\psi|_V(\omega_j) \to 0 .$$

Then $\{\omega_j\}$ is precompact in H^1, therefore it contains a subsequence, which we continue to denote by $\{\omega_j\}$, such that

$$\omega_j \to \omega \text{ strongly in } H^1 \text{ (and in } L^{2^*}) .$$

Then, since $|\omega_j|_{2^*} = 1$, we have $|\omega|_{2^*} = 1$. On the other hand $|\omega|_2 = 0$. Therefore we get a contradiction. Q.E.D.

Also manifolds which are not critical do not satisfy the (P-S) condition at $c = \gamma S$. More precisely the following Theorem holds

THEOREM 2.2. *Let M be a C^∞ Riemannian compact manifold of dimension $n \geqslant 3$. Then the functionl $\psi|_V$ does not satisfy the* (P-S) *condition in $c = \gamma S$.*

In order to prove this result we need to introduce some notations.

Let δ be a positive number less than the injectivity radius of M. If $P \in M$ we set $B(P, \delta)$ the ball of M centered at P and radius δ. Using geodesic local coordinates, if $Q \in B(P, \delta)$ we have

$$Q = (r, \theta) \in [0, \delta] \times S_{n-1}, \qquad r = r(P, Q) = d(P, Q) . \tag{2.2}$$

The metric tensor g can be expressed by

$$ds^2 = dr^2 + r^2 g_{\theta i \theta j}(r, \theta)\, d\theta^i d\theta^j . \tag{2.3}$$

Set

$$|g| = \det (g_{\theta i \theta j}) . \tag{2.4}$$

Fix $\alpha > 0$. Suppose $R(P) > 0$ we choose

$$\alpha^2 = R(P)/n(n-1) .$$

It is not restrictive to assume also

$$\alpha\delta < \pi . \tag{2.5}$$

In order to prove Theorem 2.2 we need to introduce a suitable sequence in V.

Following Aubin [1] we consider the sequence of functions

$$k \in \mathbb{N} , \quad v_k(Q) =$$
$$= \begin{cases} \left(\dfrac{1}{k} + \dfrac{1 - \cos \alpha r}{\alpha^2}\right)^{1-n/2} - \left(\dfrac{1}{k} + \dfrac{1 - \cos \alpha\delta}{\alpha^2}\right)^{1-n/2} & \text{if } r < \delta , \\ 0 \quad \text{if } r > \delta . \end{cases}$$

Observe that the functions $r \to [1/k + (1 - \cos \alpha r)/\alpha^2]^{1-n/2}$ solve the Yamabe equation for $M = S_n$.

Obviously v_k have supports contained in $\overline{B(P, \delta)}$. We set

$$u_k = \frac{v_k}{|v_k|_{2^*}} . \tag{26}$$

In order to prove Theorem 2.2 we need the following lemma

LEMMA 2.3. *For $k \to +\infty$ we have the following asymptotic estimates*

$$\psi(u_k) = \gamma S + \sigma_k \tag{2.7}$$

where

$$\sigma_k = \begin{cases} O(k^{-2}) & \text{if } n > 6 , \\ O(\lg k \cdot k^{-2}) & \text{if } n = 6 , \\ O(k^{-(n-2)/2}) & \text{if } n = 3, 4, 5 . \end{cases} \tag{2.8}$$

Moreover

$$|u_k|_2^2 = \begin{cases} \frac{\gamma}{2} \omega_n^{2/n} k^{-1} + o(k^{-1}) & \text{if } n \geq 5 , \ \omega_n = \text{meas } S_n , \\ \frac{\gamma}{2} \omega_4^{\frac{1}{4}} \frac{\lg k}{k} + o(k^{-1} \lg k) & \text{if } n = 4 , \\ \text{const } k^{-\frac{1}{2}} + o(k^{-\frac{1}{2}}) & \text{if } n = 3 , \end{cases} \tag{2.9}$$

and the same asymptotic estimates hold for $\big|r|\nabla u_k|\big|_2^2$ with different constants.

The proof of lemma 2.3 is quite technical and it will be sketched in Appendix.

PROOF OF THEOREM 2.2. Consider the sequence defined in (2.6). Obviously $|u_k|_{2^*} = 1$.

