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A Note on Certain Classes of Starlike Functions.

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Summary - A number of distortion theorems are proved for the classes $\mathcal{G}_a(n)$ and $\mathcal{C}_a(n)$ of analytic and univalent functions with negative coefficients, studied recently by S. K. Chatterjea [1]. We also present several interesting results for the convolution product of functions belonging to the classes $\mathcal{G}_a(n)$ and $\mathcal{C}_a(n)$.

1. Introduction.

Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic in the unit disk $\mathbb{U} = \{z: |z| < 1\}$. Further, let $\mathcal{S}(n)$ be the subclass of $\mathcal{A}(n)$ consisting of analytic and univalent functions in the unit disk $\mathbb{U}$. Then a function $f(z) \in \mathcal{S}(n)$ is said to be...
in the class $S_\alpha(n)$ if and only if

\begin{equation}
\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U}),
\end{equation}

for $0 \leq \alpha < 1$. Also a function $f(z) \in S(n)$ is said to be in the class $K_\alpha(n)$ if and only if

\begin{equation}
\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{U}),
\end{equation}

for $0 \leq \alpha < 1$.

We note that $f(z) \in K_\alpha(n)$ if and only if $zf'(z) \in S_\alpha(n)$, and that

$$S_\alpha(n) \subseteq S_0(n), \quad K_\alpha(n) \subseteq K_0(n), \quad \text{and} \quad K_\alpha(n) \subset S_\alpha(n).$$

for $0 \leq \alpha < 1$.

The class of functions of the form (1.1) was considered by Chen [2]. The classes $S_\alpha(1)$ and $K_\alpha(1)$ were first introduced by Robertson [9], and were studied subsequently by Schild [10], Pinchuk [8], Owa and Srivastava ([6], [7]), and others (see, for example, Duren [3] and Goodman [4]).

Let $\mathcal{C}(n)$ denote the subclass of $S(n)$ consisting of functions of the form

\begin{equation}
f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0).
\end{equation}

Denote by $\mathcal{C}_\alpha(n)$ and $C_\alpha(n)$ the classes obtained by taking intersections, respectively, of the classes $S_\alpha(n)$ and $K_\alpha(n)$ with $\mathcal{C}(n)$, that is,

\begin{equation}
\mathcal{C}_\alpha(n) = S_\alpha(n) \cap \mathcal{C}(n) \quad (0 \leq \alpha < 1; \; n \in \mathbb{N})
\end{equation}

and

\begin{equation}
C_\alpha(n) = K_\alpha(n) \cap \mathcal{C}(n) \quad (0 \leq \alpha < 1; \; n \in \mathbb{N}).
\end{equation}

The classes $\mathcal{C}_\alpha(n)$ and $C_\alpha(n)$ were studied recently by Chatterjea [1]. In particular, $\mathcal{C}_\alpha(1)$ and $C_\alpha(1)$ are the classes $\mathcal{C}^*(\alpha)$ and $C(\alpha)$, respectively, which were introduced by Silverman [12].

In order to prove our results for functions belonging to the general
classes $\mathcal{C}_\alpha(n)$ and $\mathbb{C}_\alpha(n)$, we shall require the following lemmas given by Chatterjea [1]:

**Lemma 1.** Let the function $f(z)$ be defined by (1.4). Then $f(z)$ is in the class $\mathcal{C}_\alpha(n)$ if and only if

$$\sum_{k=n+1}^{\infty} \left( k \frac{n}{n+1} \right) a_k \leq 1, \quad (n \geq 1). \quad (1.7)$$

**Lemma 2.** Let the function $f(z)$ be defined by (1.4). Then $f(z)$ is in the class $\mathbb{C}_\alpha(n)$ if and only if

$$\sum_{k=n+1}^{\infty} \left( k(k-1) \frac{n}{n+1} \right) a_k \leq 1, \quad (n \geq 1). \quad (1.8)$$

**Remark.** Lemma 1 follows immediately from a result due to Silverman [12, p. 110, Theorem 2] upon setting

$$a_k = 0, \quad k = 2, 3, \ldots, n. \quad (1.9)$$

Lemma 2, on the other hand, is a similar consequence of another result due to Silverman [12, p. 111, Corollary 2].

