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**Nonexistence of Solutions for Differential Inclusions
with Upper Semicontinuous
Nonconvex Right-Hand Side.**

ALBERTO BRESSAN (*)

This note is concerned with the «feedback» differential inclusion

$$(1) \quad \dot{x}(t) = f(x(t), u(t)), \quad x(0) = 0 \in \mathbb{R}^n,$$

$$(2) \quad u(t) \in R(x(t)),$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous and $R: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an upper semicontinuous multifunction with compact, convex values. Under various additional conditions on f and R , the existence of a Caratheodory solution for the Cauchy problem (1) with the constraints (2) was proved in [2]. The two counterexamples given in this note show that, in the general case, the problem (1), (2) need not have solutions. In the first example, f is a quadratic function of u , independent of x . In the second example, the upper semicontinuous map R is «best possible» and f is C^∞ . Still, the viability conditions used in [2] fail, and no solution is found.

EXAMPLE 1. Consider the function

$$(3) \quad \varphi(x) = x \cdot \cos \frac{1}{x} \quad \text{if } x \neq 0, \quad \varphi(0) = 0.$$

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Define the map $f(u) = (1 - u^2, u)$ and the upper semicontinuous multifunction

$$(4) \quad R(x_1, x_2) = \begin{cases} \{-1\} & \text{if } x_2 > \varphi(x_1), \\ [-1, 1] & \text{if } x_2 = \varphi(x_1), \\ \{1\} & \text{if } x_2 < \varphi(x_1). \end{cases}$$

Then the problem (1), (2) has no solutions. Indeed, any trajectory $x(\cdot)$ satisfying (1), (2) must be a nonconstant, Lipschitz continuous function of t , taking values inside the graph of φ . However, this is impossible because any continuous arc on the graph of φ , connecting the origin to any other point $(x_1, \varphi(x_1))$, has infinite length.

EXAMPLE 2. Let φ be as in (3) and let $\psi: \mathbf{R}^2 \rightarrow \mathbf{R}$ be a C^∞ function such that

$$\begin{cases} \psi(x_1, x_2) > 0 & \text{if } x_2 > \varphi(x_1), \\ \psi(x_1, x_2) = 0 & \text{if } x_2 \leq \varphi(x_1). \end{cases}$$

Let $\gamma: [-2, 2] \rightarrow \mathbf{R}$ be a C^∞ function such that

$$\begin{cases} \gamma(u) < 0 & \text{if } u \in [-2, -1), \\ \gamma(u) = 0 & \text{if } u \in [-1, 1], \\ \gamma(u) > 0 & \text{if } u \in (1, 2]. \end{cases}$$

Consider the C^∞ function $f: \mathbf{R}^3 \times \mathbf{R} \rightarrow \mathbf{R}^3$,

$$\begin{aligned} f(x, u) &= (f_1, f_2, f_3) = \\ &= (1 - u^2, u, \gamma(u) + (u + 1)\psi(x_1, x_2) + (u - 1)\psi(-x_1, -x_2)), \end{aligned}$$

and define the upper semicontinuous map $R: \mathbf{R}^3 \rightarrow \mathbf{R}$,

$$R(x_1, x_2, x_3) = \begin{cases} \{-2\} & \text{if } x_3 > 0, \\ [-2, 2] & \text{if } x_3 = 0, \\ \{2\} & \text{if } x_3 < 0. \end{cases}$$

For these maps f and R , the Cauchy problem (1), (2) on \mathbf{R}^3 has no solution. Indeed, if $x_3 > 0$,

$$f(x, u) = f(x, -2) = (-3, -2, \gamma(-2) - \Psi(x_1, x_2) - 3\psi(-x_1, -x_2)),$$

hence $f_3 < 0$. If $x_3 < 0$,

$$f(x, u) = f(x, 2) = (-3, 2, \gamma(2) + 3\psi(x_1, x_2) + \psi(-x_1, -x_2)),$$

hence $f_3 > 0$. No solution can thus leave the plane $x_3 = 0$.

Any solution of (1), (2) therefore has the form $x(t) = (x_1(t), x_2(t), 0)$. This yields the two-dimensional problem

$$(5) \quad (\dot{x}_1, \dot{x}_2) = (1 - u^2, u),$$

$$(6) \quad u \in [-2, 2] \cap \{u; f_3(x, u) = 0\}.$$

The choice of the functions ψ and γ , used in the definition of f , implies that the problem (5), (6) is precisely the one considered in Example 1, hence it has no solutions.

REFERENCES

- [1] J. P. AUBIN - A. CELLINA, *Differential Inclusions*, Springer, Berlin, 1984.
- [2] J. P. AUBIN - H. FRANKOWSKA, *Trajectoires lourdes de systèmes contrôlés*, C. R. Acad. Paris Sér. I Math., **298** (1984), pp. 521-524.

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