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Remarks on the Yamabe Problem and the Palais-Smale Condition.

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0. Introduction.

Let M be a C^∞ compact Riemannian manifold of dimension $n \geq 3$. Let $g(x)$ be its metric and $R(x)$ its scalar curvature. An important problem concerning the scalar curvature is the Yamabe problem: does there exist a metric g' conformal to g , such that the scalar curvature \bar{R} of the metric is constant?

Since the pionering paper of Yamabe [11] appeared, several authors have studied this problem (cp. [1, 9, 10] and the references contained in [3]).

If we consider the conformal deformation in the form $g' = u^{4/(n-2)} g$ (with $u \in C^\infty$, $u > 0$), the Yamabe problem is reduced (cp. [1, 9, 11]) to the following eigenvalue problem:

$$(0.1) \quad \begin{cases} \text{find } u \in C^\infty, u > 0 \text{ and } \bar{R} \in \mathbb{R} \text{ s.t. ,} \\ \gamma \Delta u + R(x)u = \bar{R}u^{2^*-1}, \end{cases}$$

where $\gamma = 4(n-1)/(n-2)$, Δ denotes the Laplace-Beltrami operator corresponding to g and $2^* = 2n/(n-2)$.

We denote by H^1 the Sobolev space on M , i.e. H^1 is the completion

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of the space $C^\infty(M)$ with respect to the norm

$$\|\varphi\| = \left(\int_M (|\varphi|^2 + |\nabla\varphi|^2) dM \right)^{\frac{1}{2}}$$

where $|\nabla\varphi|^2 = g^{ij}\varphi_{x_i}\varphi_{x_j}$.

$L^\alpha(\alpha > 1)$ denotes the space of the functions on M which are α -integrable. Moreover we set

$$|\varphi|_\alpha = \left(\int_M |\varphi|^\alpha dM \right)^{1/\alpha}.$$

Clearly if u minimizes the functional

$$(0.2) \quad \psi(u) = \int_M (\gamma|\nabla u|^2 + R(x)u^2) dM$$

on the manifold in H^1

$$(0.3) \quad V = \{u \in H^1 \mid |u|_{2^*} = 1\}$$

u is a weak solution of (0.1) and \bar{R} is the minimum of ψ on V . Then by a regularity result of Trudinger (cp. [9, Th. 3]) u is C^∞ . Moreover, since $|\nabla|u|| = |\nabla u|$ (cp. [3, prop. 3.49]), we can assume that $u \geq 0$. Then (cp. [3, prop. 3.75]) we deduce that $u > 0$.

Therefore the Yamabe problem is solved if the functional ψ can be minimized on V . Since H^1 is not compactly embedded into L^{2^*} it is not easy to find a minimum directly. In [1, 9, 11] the following approximation-procedure has been used. Since the embedding $H^1 \hookrightarrow L^q$ ($q < 2^*$) is compact the problem

$$-\gamma\Delta u + R(x)u = \omega_q u^{q-1} \quad |u|_q = 1$$

can be easily solved; then, by taking the limit for $q \rightarrow 2^*$, it is possible, in some cases, to solve (0.1).

If Ω is a bounded domain in \mathbb{R}^n , set

$$S = \inf \left\{ \int_\Omega |\nabla\varphi|^2 dx : |\varphi|_{2^*} = 1, \quad \varphi \in C_0^\infty(\mathbb{R}^n) \right\} \quad (n \geq 3).$$

It is known (cp. [2, 8]) that S is independent of Ω and

$$S = \frac{\omega_n^{2/n}}{4} n(n-2)$$

where ω_n denotes the area of the unit n -dimensional sphere S_n .

Using the « approximation-method » Aubin [1, Th. 7] proved the following Theorem.

THEOREM 0.1. *Suppose that*

$$\inf \{ \psi(u) : u \in V \} < S \cdot \gamma .$$

Then there exists $u \in V$ which minimized $\psi|_V$.

In [1] Aubin has also pointed out that

$$\mu = \inf \{ \psi(u) : u \in V \}$$

is a conformal invariant and that the inequality

$$\mu \leq \gamma S$$

holds for any compact n -dimensional Riemannian manifold.

In this paper we prove that $\psi|_V$ satisfies the Palais-Smale (P-S) condition in the range $]-\infty, \gamma S[$ (cp. lemma 1.1). Then Theorem 0.1 can be proved directly using standard variational arguments.

Moreover we show that $\psi|_V$ does not satisfy the (P-S) condition at $c = \gamma S$ (cp. Th. 2.1, 2.2).