Moreover by lemma 2.3

$$\psi(u_k) \to \gamma S \quad \text{and} \quad |u_k|_2 \to 0 .$$

Then $\{u_k\}$ is not precompact in H^1. Therefore, in order to verify that $\psi|_V$ does not satisfy the (P-S) condition in γS, we need only to prove that

$$d\psi|_V(u_k) \to 0 \text{ strongly in } H^{-1} .$$

Now

$$\Delta u_k(r) = u_k''(r) + \frac{n-1}{r} u_k'(r) + u_k'(r) \frac{\partial}{\partial r} \lg \sqrt{|g|}$$

and there exists $A > 0$ such that

$$\left| \frac{\partial}{\partial r} \lg \sqrt{|g|} \right| < Ar . \tag{2.10}$$

By the above expression for $\Delta u_k(r)$, we have

$$\begin{aligned} -\gamma\, \Delta u_k + R(x) u_k - n(n-1)\omega_n^{2/n} \cdot u_k^{(n+2)/(n-2)} = \\ = Lu_k - n(n-1)\omega_n^{2/n} u_n^{(n+2)/(n-2)} + \\ + \gamma u_k' \frac{\partial}{\partial r}\left[\lg\left(\frac{\sin \alpha r}{r}\right)^{n-1} - \lg\sqrt{|g|}\right] + u_k[R(x) - \alpha^2 n(n-1)] = \chi_k \end{aligned} \tag{2.11}$$

where

$$Lu_k = -\gamma\left(u_k'' + \frac{n-1}{r} u_k' + u_k' \frac{\partial}{\partial r} \lg\left(\frac{\sin \alpha r}{r}\right)^{n-1}\right) + \alpha^2 n(n-1) u_k . \tag{2.12}$$

Moreover it can be verifed that

$$\begin{aligned} -\gamma\, \Delta u_k + R(P) u_k = \frac{n(n-1)}{|v_k|_{2^*}} \left[\left(\frac{1}{k} + \frac{1-\cos \alpha r}{\alpha^2}\right)^{-1-n/2} \cdot \right. \\ \left. \cdot \frac{2}{k}\left(1 + \frac{\alpha^2}{2k}\right) - \alpha^2\left(\frac{1}{k} + \frac{1-\cos \alpha\delta}{\alpha^2}\right)^{1-n/2}\right] \end{aligned} \tag{2.13}$$

using (A.10) in the Appendix

$$|v_k|_{2^*}^{4/(n-2)} = k\left[2^{-1+n/2}\omega_{n-1} I_n^{-1+n/2} + O\left(\frac{1}{k}\right)\right]^{2/n} .$$

Moreover

$$\omega_n = 2^{n-1}\omega_{n-1} I_n^{-1+n/2}.$$

where

$$I_n^{-1+n/2} = \int_0^{+\infty} (1+t)^{-n} t^{-1+n/2}\, dt .$$

Then

$$\frac{2}{k|v_k|_{2^*}} = \left(\frac{1}{|v_k|_{2^*}}\right)^{(n+2)/(n-2)} \cdot \left(\omega_n + O\left(\frac{1}{k}\right)\right)^{2/n} . \tag{2.14}$$

Inserting (2.14) into (2.13) and setting

$$\nu_k = \left(\frac{1}{k} + \frac{1-\cos\alpha\delta}{\alpha^2}\right)^{1-n/2},$$

we get

$$Lu_k = n(n-1)\left(1+\frac{\alpha^2}{2k}\right)\left(\omega_n + O\left(\frac{1}{k}\right)\right)^{2/n}\left[u_k + \frac{\nu_k}{|v_k|_{2^*}}\right]^{(n+2)/(n-2)} + \\ - \alpha^2 n(n-1)\frac{\nu_k}{|v_k|_{2^*}} \tag{2.15}$$

by (2.15), (2.10), (2.11) and lemma 2.3 it is easy to verify that $\chi_k \to 0$ in $L^{2n/(n+2)} = (L^{2^*})' \hookrightarrow H^{-1}$; in fact

$$\big(R(x) - \alpha^2 n(n-1)\big)u_k \to 0 \qquad \text{in } L^2 \hookrightarrow L^{2n/(n+2)}$$

$$\nu_k/|v_k|_{2^*} \to 0 \qquad \text{in } L^\infty \text{ by (A.10)}$$

and also

$$\int_M [(u_k+\nu_k)^{(n+1)/(n-2)} - u_k^{(n+1)/(n-2)}]^{2n/(n+2)}\, dM \to 0\,,$$

$$\int_M \left| u_k' \cdot \frac{\partial}{\partial r}\left(\lg\left(\frac{\sin\alpha r}{r}\right)^{n-1} - \lg\sqrt{|g|}\right)\right|^2 dM \to 0$$

Q.E.D.

