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On Differential Systems with Impulsive Controls.

ALBERTO BRESSAN (*)

SUNTO - Si dimostra un teorema di buona posizione per un problema di Cauchy con controlli impulsivi. Ciò conduce ad una nuova definizione di soluzioni generalizzate, per cui sussiste un generale risultato di unicità.

1. Introduction.

Let f, g be continuously differentiable mappings and consider the control system

$$(1.1) \qquad \begin{cases} \dot{x}(t) = f(t, x, u) + g(t, x, u)\dot{u}(t) \\ x(0) = \bar{x} \in \mathbf{R}^n, \end{cases}$$

where, as usual, dots denote differentiation w.r.t. time. The presence of the derivative of the control u on the right-hand side of (1.1), which is motivated by several applications [1, 2, 4], requires a careful definition of solutions of (1.1). Indeed, as long as the control u is C^1, the classical theory on O.D.E. applies. However, if u is assumed to be a bounded measurable function, its derivative can only be interpreted as a distribution. Two main approaches to the Cauchy Problem (1.1) are then possible. In [4], solutions are defined in the distributional sense, and existence theorems are proven for scalar controls with bounded variations. In [5], Sussmann considered the input-output functional φ that maps a smooth control $u(\cdot)$ into the corresponding

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trajectory $x(u, \cdot)$ of (1.1) on the time interval $[0, T]$. In the case of scalar controls, he showed that the map φ admits a unique continuous extension $\hat{\varphi}$ defined for controls u which are merely continuous, possibly with unbounded variation. This provided a new method to construct Stratonovich solutions for stochastic differential equations driven by scalar white noise.

Aim of the present paper is to push this second approach further, in order to include discontinuous controls as well. For scalar controls, we prove that the functional φ is Lipschitz continuous with respect to suitable $\mathcal{L}^1$ norms on the spaces of controls and trajectories, hence it admits a unique extension to a functional $\tilde{\varphi}$ that maps $\mathcal{L}^1$-equivalence classes of controls into $\mathcal{L}^1$-equivalence classes of trajectories. This correspondence can be further refined by constructing a version of $\tilde{\varphi}$ which is Lipschitz continuous w.r.t. the norms of uniform convergence on $[0, T]$. A new definition of generalized solution, similar to the one in [5], is given in § 3. We conclude with an example of a Cauchy Problem which, according to [4], has infinitely many solutions. The present definition, on the contrary, singles out a unique acceptable trajectory.

2. The basic estimates.

Let V be an open set in $\mathbb{R}^n$ and let f, g be $\mathcal{C}^1$ and $\mathcal{C}^2$ functions respectively, from V into $\mathbb{R}^n$. Given a scalar control $u(\cdot) \in \mathcal{C}^1[0, T]$, we denote by $x(u, \cdot)$ the solution (if it exists) of the Cauchy Problem

$$(2.1) \qquad \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))\,\dot{u}(t)\,, \\ x(0) = \bar{x} \in V, \end{cases}$$

on the time interval $[0, T]$. Using the coordinates $x = (x_1, \ldots, x_n)$, (2.1) becomes

$$(2.2) \qquad \begin{cases} \dot{x}_1 = f_1(x) + g_1(x)\,\dot{u}\,, & x_1(0) = \bar{x}_1\,, \\ \qquad \cdots\cdots\cdots\cdots & \\ \dot{x}_n = f_n(x) + g_n(x)\,\dot{u}\,, & x_n(0) = \bar{x}_n\,. \end{cases}$$

In order to extend the input-output map $\varphi\colon u(\cdot) \to x(u, \cdot)$ from $\mathcal{C}^1[0, T]$ to a broader class of controls, it is necessary to investigate the continuity of φ w.r.t. weaker norms on the spaces of controls and trajectories.

THEOREM 1. Let $U \subset C^1[0, T]$ and let $K \subset V$, $K' \subset \mathbf{R}$ be compact sets such that

i) all controls $u \in U$ take values inside K',

ii) for every $u \in U$, the solution $x(u, \cdot)$ of (2.1) exists on $[0, T]$ and takes values inside K.

Then there exists a constant M such that

$$\begin{aligned}(2.3) \qquad |x(u, \tau) - x(v, \tau)| + \int_0^T |x(u, t) - x(v, t)|\, dt \leqslant \\ \leqslant M\Big[|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^T |u(t) - v(t)|\, dt\Big]\end{aligned}$$

for all $u, v \in U$, $\tau \in [0, T]$.