2. Distortion theorems.

We first prove

**Theorem 1.** Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{C}_\alpha(n)$. Then

$$|z| - \left( \frac{1-\alpha}{n+1-\alpha} \right) |z|^{n+1} \leq |f(z)| \leq |z| + \left( \frac{1-\alpha}{n+1-\alpha} \right) |z|^{n+1} \quad (2.1)$$

and

$$1 - \frac{(n+1)(1-\alpha)}{n+1-\alpha} |z|^n \leq |f'(z)| \leq 1 + \frac{(n+1)(1-\alpha)}{n+1-\alpha} |z|^n \quad (2.2)$$

for $z \in \mathbb{U}$. The results (2.1) and (2.2) are sharp.
Proof. By virtue of Lemma 1, we have

\begin{equation}
\left( \frac{n + 1 - \alpha}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right) a_k \leq 1,
\end{equation}

or

\begin{equation}
\sum_{k=n+1}^{\infty} a_k \leq \frac{1 - \alpha}{n + 1 - \alpha}.
\end{equation}

It follows from (2.4) that

\begin{equation}
|f(z)| \geq |z| - |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \geq |z| - \left( \frac{1 - \alpha}{n + 1 - \alpha} \right) |z|^{n+1}
\end{equation}

and

\begin{equation}
|f(z)| \leq |z| + |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \leq |z| + \left( \frac{1 - \alpha}{n + 1 - \alpha} \right) |z|^{n+1}.
\end{equation}

Note that

\begin{equation}
\sum_{k=n+1}^{\infty} ka_k \leq 1 - \alpha + \frac{\alpha(1 - \alpha)}{n + 1 - \alpha} = \frac{(n + 1)(1 - \alpha)}{n + 1 - \alpha}.
\end{equation}

Hence

\begin{equation}
|f'(z)| \geq 1 - |z|^n \sum_{k=n+1}^{\infty} ka_k \geq 1 - \frac{(n + 1)(1 - \alpha)}{n + 1 - \alpha} |z|^n
\end{equation}

and

\begin{equation}
|f'(z)| \leq 1 + |z|^n \sum_{k=n+1}^{\infty} ka_k \leq 1 + \frac{(n + 1)(1 - \alpha)}{n + 1 - \alpha} |z|^n.
\end{equation}

Further, by taking the function given by

\begin{equation}
f(z) = z - \frac{1 - \alpha}{n + 1 - \alpha} z^{n+1},
\end{equation}

we can show that the results (2.1) and (2.2) are sharp.

**Corollary 1.** Let the function \( f(z) \) defined by (1.4) be in the class \( \mathcal{C}_n(n) \). Then the unit disk \( \mathbb{U} \) is mapped onto a domain that contains the disk \( |w| < n/(n + 1 - \alpha) \).
Next we state

**Theorem 2.** Let the function $f(z)$ defined by (1.4) be in the class $C_\alpha(n)$. Then

$$|z| \leq \frac{1 - \alpha}{(n + 1)(n + 1 - \alpha)} |z|^{n+1} \leq |f(z)| \leq$$

$$\leq |z| + \frac{1 - \alpha}{(n + 1)(n + 1 - \alpha)} |z|^{n+1}$$

and

$$1 - \left( \frac{1 - \alpha}{n + 1 - \alpha} \right) |z|^n \leq |f'(z)| \leq 1 + \left( \frac{1 - \alpha}{n + 1 - \alpha} \right) |z|^n$$

for $z \in \mathcal{U}$. The results (2.11) and (2.12) are sharp.

**Proof.** In view of Lemma 2, we have

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{1 - \alpha}{(n + 1)(n + 1 - \alpha)}$$

and

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{1 - \alpha}{n + 1 - \alpha},$$

and the assertions (2.11) and (2.12) of Theorem 2 follow from (2.13) and (2.14), respectively.

Noting that the results (2.11) and (2.12) are sharp for the function given by

$$f(z) = z - \frac{1 - \alpha}{(n + 1)(n + 1 - \alpha)} z^{n+1},$$

the proof of Theorem 2 is completed.

**Corollary 2.** Let the function $f(z)$ defined by (1.4) be in the class $C_\alpha(n)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < r_0$, where $r_0$ is given by

$$r_0 = 1 - \frac{1 - \alpha}{(n + 1)(n + 1 - \alpha)}.$$
3. Convolution product.

Let \( f_i(z) \) (\( i = 1, 2 \)) be defined by

\[
 f_i(z) = z - \sum_{k=n+1}^{\infty} a_{i,k} z^k \quad (a_{i,k} \geq 0).
\]

We denote by \( f_1 \ast f_2(z) \) the convolution product of the functions \( f_1(z) \) and \( f_2(z) \) defined by

\[
 f_1 \ast f_2(z) = z - \sum_{k=n+1}^{\infty} a_{1,k} a_{2,k} z^k,
\]

and we prove

**Theorem 3.** Let the functions \( f_i(z) \) (\( i = 1, 2 \)) be in the class \( \mathcal{C}_\alpha(n) \). Then \( f_2(z) \) belongs to the class \( \mathcal{C}_\beta(n) \), where

\[
 \beta = \frac{n + 1 - \alpha^2}{n + 2 - 2\alpha}.
\]

The result is sharp.