It can be proved also that slight perturbations of (0.1) have always a solution. More precisely we consider the following perturbed eigenvalue problem:

$$(0.4) \quad \begin{cases} \text{find } u \in C^\infty, u > 0 \text{ and } \bar{R} \in \mathbb{R} \text{ s.t. ,} \\ -\gamma \Delta u + (R(x) - \varepsilon)u = \bar{R}u^{2^*-1}, & \varepsilon > 0, \end{cases}$$

and we prove the following

THEOREM 0.2. *Let $n \geq 4$, than for any $\varepsilon > 0$, (0.4) possesses a solution.*

Observe that Theorem 0.2 is similar to a result obtained by Brezis and Nirenberg (cp. [4, Th. 11]) in the context of elliptic boundary value problems on a bounded domain $\Omega \subset \mathbb{R}^n$.

Let us finally recall that another direct proof of Theorem 0.1 is contained in [10].

1. Proof of Theorem 0.1, 0.2.

The following lemma plays a fundamental role in proving Theorems 0.1, 0.2.

LEMMA 1.1. *The functional $\psi|_V$ defined by (0.2) and (0.3) satisfies the Palais-Smale condition in $]-\infty, \gamma S[$ in the following sense:*

(P-S) *If $\{u_j\} \subset V$ s.t. as $j \rightarrow \infty$*

$$(1.1) \quad d\psi|_V(u_j) \rightarrow 0 \text{ strongly in } H^{-1}$$

and

$$(1.2) \quad \psi(u_j) \rightarrow c, \quad c < \gamma S$$

Then $\{u_j\}$ contains a subsequence converging strongly in H^1 .

PROOF. Let $\{u_j\} \subset V$ be a sequence which satisfies (1.1) and (1.2). Then, by (1.2), we easily deduce that we can select a subsequence, which we continue to denote by $\{u_j\}$, such that

$$(1.3) \quad \begin{cases} u_j \rightarrow u \text{ weakly in } H^1, \\ u_j \rightarrow u \text{ strongly in } L^p, \end{cases} \quad 1 \leq p < 2^*.$$

Set $\varphi(u) = |u|_{2^*}^{2^*}$ $u \in H^1$.

From (1.1) we deduce that there exists a sequence $\{\lambda_j\} \subset \mathbb{R}$ s.t.

$$(1.4) \quad d\psi(u_j) - \lambda_j d\varphi(u_j) \rightarrow 0 \quad \text{in } H^{-1} = (H^1)'$$

Then, since $|u_j|_{2^*} = 1$ and by using (1.2), we obtain

$$(1.5) \quad \lambda_j = \frac{\langle d\psi(u_j), u_j \rangle}{2^*} + o(1) = \frac{2}{2^*} \psi(u_j) + o(1) = \frac{2}{2^*} c + o(1).$$

We show now that u solves the equation

$$(1.6) \quad -\gamma \Delta u + R(x)u - cu|u|^{2^*-2} = 0.$$

Let $\zeta \in C^\infty(M)$, then by (1.4), (1.3), (1.5) we deduce that

$$\begin{aligned} o(1) &= \langle d\varphi(u_j) - \lambda_j d\varphi(u_j), \zeta \rangle = \langle d\varphi(u_j), \zeta \rangle - \frac{2}{2^*} c \langle d\varphi(u_j), \zeta \rangle + o(1) = \\ &= 2 \langle -\gamma \Delta u + R(x)u, \zeta \rangle - 2c \int_M u|u|^{2^*-2} \cdot \zeta \, dM + o(1). \end{aligned}$$

Then u is a weak solution of (1.6). Then (cp. [9, Th. 3]) u is a C^∞ solution of (1.6).

To show that $u_j \rightarrow u$ strongly in $H^1(M)$, let

$$v_j = u_j - u.$$

Testing (1.4) with v_j , we obtain

$$\begin{aligned} (1.7) \quad o(1) &= \langle d\varphi(u_j), v_j \rangle - \lambda_j \langle d\varphi(u_j), v_j \rangle = \\ &= 2 \int_M [\gamma(|\nabla u| |\nabla v_j| + |\nabla v_j|^2) + R(x)(uv_j + v_j^2)] \, dM - \\ &\quad - \lambda_j \langle d\varphi(u_j), v_j \rangle. \end{aligned}$$