Finally we can argue in an analogous manner if $R(P) \leqslant 0$ for any P.

PROOF OF THEOREM 0.2. We shall prove that for any $\varepsilon > 0$ the functional

$$\psi_\varepsilon(u) = \int_M [\gamma|\nabla u|^2 + \big(R(x)-\varepsilon\big)u^2]\, dM$$

has a minimum on the manifold

$$V = \{u \in H^1 \colon |u|_{2^*} = 1\} .$$

By theorem 0.1 (see also remark 1.2) we need only to show that

$$\mu = \inf \psi_\varepsilon|_V < \gamma S . \tag{2.16}$$

If $R(x) \leqslant 0$ for any $x \in M$, (2.16) is obviously satisfied.

Suppose now that there exists $P \in M$ such that $R(P) > 0$ and consider the sequence $\{u_k\}$ defined by (2.6).

By lemma 2.3 we have

$$\psi_\varepsilon(u_k) = \psi(u_k) - \varepsilon |u_k|_2^2 = \begin{cases} \gamma S - \varepsilon \dfrac{\gamma}{2} \omega_n^{2/n} \dfrac{1}{k} + o\left(\dfrac{1}{k}\right) & \text{if } n \geqslant 5 , \\ \gamma S - \varepsilon \dfrac{\gamma}{2} \omega^{\frac{1}{2}} \dfrac{\lg k}{k} + o(k^{-1} \lg k) & \text{if } n = 4 . \end{cases}$$

Then we can choose $k \in \mathbf{N}$ s.t.

$$\inf \psi_\varepsilon|_V \leqslant \psi_\varepsilon(u_k) < \gamma S \qquad \text{Q.E.D.} .$$

Appendix.

In proving Theorem 2.1 we have used the following result:

PROPOSITION A.1. *Let f be a C^1 functional on an Hilbert manifold V. Suppose that f is bounded from below and that it satisfies the Palais-Smale condition at $\mu = \inf f$, in the following sense:*

If $\{u_j\} \subset V$ s.t. as $j \to \infty$

(i) $df(u_j) \to 0$

(ii) $f(u_j) \to \mu$

then $\{u_j\}$ is precompact. Under the above assumption for every minimizing sequence $\{w_n\}$ for f we have $df(w_n) \to 0$.

The proof of proposition A.1 is deduced by following standard variational arguments. For completeness we shall give a sketch of the proof.

Since f satisfies the (P-S) condition at μ, the set

$$K_\mu = \{u \in V : f(u) = \mu,\ df(u) = 0\}$$

is compact.

Let $k \in \mathbf{N}$ and consider the neighborhood of K_μ defined by

$$N_k = \left\{x \in V:\ d(x, K_\mu) < \frac{1}{k}\right\}.$$

In correspondence of N_k, there exists $\varepsilon > 0$ and an homeomorphism $\eta: V \to V$ s.t.

$$\eta\big(f^{-1}(]-\infty, \mu + \varepsilon])\setminus N_k\big) \subset f^{-1}(]-\infty, \mu - \varepsilon]) = \emptyset .$$

Then

$$f^{-1}(]-\infty, \mu + \varepsilon]) \subset N_k . \tag{A.0}$$

Now consider $\{\omega_n\} \subset V$ s.t.

$$f(w_n) \to \mu .$$

Then by (A.0) there exists n_k s.t. for $n > n_k$

$$w_n \in N_k .$$

Therefore, since K_μ is compact there exists subsequence $\{w_{n_k}\}$ converging to $u \in K_\mu$, then $df(w_{n_k}) \to 0$. Q.E.D.

