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Product measurability and Scorza-Dragoni’s type property


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and Scorza-Dragoni’s Type Property.

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Summary - A new proof is given that if \( f: T \times X \to \mathbb{R} \) is normal integrand if and only if \( f \) verifies Scorza-Dragoni’s property in the semicontinuous case.

1. Introduction.

First we recall the notions of measurability and semicontinuity for a real function. So, if \( (S, \Sigma) \) is a measurable space then a function \( h: S \to \mathbb{R} \), where \( \mathbb{R} \) denotes a real line with natural topology, is called \( \Sigma \)-measurable if \( h^{-1}(A) \in \Sigma \) for each open subset \( A \) of \( \mathbb{R} \). It is well known that \( h: S \to \mathbb{R} \) is \( \Sigma \)-measurable if and only if \( h^{-1}(A) \in \Sigma \) for each closed subset \( A \) of \( \mathbb{R} \) or \( \mathbb{R} \in \Sigma \), where \( h^{-1}(A) \) may have one of the following forms: \((a, \infty)\), \([a, \infty)\), \((\infty, a)\) or \((\infty, a]\). A function \( g: E \to \mathbb{R} \), where \( E \) is a topological space, is called lower semicontinuous \((lsc)\) (resp. upper semicontinuous \((usc)\)) if the set \( g^{-1}((-\infty, a]) \) (\( g^{-1}([a, \infty)) \)) is closed for each \( a \in \mathbb{R} \).

Now let \( (T, \mathcal{A}, \mu) \) be a measure space, where \( T \) is a metric compact Hausdorff space and \( \mu \) a Borel, \( \sigma \)-finite, regular and complete measure defined on a \( \sigma \)-field \( \mathcal{A} \) of subsets of \( T \). Let \( X \) be a Polish space (i.e. separable, complete, metric) and let \( \mathcal{B}(X) \) be the \( \sigma \)-field of Borel sets in \( X \). By \( \mathcal{A} \times \mathcal{B}(X) \) we denote the \( \sigma \)-field on \( T \times X \) generated by sets \( A \times B \) where \( A \in \mathcal{A} \), \( B \in \mathcal{B}(X) \). Consider a function

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\( f : T \times X \rightarrow \mathbb{R} \). We say that \( f \) is \( C_* \) type (resp. \( C^* \) type) if \( f(t, \cdot) \) is lsc (resp. usc) for each \( t \in T \) and \( f(\cdot, x) \) is \( \mathcal{A} \)-measurable for each \( x \in X \).

Furthermore we call \( f : T \times X \rightarrow \mathbb{R} \) Scorza-Dragonian if it satisfies the following condition: for every \( \epsilon > 0 \) there exists a closed subset \( T_\epsilon \) of \( T \), with \( \mu(T \setminus T_\epsilon) < \epsilon \), such that the restriction of \( f \) to the set \( T_\epsilon \times X \) is jointly semicontinuous (this condition is analogous to that of Scorza-Dragoni which had been introduced in [SD] for Caratheodory's functions). More precisely we shall say that \( f \) has \( SD_* \) property (resp. \( SD^* \) property) if \( f|_{T \times X} \) is lsc (resp. usc).

The properties of Scorza-Dragonians have been investigated in [BA] and [Z]. Specially worthy of notice is the paper [BA] in which the author examined in detail the relationship between \( \mathcal{A} \times \mathcal{B}(X) \)-measurability and \( SD_* \) property. There Bottaro Aruffo showed, among others, that for \( C_* \) type functions the \( \mathcal{A} \times \mathcal{B}(X) \)-measurability and \( SD_* \) property are equivalent. The \( \mathcal{A} \times \mathcal{B}(X) \)-measurable and \( C_* \) type function is also called the normal integrand (cf. [BA, Definizione 3.0]).

In the present note, based on [Z], we give another proof of the same fact.

2. The result.

Before stating our main theorem let us introduce some definitions.