By (1.3) we have

$$(1.8) \quad \int_M [\gamma(|\nabla u| |\nabla v_j| + |\nabla v_j|^2) + R(x)(uv_j + v_j^2)] \, dM = o(1).$$

Whence from (1.7), (1.8) we deduce that

$$\begin{aligned} (1.9) \quad 2\gamma |\nabla v_j|_2^2 &= \lambda_j \langle d\varphi(u_j), v_j \rangle + o(1) = \\ &= 2^* \lambda_j \int_M |u + v_j|^{2^*-2} (u + v_j) v_j \, dM + o(1) \end{aligned}$$

Now we claim that

$$(1.10) \quad \gamma \|v_j\|^2 = \gamma |\nabla v_j|_2^2 + o(1) = c |v_j|_2^{2^*} + o(1).$$

In fact, following an analogous argument as in [5, lemma 2.1], we obtain:

$$\begin{aligned}
 \left| \int_M [(u + v_j)|u + v_j|^{2^*-2}v_j - |v_j|^{2^*}] dM \right| &= \\
 &= \left| \int_M \int_0^{u(x)} \frac{\partial}{\partial \xi} [(v_j + \xi)|v_j + \xi|^{2^*-2}] v_j d\xi dM \right| = \\
 &= (2^* - 1) \left| \int_M \int_0^1 |v_j + tu|^{2^*-2} v_j u dt dM \right| \leq \\
 &\leq \text{const} \left(\int_M (|u| |v_j|^{2^*-1} + |v_j| |u|^{2^*-1}) dM \right).
 \end{aligned}$$

Then, by (1.3) and since $u \in L^{2^*}(M)$, we have

$$(1.11) \quad \int_M (u + v_j) |u + v_j|^{2^*-2} v_j dM = |v_j|_{2^*}^{2^*} + o(1).$$

So (1.10) easily follows from (1.9), (1.11) and (1.5). If $c \leq 0$ from (1.10) we deduce that

$$\|v_j\| = o(1).$$

Whence

$$u_j \rightarrow u \text{ strongly in } H^1.$$

Now suppose that

$$(1.12) \quad 0 < c < \gamma S.$$

Using Theorem 2.21 in [3] (cp. also [5, lemma 2.5]) we have

$$(1.13) \quad \|v_j\|^2 \geq S |v_j|_{2^*}^{2^*} + o(1).$$

By (1.10), (1.13)

$$\begin{aligned}
 \|v_j\|^{2^*} &\geq S^{2^*/2} \frac{\gamma}{c} \|v_j\|^2 + o(1), \\
 (1.14) \quad \|v_j\|^2 &\left(\frac{\gamma}{c} S^{2^*/2} - \|v_j\|^{2^*-2} \right) \leq o(1).
 \end{aligned}$$

Let us now prove that

$$(1.15) \quad \|v_j\|^2 \leq \frac{c}{\gamma},$$

Easy calculations show that

$$(1.16) \quad \begin{aligned} \psi(u_j) = \psi(u + v_j) &= \gamma \int_M (|\nabla u|^2 + |\nabla v_j|^2) dM + \\ &+ \int_M R(x) u^2 dM + o(1). \end{aligned}$$

By (1.16) and (1.2) we deduce that

$$(1.17) \quad \gamma \|v_j\|^2 = c - \int_M (\gamma |\nabla u|^2 + R(x) u^2) dM + o(1).$$

Moreover, since u solves (1.6), we have

$$(1.18) \quad \int_M (\gamma |\nabla u|^2 + R(x) u^2) dM = c |u|_2^{2^*}.$$

From (1.17) and (1.18) we easily deduce (1.15).

By (1.14) and (1.15) we have

$$(1.19) \quad \|v_j\|^2 \left(\frac{\gamma}{c} S^{2^*/2} - \left(\frac{c}{\gamma} \right)^{(2^*-2)/2} \right) \leq o(1).$$

Since $c < \gamma S$, we have

$$(1.20) \quad \frac{\gamma}{c} S^{2^*/2} > \left(\frac{c}{\gamma} \right)^{(2^*-2)/2}.$$

Finally from (1.19) and (1.20) we deduce that

$$\|v_j\| \rightarrow 0 \quad \text{Q.E.D. .}$$

PROOF OF THEOREM 0.1. The proof of Theorem 0.1 can be deduced from lemma 1.1 using standard variational arguments. Suppose

$$\mu = \inf \psi|_{\mathcal{V}} < \gamma S$$

and arguing by contradiction assume that μ is not a critical value for $\psi|_V$. Since the (P-S) condition holds in $]-\infty, \gamma\mathcal{S}[$ by well known results (cp. [6, 7]) there exists $\varepsilon > 0$ and an homeomorphism

$$\eta: V \rightarrow V \text{ s.t.}$$

$$(u \in V, \psi(u) \leq \mu + \varepsilon) \Rightarrow (\psi(\eta(u)) \leq \mu - \varepsilon).$$

Obviously this contradicts $\mu = \inf \psi|_V$. **Q.E.D.**

REMARK 1.2. From the proofs given before it is easy to see that lemma 1.1 and Theorem 0.1 still hold if $R(x)$ is any smooth function which does not represent necessarily the scalar curvature of M .