The theorem will be proven first for control systems of the form

$$(2.4) \qquad \begin{cases} \dot{x} = f(x) + e\dot{u}(t)\,, \\ x(0) = \bar{x} \in \mathbf{R}^n\,, \end{cases}$$

where f is a C^1 vector field with compact support in $\mathbf{R}^n$ and e is a unit vector, then in the general case. For any $u \in C^1[0, T]$, (2.4) is equivalent to the integral equation

$$(2.5) \qquad x(t) = \bar{x} + \int_0^t f(x(s))\, ds + e[u(t) - u(0)]\,,$$

which can be written in the more compact form

$$(2.6) \qquad x(u) = \psi(u, x(u))$$

with

$$(2.7) \qquad \psi(u, x)(t) = \bar{x} + \int_0^t f(x(s))\, ds + e[u(t) - u(0)]\,.$$

In order to show that the functional $u \to x(u, \cdot)$, implicitly defined by (2.6), is Lipschitz continuous (w.r.t. suitable norms), we rely upon the following corollary of the Contraction Mapping Theorem [3].

Lemma 1. Let E, F be Banach spaces, $\psi\colon E\times F\to F$ be a map such that $\forall u,\ v\in E$, $\forall x,\ y\in F$ one has

$$\left\{\begin{array}{l}\|\psi(u,x)-\psi(u,y)\|_F\leqslant \frac{1}{2}\|x-y\|_F\\ \|\psi(u,x)-\psi(v,x)\|_F\leqslant L\|u-v\|_E\end{array}\right. \tag{2.8}$$

for some constant L. Then for each $u\in E$ there exists a unique $x = x(u)\in F$ such that $x(u)=\psi(u,x(u))$. Moreover

$$\|x(u)-x(v)\|_F\leqslant 2L\|u-v\|_E\,. \tag{2.9}$$

Proof of the Lemma. For each $u\in E$, $x(u)$ exists and is unique, being the fixed point of the strict contraction $x\to\psi(u,x)$ in F. Moreover

$$\|x(u)-x(v)\|_F\leqslant\|\psi(u,x(u))-\psi(u,x(v))\|_F+$$
$$+\|\psi(u,x(v))-\psi(v,x(v))\|_F\leqslant\tfrac{1}{2}\|x(u)-x(v)\|_F+L\|u-v\|_E$$

from which (2.9) follows.

To prove (2.3) for the special system (2.4), choose a constant $N\geqslant 1$ such that the operator norm of the derivative of f satisfies

$$\|f_x(x)\|\leqslant N\qquad\forall x\in\mathbb{R}^n\,. \tag{2.10}$$

We will apply Lemma 1 to the functional ψ defined by (2.7) on the spaces $E=\{u;\,u\in\mathfrak{L}^1[0,T]\}$ with norm

$$\|u\|_E=|u(0)|+|u(\tau)|+\int_0^T|u(t)|\,dt$$

and $F=\{x\in\mathfrak{L}^1([0,T];\mathbb{R}^n)\}$ with norm

$$\|x\|_F=\frac{\exp(-4NT)}{4N}\,|x(\tau)|+\int_0^T\exp(-4Nt)|x(t)|\,dt\,.$$

The assumptions (2.8) are both satisfied. Indeed, if $u \in E$, $x, y \in F$, recalling (2.10) one has

$$\|\psi(u, x) - \psi(u, y)\|_F = \left|\int_0^\tau [f(x(t)) - f(y(t))]\, dt\right| \cdot \frac{\exp(-4NT)}{4N} +$$

$$+ \int_0^T \exp(-4NT) \left|\int_0^t [f(x(s)) - f(y(s))]\, ds\right| dt \leqslant$$

$$\leqslant \int_0^\tau N|x(t) - y(t)|dt \cdot \frac{\exp(-4NT)}{4N} + \int_0^T \exp(-4Nt) \int_0^t N|x(s) - y(s)|ds\, dt \leqslant$$

$$\leqslant \frac{1}{4} \int_0^\tau \exp(-4Nt)|x(t) - y(t)|dt + \int_0^T |x(s) - y(s)| \int_0^T N \exp(-4Nt)\, dt\, ds \leqslant$$