Let us give a sketch of the proof of Lemma 2.3. Set

$$S(r) = \{x \in M : d(x, P) = r\} = \partial B(P, r)$$

Then (cp. [1, lemma 1]) we have

$$\text{(A.1)}\qquad \int_{S(r)}\sqrt{|g|}\,d\Omega=\omega_{n-1}\left\{\left[1+a\left(\frac{1-\cos\alpha r}{\alpha^2}\right)^2\right]\left(\frac{\sin\alpha r}{\alpha r}\right)^{n-1}+o(r^5)\right\}$$

where a is a suitable constant depending on the metric. Set

$$\text{(A.2)}\qquad I_p^q=\int_0^{+\infty}(1+t)^{-p}t^q\,dt\,,$$

$$\text{(A.3)}\qquad \nu_k=\left(\frac{1-\cos\alpha\delta}{\alpha^2}+\frac{1}{k}\right)^{1-n/2}.$$

Then

$$\text{(A.4)}\qquad |v_k|_2^2=\int_0^\delta\left[\left(\frac{1}{k}+\frac{1-\cos\alpha r}{\alpha^2}\right)^{1-n/2}-\nu_k\right]^2 r^{n-1}\int_{S(r)}\sqrt{|g|}\,d\Omega\,.$$

Inserting (A.1) in (A.4) we have

$$\text{(A.5)}\qquad |v_k|_2^2=\omega_{n-1}\int_0^\delta\left\{\left[\left(\frac{1}{k}+t\right)^{1-n/2}-\nu_k\right]^2\cdot\right.$$
$$\left.\cdot(2t)^{(n-2)/2}\left(1-\frac{\alpha^2 t}{2}\right)^{(n-2)/2}(1+at^2)+O(t^{(4+n)/2})\right\}dt$$

where

$$t=\frac{1-\cos\alpha r}{\alpha^2}\,,\qquad \varrho=\frac{1-\cos\alpha\delta}{\alpha^2}\,.$$

Then by using the Taylor formula for the factor

$$(2t)^{(n-2)/2}\left(1-\frac{\alpha^2 t}{2}\right)^{(n-2)/2}(1+at)^2$$

and observing that for p and q positives

$$\int_0^\varrho\left(\frac{1}{k}+t\right)^{-p}t^q\,dt=\begin{cases} I_p^q k^{p-q-1}+O(1) & \text{if } p-q-1>0\,,\\ \log k+O(1) & \text{if } p-q-1=0\,,\\ O(1) & \text{if } p-q-1<0\,,\end{cases}$$

we get

$$\text{(A.6)}\qquad |v_k|_2^2=\begin{cases} 2^{(n-2)2}\omega_{n-1}[k^{(n-4)/2}I_{n-2}^{(n-2)/2}+\sigma_n^1] & \text{if } n\geqslant 5\,,\\ 2^{(n-2)/2}\omega_{n-1}\{\log k+O(1)\} & \text{if } n=4\,,\\ O(1) & \text{if } n=3\,,\end{cases}$$

where

$$\text{(A.7)}\qquad \sigma_n^1=\begin{cases} \dfrac{2-n}{4}\alpha^2 k^{(n-6)/2}I_{n-2}^{n/2}+o(k^{(n-6)/2}) & \text{if } n>6\,,\\ \dfrac{2-n}{4}\alpha^2\log k+o(\log k) & \text{if } n=6\,,\\ O(1) & \text{if } n=5\,.\end{cases}$$

Moreover it can be verified that the following asymptotic expansions hold:

$$\text{(A.8)}\qquad |\nabla v_k|_2^2=\left(\frac{n-2}{2}\right)^2 2^{n/2}\omega_{n-1}\,[I_n^{n/2}k^{(n-2)/2}+\sigma_n^2]$$

where

$$\text{(A.9)}\qquad \sigma_n^2=\begin{cases} -\dfrac{n\alpha^2}{4}\dfrac{n+2}{n-4}I_n^{n/2}k^{n/2-2}+o(k^{(n-4)/2}) & \text{if } n\geqslant 5\,,\\ -\dfrac{n\alpha^2}{4}\log k+O(1) & \text{if } n=4\,,\\ O(1) & \text{if } n=3\,,\end{cases}$$