Moreover lemma 1.1 and Theorem 0.1 hold also, with obvious changes, if we replace M with a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and H^1 with the Sobolev space $H_0^1(\Omega)$. In this setting Theorem 0.1 becomes a variant of Lemma 1.2 in [4].

REMARK 1.3. In order to apply Theorem 0.1 it is important to find conditions which guarantee

$$(1.21) \quad \mu = \inf \psi|_V < \gamma\mathcal{S}.$$

It is easy to see that if we assume $\int_M R(x) dM < (\text{meas } M)^{(n-2)/n} \gamma\mathcal{S}$, then (1.21) is verified (cp. also [1, 3, 9]).

Other classes of manifolds which satisfy (1.21) can be found in [1, 3].

2. Remarks about the (P-S) condition and Proof of Theorem 0.2.

For any Riemannian manifold [1, lemma 4] we have

$$\mu = \inf \psi|_V \leq \gamma\mathcal{S}.$$

Moreover there exist manifolds, which we call critical, for which

$$\mu = \gamma\mathcal{S}.$$

An example of critical manifold is the n -dimensional sphere $S_n (n \geq 3)$ [1, cor. 4].

For this class of manifolds the (P-S) condition does not hold in $\mu = \gamma S$.

In fact the following Theorem holds

THEOREM 2.1. *Let M be a critical Riemannian manifold i.e.*

$$\mu = \inf \psi|_V = \gamma S .$$

Then $\psi|_V$ does not satisfy the (P-S) condition in $\mu = \gamma S$, i.e. there exists a sequence $\{u_j\} \subset V$ s.t. $\psi(u_j) \rightarrow \gamma S$ and $d\psi|_V(u_j) \rightarrow 0$ and which is not precompact in H^1 .

PROOF. By a result of Aubin [3, Th. 2.21] there exists a sequence $\{\omega_j\} \subset V$ s.t.

$$|\omega_j|_2 \rightarrow 0 \quad \text{and} \quad |\nabla \omega_j|_2^2 \rightarrow S .$$

Then

$$(2.1) \quad \psi(\omega_j) \rightarrow \gamma S .$$

Since the manifold M is critical, by (2.1) we deduce that $\{\omega_j\}$ is a minimizing sequence for $\psi|_V$.

Now we argue by contradiction and suppose that $\psi|_V$ satisfies the (P-S) condition at $\mu = \gamma S$. Then by using standard variational arguments (cp. prop. A.1 in Appendix)

$$d\psi|_V(\omega_j) \rightarrow 0 .$$

Then $\{\omega_j\}$ is precompact in H^1 , therefore it contains a subsequence, which we continue to denote by $\{\omega_j\}$, such that

$$\omega_j \rightarrow \omega \text{ strongly in } H^1 \text{ (and in } L^{2^*}) .$$

Then, since $|\omega_j|_{2^*} = 1$, we have $|\omega|_{2^*} = 1$. On the other hand $|\omega|_2 = 0$. Therefore we get a contradiction. **Q.E.D.**

Also manifolds which are not critical do not satisfy the (P-S) condition at $c = \gamma S$. More precisely the following Theorem holds

THEOREM 2.2. *Let M be a C^∞ Riemannian compact manifold of dimension $n \geq 3$. Then the functional $\psi|_V$ does not satisfy the (P-S) condition in $c = \gamma S$.*

In order to prove this result we need to introduce some notations.