We say that the function \( f : T \times X \rightarrow \mathbb{R} \) has property Cl if the multifunction \( F'_A : T \rightarrow \mathcal{P}(X) \) (\( \mathcal{P}(X) \) is the family of all subsets of \( X \) including the empty set) defined by \( F'_A(t) = \{ x \in X : f(t, x) \in A \} \) is weakly \( \mathcal{A} \)-measurable for each closed subset \( A \) of \( \mathbb{R} \). We recall that a multifunction \( F'_A : T \rightarrow \mathcal{P}(X) \) is weakly \( \mathcal{A} \)-measurable if the set \( F'_A(B) = \{ t \in T : F'_A(t) \cap B \neq \emptyset \} \) belongs to \( \mathcal{A} \) for each open subset \( B \) of \( X \) (cf. [H, p. 54]).

The following theorem has been proved in [Z].

**Theorem 1.** Let \( f : T \times X \rightarrow \mathbb{R} \) be \( C_* \) type (resp. \( C^* \) type). Then \( f \) has property \( SD_* \) (resp. property \( SD^* \)) if and only if \( f \) has property Cl.

Now we can establish our main result

**Theorem 2.** Let \( f : T \times X \rightarrow \mathbb{R} \) be \( C_* \) type (resp. \( C^* \) type). Then \( f \) has property \( SD_* \) (resp. property \( SD^* \)) if and only if \( f \) is \( \mathcal{A} \times \mathcal{B}(X) \)-measurable.
PROOF. Let us assume that $f$ has property $SD_*$ (resp. property $SD^*$) and choose an arbitrary $a \in \mathbb{R}$. Since the interval $(-\infty, a]$ (resp. $[a, \infty)$) is closed in $\mathbb{R}$, the multifunction $F_{(-\infty,a]}$ (resp. $F_{[a,\infty)}$), by Theorem 1, is weakly $\mathcal{A}$-measurable. Moreover in view of lower-semicontinuity (resp. upper-semicontinuity) of functions $f(t, \cdot)$, $t \in T$, the multifunction $F_{(-\infty,a]}$ (resp. $F_{[a,\infty)}$) has closed values. Thus by [H, Theorem 3.5] its graph $\text{Gr} F_{(-\infty,a]}$ (resp. $\text{Gr} F_{[a,\infty)}$) belongs to $\mathcal{A} \times \mathcal{B}(X)$.

But

$$f^{-1}((-\infty,a]) = \{(t,x) \in T \times X : f(t,x) < a\} =$$

$$= \{(t,x) \in T \times X : x \in F_{(-\infty,a]}(t)\} = \text{Gr} F_{(-\infty,a]} \in \mathcal{A} \times \mathcal{B}(X)$$

(resp. we have $f^{-1}([a,\infty)) = \text{Gr} F_{[a,\infty)}$ for $f$ having property $SD^*$).

Conversely, let $f : T \times X \to \mathbb{R}$ be $\mathcal{A} \times \mathcal{B}(X)$-measurable. Then $f^{-1}(A) \in \mathcal{A} \times \mathcal{B}(X)$ for each closed $A \subset \mathbb{R}$ and hence, in particular, $f^{-1}(A) \cap T \times B \in \mathcal{A} \times \mathcal{B}(X)$ for each open $B \subset X$. By projection theorem [CV, Theorem III.23] we have $\text{pr}_T(f^{-1}(A) \cap T \times B) \in \mathcal{A}$, where $\text{pr}_T$ denotes the projection map from the product $T \times X$ onto $T$. Now let us observe that

$$F_A^{-1}(B) = \{t \in T : F_A(t) \cap B \neq \emptyset\} = \{t \in T : \forall x \in F_A(t) \cap B\} =$$

$$= \{t \in T : \forall f(t,x) \in A \& (t,x) \in T \times B\} =$$

$$= \{t \in T : \forall (t,x) \in f^{-1}(A) \cap T \times B\} = \text{pr}_T(f^{-1}(A) \cap T \times B) \in \mathcal{A}.$$}

So we see that $F_A^{-1}$ is weakly $\mathcal{A}$-measurable and this $f$ has property Cl. Therefore by Theorem 1 $f$ has property $SD_*$ (resp. $SD^*$) which completes the proof of Theorem 2.

REFERENCES