$$\text{(A.10)}\qquad |v_k|_{2^*}^{2^*}=2^{(n-2)/2}\omega_{n-1}k^{n/2}I_n^{(n-2)/2}\left(1-\frac{n\alpha^2}{4k}\right)+\sigma_n^3$$

where

$$\text{(A.11)}\qquad \sigma_n^3=\begin{cases} O(k^{n/2-2}) & \text{if } n\geqslant 6\,,\\ O(k) & \text{if } n=5\,,\\ -4kv_kI_3^1+O(\log k) & \text{if } n=4\,,\\ -6kv_kI_{5/2}^{1/2}+O(\log k) & \text{if } n=3\,,\end{cases}$$

$$\text{(A.12)}\qquad \int_M Rv_k^2\,dM=$$

$$=\begin{cases} \omega_{n-1}2^{(n+2)/2}\dfrac{(n-2)(n-1)^2}{n-4}\alpha^2 I_n^{n/2}k^{(n-4)/2}+\sigma_n^4 & \text{if } n\geqslant 5\,,\\ \omega_3 2^3\cdot 3\cdot\alpha^2\log k+O(1) & \text{if } n=4\,,\\ O(1) & \text{if } n=3\,,\end{cases}$$

where

$$\text{(A.13)} \qquad \sigma_n^4 = \begin{cases} O(k^{(n-6)/2}) & \text{if } n > 6\,, \\ O(\log k) & \text{if } n = 6\,, \\ O(1) & \text{if } n = 5\,. \end{cases}$$

Moreover

$$\text{(A.14)} \qquad \int_M r^2 |\nabla v_k|^2 \, dM =$$

$$= \begin{cases} 2^{(n+2)/2}\left(\dfrac{n-2}{2}\right)^2 \omega_{n-1} I_n^{(n+2)/2} k^{(n-4)/2} + \sigma_k^5 & \text{if } n \geqslant 5\,, \\ 2^3 \omega_3 \log k + O(1) & \text{if } n = 4\,, \\ O(1) & \text{if } n = 3\,, \end{cases}$$

where

$$\text{(A.15)} \qquad \sigma_k^5 = \begin{cases} O(k^{(n-6)/2}) & \text{if } n > 6\,, \\ O(\log k) & \text{if } n = 6\,, \\ O(1) & \text{if } n = 5\,. \end{cases}$$

Then, since $u_k = v_k/|v_k|_{2^*}$, the asymptotic expansions (2.7), (2.9), are easily derived. Q.E.D.

Notes added in proofs. After submission of this paper we have known that R. Schoen (*Conformal deformation of a Riemannian metric to constant scalar curvature*, preprint) has proved that for a manifold conformally different from S^n, the conformal invariant μ is (strictly) less then γS; then the Yamabe conjecture is positively solved.

REFERENCES

[1] T. Aubin, *Equations différentielles non linéaires et Problème de Yamabe concernant la courbure scalaire*, J. Math. Pure Appl., **55** (1976), pp. 269-296.

[2] T. Aubin, *Espace de Sobolev sur le variétés Riemanniennes*, Bull. Sc. Math., **100** (1976), pp. 149-173.

[3] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*, Springer-Verlag (1982).

[4] H. Brezis - L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36** (1983), pp. 437-477.

[5] G. Cerami - D. Fortunato - M. Struwe, *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré - Nonlinear Analysis, (1984), pp. 341-350.

[6] R. Palais, *Ljusternik-Schnirelman theory on Banach manifold*, Topology, **5** (1966), pp. 115-132.

[7] P. H. Rabinowitz, *Variational methods for nonlinear eigenvalue problems*, Proceedings of C.I.M.E. (1974), pp. 139-195, G. Prodi Editor.

[8] G. Talenti, *Best constant in Sobolev inequality*, Ann. Math. Pura Appl., **110** (1976), pp. 353-372.

[9] N. Trüdinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Sc. Norm. Sup. Pisa, **22** (1968), pp. 265-274.

[10] M. Vaugon, *Equations différentielles non linéaires sur les variétés riemanniennes compactes*, Bull. Sci. Math., II Sér. 107 (1983) pp, 371-391.

[11] H. Yamabe, *On the deformation of Riemannian structures on compact manifolds*, Osaka Math. J., **12** (1960), pp. 21-37.

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