Let δ be a positive number less than the injectivity radius of M . If $P \in M$ we set $B(P, \delta)$ the ball of M centered at P and radius δ . Using geodesic local coordinates, if $Q \in B(P, \delta)$ we have

$$(2.2) \quad Q = (r, \theta) \in [0, \delta] \times S_{n-1}, \quad r = r(P, Q) = d(P, Q).$$

The metric tensor g can be expressed by

$$(2.3) \quad ds^2 = dr^2 + r^2 g_{\theta_i \theta_j}(r, \theta) d\theta^i d\theta^j.$$

Set

$$(2.4) \quad |g| = \det(g_{\theta_i \theta_j}).$$

Fix $\alpha > 0$. Suppose $R(P) > 0$ we choose

$$\alpha^2 = R(P)/n(n-1).$$

It is not restrictive to assume also

$$(2.5) \quad \alpha\delta < \pi.$$

In order to prove Theorem 2.2 we need to introduce a suitable sequence in V .

Following Aubin [1] we consider the sequence of functions

$$k \in \mathbb{N}, \quad v_k(Q) = \begin{cases} \left(\frac{1}{k} + \frac{1 - \cos \alpha r}{\alpha^2} \right)^{1-n/2} - \left(\frac{1}{k} + \frac{1 - \cos \alpha \delta}{\alpha^2} \right)^{1-n/2} & \text{if } r < \delta, \\ 0 & \text{if } r > \delta. \end{cases}$$

Observe that the functions $r \rightarrow [1/k + (1 - \cos \alpha r)/\alpha^2]^{1-n/2}$ solve the Yamabe equation for $M = S_n$.

Obviously v_k have supports contained in $\overline{B(P, \delta)}$. We set

$$(26) \quad u_k = \frac{v_k}{|v_k|_{2^*}}.$$

In order to prove Theorem 2.2 we need the following lemma

LEMMA 2.3. *For $k \rightarrow +\infty$ we have the following asymptotic estimates*

$$(2.7) \quad \psi(u_k) = \gamma S + \sigma_k$$

where

$$(2.8) \quad \sigma_k = \begin{cases} O(k^{-2}) & \text{if } n > 6, \\ O(\lg k \cdot k^{-2}) & \text{if } n = 6, \\ O(k^{-(n-2)/2}) & \text{if } n = 3, 4, 5. \end{cases}$$

Moreover

$$(2.9) \quad |u_k|_2^2 = \begin{cases} \frac{\gamma}{2} \omega_n^{2/n} k^{-1} + o(k^{-1}) & \text{if } n \geq 5, \omega_n = \text{meas } S_n, \\ \frac{\gamma}{2} \omega_4^{\frac{1}{2}} \frac{\lg k}{k} + o(k^{-1} \lg k) & \text{if } n = 4, \\ \text{const } k^{-\frac{1}{2}} + o(k^{-\frac{1}{2}}) & \text{if } n = 3, \end{cases}$$

and the same asymptotic estimates hold for $|r|\nabla u_k|_2^2$ with different constants.

The proof of lemma 2.3 is quite technical and it will be sketched in Appendix.

PROOF OF THEOREM 2.2. Consider the sequence defined in (2.6). Obviously $|u_k|_{2^*} = 1$.

Moreover by lemma 2.3

$$\psi(u_k) \rightarrow \gamma S \quad \text{and} \quad |u_k|_2 \rightarrow 0.$$

Then $\{u_k\}$ is not precompact in H^1 . Therefore, in order to verify that $\psi|_{\mathcal{V}}$ does not satisfy the (P-S) condition in γS , we need only to prove that

$$d\psi|_{\mathcal{V}}(u_k) \rightarrow 0 \text{ strongly in } H^{-1}.$$

Now

$$\Delta u_k(r) = u_k''(r) + \frac{n-1}{r} u_k'(r) + u_k'(r) \frac{\partial}{\partial r} \lg \sqrt{|g|}$$

and there exists $A > 0$ such that

$$(2.10) \quad \left| \frac{\partial}{\partial r} \lg \sqrt{|g|} \right| < Ar.$$

By the above expression for $\Delta u_k(r)$, we have

$$(2.11) \quad \begin{aligned} -\gamma \Delta u_k + R(x) u_k - n(n-1) \omega_n^{2/n} \cdot u_k^{(n+2)/(n-2)} = \\ = Lu_k - n(n-1) \omega_n^{2/n} u_n^{(n+2)/(n-2)} + \\ + \gamma u_k' \frac{\partial}{\partial r} \left[\lg \left(\frac{\sin \alpha r}{r} \right)^{n-1} - \lg \sqrt{|g|} \right] + u_k [R(x) - \alpha^2 n(n-1)] = \chi_k \end{aligned}$$

where

$$(2.12) \quad Lu_k = -\gamma \left(u_k'' + \frac{n-1}{r} u_k' + u_k' \frac{\partial}{\partial r} \lg \left(\frac{\sin \alpha r}{r} \right)^{n-1} \right) + \alpha^2 n(n-1) u_k.$$