**Proof.** Employing the technique used earlier by Schild and Silverman [11], we need to find the largest \( \beta \) such that

\[
 \sum_{k=n+1}^{\infty} \left( \frac{k - \beta}{1 - \beta} \right) a_{1,k} a_{2,k} \leq 1.
\]

Since

\[
 \sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right) a_{1,k} \leq 1
\]

and

\[
 \sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right) a_{2,k} \leq 1,
\]

by the Cauchy-Schwarz inequality we have

\[
 \sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right) \sqrt{a_{1,k} a_{2,k}} \leq 1.
\]
Thus it is sufficient to show that

\[(3.8) \quad \left(\frac{k-\beta}{1-\beta}\right)a_{1,k}a_{2,k} \leq \left(\frac{k-\alpha}{1-\alpha}\right)\sqrt{a_{1,k}a_{2,k}}, \quad (k \geq n + 1),\]

that is, that

\[(3.9) \quad \sqrt{a_{1,k}a_{2,k}} \leq \frac{(k-\alpha)(1-\beta)}{(1-\alpha)(k-\beta)}.\]

Note that

\[(3.10) \quad \sqrt{a_{1,k}a_{2,k}} \leq \frac{1-\alpha}{k-\alpha}, \quad (k \geq n + 1).\]

Consequently, we need only to prove that

\[(3.11) \quad \frac{1-\alpha}{k-\alpha} \leq \frac{(k-\alpha)(1-\beta)}{(1-\alpha)(k-\beta)}, \quad (k \geq n + 1),\]

or, equivalently, that

\[(3.12) \quad \beta \leq \frac{(k-\alpha)^2 - k(1-\alpha)^2}{(k-\alpha)^2 - (1-\alpha)^2}, \quad (k \geq n + 1).\]

Since

\[(3.13) \quad \Phi(k) = \frac{(k-\alpha)^2 - k(1-\alpha)^2}{(k-\alpha)^2 - (1-\alpha)^2}\]

is an increasing function of \(k\), letting \(k = n + 1\) in (3.12) we obtain

\[(3.14) \quad \beta \leq \Phi(n + 1) = \frac{n + 1-\alpha^2}{n + 2 - 2\alpha^2},\]

which completes the proof of the theorem.

Finally, by taking the functions given by

\[(3.15) \quad f_i(z) = z - \frac{1-\alpha}{n + 1-\alpha} z^{n+1}, \quad (i = 1, 2),\]

we can see that the result is sharp.
Similarly, we have

**THEOREM 4.** Let the functions \( f_i(z) \) \((i = 1, 2)\) defined by (3.1) be in the class \( C_\alpha(n) \). Then \( f_1 \ast f_2(z) \) belongs to the class \( C_\beta(n) \), where

\[
\beta = \frac{n(n + 1)(n + 2 - 2\alpha)}{(n + 1)^2 - 2n(n + 2)\alpha + n\alpha^2 - 1}.
\]

The result is sharp for the functions given by

\[
f_i(z) = z - \frac{1 - \alpha}{(n + 1)(n + 1 - \alpha)} z^{n+1}, \quad (i = 1, 2).
\]

Finally, we prove

**THEOREM 5.** Let the functions \( f_i(z) \) \((i = 1, 2)\) defined by (3.1) be in the class \( C_\alpha(n) \). Then the function

\[
h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2) z^k
\]

belongs to the class \( C_\beta(n) \), where

\[
\beta = \frac{n^2 + 2n\alpha(1 - \alpha) - (1 - \alpha)^2}{n^2 + 2n(1 - \alpha) - (1 - \alpha)^2}.
\]

The result is sharp for the functions \( f_i(z) \) \((i = 1, 2)\) defined by (3.15).

**Proof.** By virtue of Lemma 1, we obtain

\[
\sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right)^2 a_{1,k}^2 \leq \left[ \sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right) a_{1,k} \right]^2 \leq 1,
\]

and

\[
\sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right)^2 a_{2,k}^2 \leq \left[ \sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right) a_{2,k} \right]^2 \leq 1.
\]

It follows from (3.20) and (3.21) that

\[
\sum_{k=n+1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} \right)^2 (a_{1,k}^2 + a_{2,k}^2) \leq 2.
\]
Therefore, we need to find the largest $\beta$ such that

$$
\frac{k - \beta}{1 - \beta} \leq \frac{1}{2} \left( \frac{k - \alpha}{1 - \alpha} \right)^2, \quad (k \geq n + 1),
$$

that is,

$$
\beta \leq \frac{(k - \alpha)^2 - 2k(1 - \alpha)^2}{(k - \alpha)^2 - 2(1 - \alpha)^2}, \quad (k \geq n + 1).
$$

Since

$$
\Psi(k) = \frac{(k - \alpha)^2 - 2k(1 - \alpha)^2}{(k - \alpha)^2 - 2(1 - \alpha)^2}
$$

is an increasing function of $k$, we readily have

$$
\beta \leq \Psi(n + 1) = \frac{n^2 + 2n\alpha(1 - \alpha) - (1 - \alpha)^2}{n^2 + 2n(1 - \alpha) - (1 - \alpha)^2},
$$

and Theorem 5 follows at once.

REFERENCES


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