$$\leqslant \frac{1}{4} \|x - y\|_F + \int_0^T |x(s) - y(s)| \cdot \frac{1}{4} [\exp(-4Ns) - \exp(-4NT)]\, ds \leqslant$$

$$\leqslant \frac{1}{2} \|x - y\|_F ,$$

hence $(2.8)_1$ holds. As to $(2.8)_2$ we have, for any $u, v \in E$, $x \in F$:

$$\|\psi(u, x) - \psi(v, x)\|_F \leqslant |u(\tau) - v(\tau)| +$$

$$+ \int_0^T \exp(-4Nt)\, |(u(t) - u(0)) - (v(t) - v(0))|\, dt \leqslant \|u - v\|_E .$$

This yields $(2.8)_2$ with $L = 1$.

By Lemma 1, the map $u \to x(u, \cdot)$, implicity defined by (2.6), (2.7), is Lipschitz continuous with constant 2. This means that, for all $u, v \in C^1[0, T]$,

$$\frac{|x(u, \tau) - x(v, \tau)|}{4N \exp(4NT)} + \int_0^T \exp(-4Nt)|x(u, t) - x(v, t)|dt \leqslant$$

$$\leqslant 2 \left[|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^T |u(t) - v(t)| dt \right],$$

and yields (2.3) with $M = 8N \exp(4NT)$.

To achieve the proof in the general case, notice first that if f, g in (2.1) are replaced by vector fields f^*, g^* with compact support such that

$$f^*(x) = f(x), \; g^*(x) = g(x) \qquad \forall x \in K,$$

then the input-output map $\varphi\colon u \to x(u, \cdot)$ does not change on U. We can thus assume that f and g have already compact support.

Consider the system on $\mathbb{R}^{n+1}$ obtained by adjoining to (2.2) the trivial equation $\dot{x}_0 = \dot{u}$, $x_0(0) = 0$, which yields $x_0(t) = u(t) - u(0)$. This can be written in the form

$$\text{(2.11)} \qquad \begin{cases} \dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\,\dot{u}\,, \\ \tilde{x}(0) = (0, \bar{x}_1, \ldots, \bar{x}_n)\,, \end{cases}$$

with $\tilde{x} = (x_0, x_1, \ldots, x_n) = (x_0, x) \in \mathbb{R}^{n+1}$,

$$\tilde{f}(\tilde{x}) = (0, f_1, \ldots, f_n), \qquad \tilde{g}(\tilde{x}) = \big(1, g_1(x), \ldots, g_n(x)\big)\,.$$

Construct on $\mathbb{R}^{n+1}$ a new set of coordinates $\tilde{y} = (y_0, \ldots, y_n)$ as follows, Given the $n+1$-tuple $(y_0, \ldots, y_n)$, let $s \to \big(x_0(s), \ldots, x_n(s)\big)$ be the solution of the Cauchy problem

$$\text{(2.12)} \qquad \begin{cases} \dot{x}_0(0) = 1 & x_0(0) = 0 \\ \dot{x}_1(s) = g_1\big(x(s)\big) & x_1(0) = y_1 \\ \cdots & \cdots \\ \dot{x}_n(s) = g_n\big(x(s)\big) & x_n(0) = y_n\,. \end{cases}$$

Define $(y_0, \ldots, y_n)$ as the new coordinates of the point $P = (x_0, \ldots, x_n)$ in $\mathbb{R}^{n+1}$ reached by the solution of (2.12) at time $s = y_0$. It is now easy to verify that the coordinate transformation

$$(y_0, \ldots, y_n) \to \big(x_0(\tilde{y}), \ldots, x_n(\tilde{y})\big)$$

is a C^2 homeomorphism of $\mathbb{R}^{n+1}$ into itself, and that in the new coordinates the vector field $\tilde{g}$ has the constant expression $\tilde{g}(\tilde{y}) = (1, 0, \ldots, 0)$, while the components of f are still given by C^1 functions with compact support.

By the first part of the proof, Theorem 1 holds for the system (2.11). Therefore it holds for (2.1) as well.

3. A class of generalized solutions.

In analogy with [5], a notion of generalized solution for (2.1) can now be introduced.