Moreover it can be verified that

$$(2.13) \quad \begin{aligned} -\gamma \Delta u_k + R(P) u_k = \frac{n(n-1)}{|v_k|_2^*} \left[\left(\frac{1}{k} + \frac{1 - \cos \alpha r}{\alpha^2} \right)^{-1-n/2} \cdot \right. \\ \left. \cdot \frac{2}{k} \left(1 + \frac{\alpha^2}{2k} \right) - \alpha^2 \left(\frac{1}{k} + \frac{1 - \cos \alpha r}{\alpha^2} \right)^{1-n/2} \right] \end{aligned}$$

using (A.10) in the Appendix

$$|v_k|_2^{4/(n-2)} = k \left[2^{-1+n/2} \omega_{n-1} I_n^{-1+n/2} + O\left(\frac{1}{k}\right) \right]^{2/n}.$$

Moreover

$$\omega_n = 2^{n-1} \omega_{n-1} I_n^{-1+n/2}.$$

where

$$I_n^{-1+n/2} = \int_0^{+\infty} (1+t)^{-n} t^{-1+n/2} dt.$$

Then

$$(2.14) \quad \frac{2}{k|v_k|_{2^*}} = \left(\frac{1}{|v_k|_{2^*}} \right)^{(n+2)/(n-2)} \cdot \left(\omega_n + O\left(\frac{1}{k}\right) \right)^{2/n}.$$

Inserting (2.14) into (2.13) and setting

$$v_k = \left(\frac{1}{k} + \frac{1 - \cos \alpha \delta}{\alpha^2} \right)^{1-n/2},$$

we get

$$(2.15) \quad \begin{aligned} Lu_k = n(n-1) \left(1 + \frac{\alpha^2}{2k} \right) \left(\omega_n + O\left(\frac{1}{k}\right) \right)^{2/n} \left[u_k + \frac{v_k}{|v_k|_{2^*}} \right]^{(n+2)/(n-2)} + \\ - \alpha^2 n(n-1) \frac{v_k}{|v_k|_{2^*}} \end{aligned}$$

by (2.15), (2.10), (2.11) and lemma 2.3 it is easy to verify that $\chi_k \rightarrow 0$ in $L^{2n/(n+2)} = (L^{2^*})' \hookrightarrow H^{-1}$; in fact

$$(R(x) - \alpha^2 n(n-1)) u_k \rightarrow 0 \quad \text{in } L^2 \hookrightarrow L^{2n/(n+2)}$$

$$v_k/|v_k|_{2^*} \rightarrow 0 \quad \text{in } L^\infty \text{ by (A.10)}$$

and also

$$\int_M [(u_k + v_k)^{(n+1)/(n-2)} - u_k^{(n+1)/(n-2)}]^{2n/(n+2)} dM \rightarrow 0,$$

$$\int_M \left| u'_k \cdot \frac{\partial}{\partial r} \left(\lg \left(\frac{\sin \alpha r}{r} \right)^{n-1} - \lg \sqrt{|g|} \right) \right|^2 dM \rightarrow 0$$

Q.E.D.

Finally we can argue in an analogous manner if $R(P) \leq 0$ for any P .

PROOF OF THEOREM 0.2. We shall prove that for any $\varepsilon > 0$ the functional

$$\psi_\varepsilon(u) = \int_M [\gamma |\nabla u|^2 + (R(x) - \varepsilon) u^2] dM$$

has a minimum on the manifold

$$V = \{u \in H^1: |u|_{2^*} = 1\}.$$

By theorem 0.1 (see also remark 1.2) we need only to show that

$$(2.16) \quad \mu = \inf \psi_\varepsilon|_V < \gamma S.$$

If $R(x) \leq 0$ for any $x \in M$, (2.16) is obviously satisfied.

Suppose now that there exists $P \in M$ such that $R(P) > 0$ and consider the sequence $\{u_k\}$ defined by (2.6).

By lemma 2.3 we have

$$\psi_\varepsilon(u_k) = \psi(u_k) - \varepsilon |u_k|_2^2 = \begin{cases} \gamma S - \varepsilon \frac{\gamma}{2} \omega_n^{2/n} \frac{1}{k} + o\left(\frac{1}{k}\right) & \text{if } n \geq 5, \\ \gamma S - \varepsilon \frac{\gamma}{2} \omega^{\frac{1}{2}} \frac{\lg k}{k} + o(k^{-1} \lg k) & \text{if } n = 4. \end{cases}$$

Then we can choose $k \in \mathbf{N}$ s.t.

$$\inf \psi_\varepsilon|_V \leq \psi_\varepsilon(u_k) < \gamma S \quad \text{Q.E.D.}$$

Appendix.

In proving Theorem 2.1 we have used the following result:

PROPOSITION A.1. *Let f be a C^1 functional on an Hilbert manifold V . Suppose that f is bounded from below and that it satisfies the Palais-Smale condition at $\mu = \inf f$, in the following sense:*

If $\{u_j\} \subset V$ s.t. as $j \rightarrow \infty$

$$(i) \quad df(u_j) \rightarrow 0$$

$$(ii) \quad f(u_j) \rightarrow \mu$$

then $\{u_j\}$ is precompact. Under the above assumption for every minimizing sequence $\{w_n\}$ for f we have $df(w_n) \rightarrow 0$.

The proof of proposition A.1 is deduced by following standard variational arguments. For completeness we shall give a sketch of the proof.

Since f satisfies the (P-S) condition at μ , the set

$$K_\mu = \{u \in V : f(u) = \mu, df(u) = 0\}$$

is compact.

Let $k \in \mathbf{N}$ and consider the neighborhood of K_μ defined by

$$N_k = \left\{ x \in V : d(x, K_\mu) < \frac{1}{k} \right\}.$$

In correspondence of N_k , there exists $\varepsilon > 0$ and an homeomorphism $\eta: V \rightarrow V$ s.t.

$$\eta(f^{-1}(]-\infty, \mu + \varepsilon]) \setminus N_k \subset f^{-1}(]-\infty, \mu - \varepsilon]) = \emptyset.$$

Then

$$(A.0) \quad f^{-1}(]-\infty, \mu + \varepsilon]) \subset N_k.$$

Now consider $\{\omega_n\} \subset V$ s.t.

$$f(w_n) \rightarrow \mu.$$

Then by (A.0) there exists n_k s.t. for $n > n_k$

$$w_n \in N_k.$$

Therefore, since K_μ is compact there exists subsequence $\{w_{n_k}\}$ converging to $u \in K_\mu$, then $df(w_{n_k}) \rightarrow 0$. Q.E.D.

Let us give a sketch of the proof of Lemma 2.3. Set

$$S(r) = \{x \in M : d(x, P) = r\} = \partial B(P, r)$$

Then (cp. [1, lemma 1]) we have

$$(A.1) \quad \int_{S(r)} \sqrt{|g|} d\Omega = \omega_{n-1} \left\{ \left[1 + a \left(\frac{1 - \cos \alpha r}{\alpha^2} \right)^2 \right] \left(\frac{\sin \alpha r}{\alpha r} \right)^{n-1} + o(r^5) \right\}$$

where a is a suitable constant depending on the metric. Set

$$(A.2) \quad I_p^a = \int_0^{+\infty} (1+t)^{-p} t^a dt,$$

$$(A.3) \quad \nu_k = \left(\frac{1 - \cos \alpha \delta}{\alpha^2} + \frac{1}{k} \right)^{1-n/2}.$$

Then

$$(A.4) \quad |v_k|_2^2 = \int_0^\delta \left[\left(\frac{1}{k} + \frac{1 - \cos \alpha r}{\alpha^2} \right)^{1-n/2} - \nu_k \right]^2 r^{n-1} \int_{S(r)} \sqrt{|g|} d\Omega.$$

Inserting (A.1) in (A.4) we have

$$(A.5) \quad |v_k|_2^2 = \omega_{n-1} \int_0^\delta \left\{ \left[\left(\frac{1}{k} + t \right)^{1-n/2} - \nu_k \right]^2 \cdot (2t)^{(n-2)/2} \left(1 - \frac{\alpha^2 t}{2} \right)^{(n-2)/2} (1 + at)^2 + O(t^{(4+n)/2}) \right\} dt$$

where

$$t = \frac{1 - \cos \alpha r}{\alpha^2}, \quad a = \frac{1 - \cos \alpha \delta}{\alpha^2}.$$