DEFINITION. Given an equivalence class of bounded controls $u \in \mathcal{L}^1[0, T]$ and an initial value $u(0)$, a trajectory $t \to x(u, t)$ is a generalized solution of (2.1) if there exists a sequence of controls $v_k \in \mathcal{C}^1[0, T]$ such that $v_k(0) = u(0)$, $v_k \to u$ in $\mathcal{L}^1$, and the corresponding trajectories $x(v_k, \cdot)$ have uniformly bounded values and tend to $x(u, \cdot)$ in the $\mathcal{L}^1$ norm.

Thanks to the estimate (2.3), any uniform a priori bound on $x(v_k, t)$, $t \in [0, T]$, for some sequence $v_k \to u$ will provide the existence of a generalized solution to (2.1). Such solution is unique up to $\mathcal{L}^1$-equivalence and depends continuously on the control. In the case where u is defined pointwise on $[0, T]$, the trajectory $x(u, \cdot)$ can also be pointwise determined. Indeed, assume that for any fixed $\tau \in [0, T]$ there exists a sequence of $\mathcal{C}^1$ controls w_k^τ such that $w_k^\tau(0) = u(0)$, $w_k^\tau(\tau) = u(\tau)$ and $w_k^\tau \to u$ in $\mathcal{L}^1[0, T]$. The estimate (2.3) then implies that, as $k \to \infty$, $x(w_k^\tau, \cdot)$ tends to $x(u, \cdot)$ in $\mathcal{L}^1$ and $x(w_k^\tau, \tau)$ has a limit, say $\tilde{x}(\tau)$. Repeating this construction for all τ, one obtains a function $\tau \to \tilde{x}(\tau)$ defined pointwise on $[0, T]$. Notice that from any sequence v_k converging to u in $\mathcal{L}^1$ one can extract a subsequence v_k' which converges pointwise to u on the complement $[0, T] \setminus \mathcal{N}$ of a set $\mathcal{N}$ of measure zero. The estimate (2.3) implies that $x(v_k', \tau)$ converges to $\tilde{x}(\tau)$ for all $\tau \notin \mathcal{N}$, hence $\tilde{x}(\cdot)$ is a generalized solution of (2.1). More generally, if the control u is pointwise determined at $t = 0$ and on some subset $I \subset [0, T]$, the same is true for the corresponding trajectory.

Theorem 1 can be extended to the case where f and g depend on t and u as well, simply by adding the new variables $x_{n+1} = t$, $x_{n+2} = u$. The Lipschitz continuity of the trajectory $x(u, \cdot)$ w.r.t. changes in the initial condition $\bar{x}$ can also be proven. It is interesting to study the behaviour of the trajectory at points τ where the control has a jump.

PROPOSITION. Assume that there exists the limits

$$\lim_{t \to \tau -} u(t) = u^-, \quad \lim_{t \to \tau +} u(t) = u^+ .$$

Then the limits

$$\lim_{t\to\tau-} x(u,t) = x^-, \qquad \lim_{t\to\tau+} x(u,t) = x^+$$

exist and

$$x^+ = \big(\exp(u^+ - u^-)g\big)(x^-)\,. \tag{3.1}$$

As customary, the right-hand side of (3.1) denotes the value at time $t = u^+ - u^-$ of the solution to the Cauchy Problem $\dot{y} = g(y)$, $y(0) = x^-$. Indeed, by the same change of variable used in § 2, it suffices to prove the result for the system (2.4), in which case (3.1) becomes simply

$$x^+ = x^- + (u^+ - u^-)\boldsymbol{e}\,,$$

and the Proposition follows from (2.5).

EXAMPLE. Consider the scalar equation

$$\dot{x}(t) = 2(t+1)^{-1}x(t)\dot{u}(t)\,, \qquad t \in [0,2], \tag{3.2}$$

with $x(0) = 0$ and $u(t) = t$ for $0 \leqslant t < 1$, $u(t) = t + 1$ for $1 \leqslant t \leqslant 2$. In [4, p. 19] the authors consider infinitely many solutions of (3.2), given by

$$x(t) = 0 \quad \text{if } 0 \leqslant t < 1\,,$$

$$x(t) = \frac{c(t+1)^2}{4} \quad \text{if } 1 \leqslant t \leqslant 2\,,$$

where c is an arbitrary constant. However, according to the definition given in the present paper, the only acceptable solution is obtained for $c = 0$, because $x(v,t) \equiv 0$ for every $v \in C^1[0,2]$.

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