Then by using the Taylor formula for the factor

$$(2t)^{(n-2)/2} \left(1 - \frac{\alpha^2 t}{2} \right)^{(n-2)/2} (1 + at)^2$$

and observing that for p and q positives

$$\int_0^e \left(\frac{1}{k} + t \right)^{-p} t^q dt = \begin{cases} I_p^a k^{p-q-1} + O(1) & \text{if } p - q - 1 > 0, \\ \log k + O(1) & \text{if } p - q - 1 = 0, \\ O(1) & \text{if } p - q - 1 < 0, \end{cases}$$

we get

$$(A.6) \quad |v_k|_2^2 = \begin{cases} 2^{(n-2)2} \omega_{n-1} [k^{(n-4)/2} I_{n-2}^{(n-2)/2} + \sigma_n^1] & \text{if } n \geq 5, \\ 2^{(n-2)/2} \omega_{n-1} \{\log k + O(1)\} & \text{if } n = 4, \\ O(1) & \text{if } n = 3, \end{cases}$$

where

$$(A.7) \quad \sigma_n^1 = \begin{cases} \frac{2-n}{4} \alpha^2 k^{(n-6)/2} I_{n-2}^{n/2} + o(k^{(n-6)/2}) & \text{if } n > 6, \\ \frac{2-n}{4} \alpha^2 \log k + o(\log k) & \text{if } n = 6, \\ O(1) & \text{if } n = 5. \end{cases}$$

Moreover it can be verified that the following asymptotic expansions hold:

$$(A.8) \quad |\nabla v_k|_2^2 = \left(\frac{n-2}{2}\right)^2 2^{n/2} \omega_{n-1} [I_n^{n/2} k^{(n-2)/2} + \sigma_n^2]$$

where

$$(A.9) \quad \sigma_n^2 = \begin{cases} -\frac{n\alpha^2}{4} \frac{n+2}{n-4} I_n^{n/2} k^{n/2-2} + o(k^{(n-4)/2}) & \text{if } n \geq 5, \\ -\frac{n\alpha^2}{4} \log k + O(1) & \text{if } n = 4, \\ O(1) & \text{if } n = 3, \end{cases}$$

$$(A.10) \quad |v_k|_2^{2*} = 2^{(n-2)/2} \omega_{n-1} k^{n/2} I_n^{(n-2)/2} \left(1 - \frac{n\alpha^2}{4k}\right) + \sigma_n^3$$

where

$$(A.11) \quad \sigma_n^3 = \begin{cases} O(k^{n/2-2}) & \text{if } n \geq 6, \\ O(k) & \text{if } n = 5, \\ -4k\nu_k I_3^1 + O(\log k) & \text{if } n = 4, \\ -6k\nu_k I_{5/2}^1 + O(\log k) & \text{if } n = 3, \end{cases}$$

$$(A.12) \quad \int_M Rv_k^2 dM = \begin{cases} \omega_{n-1} 2^{(n+2)/2} \frac{(n-2)(n-1)^2}{n-4} \alpha^2 I_n^{n/2} k^{(n-4)/2} + \sigma_n^4 & \text{if } n \geq 5, \\ \omega_3 2^3 \cdot 3 \cdot \alpha^2 \log k + O(1) & \text{if } n = 4, \\ O(1) & \text{if } n = 3, \end{cases}$$

where

$$(A.13) \quad \sigma_n^4 = \begin{cases} O(k^{(n-6)/2}) & \text{if } n > 6, \\ O(\log k) & \text{if } n = 6, \\ O(1) & \text{if } n = 5. \end{cases}$$

Moreover

$$(A.14) \quad \int_M r^2 |\nabla v_k|^2 dM = \begin{cases} 2^{(n+2)/2} \left(\frac{n-2}{2}\right)^2 \omega_{n-1} I_n^{(n+2)/2} k^{(n-4)/2} + \sigma_k^5 & \text{if } n \geq 5, \\ 2^3 \omega_3 \log k + O(1) & \text{if } n = 4, \\ O(1) & \text{if } n = 3, \end{cases}$$

where

$$(A.15) \quad \sigma_k^5 = \begin{cases} O(k^{(n-6)/2}) & \text{if } n > 6, \\ O(\log k) & \text{if } n = 6, \\ O(1) & \text{if } n = 5. \end{cases}$$

Then, since $u_k = v_k/|v_k|_2$, the asymptotic expansions (2.7), (2.9), are easily derived. Q.E.D.

Notes added in proofs. After submission of this paper we have known that R. Schoen (*Conformal deformation of a Riemannian metric to constant scalar curvature*, preprint) has proved that for a manifold conformally different from S^n , the conformal invariant μ is (strictly) less than γS ; then the Yamabe conjecture is positively solved.

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