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Postulation and gonality for projective curves


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Postulation and Gonality for Projective Curves.

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We are interested in the interplay between intrinsic and projective properties of curves. In particular we are interested in the postulation of general k-gonal curves. A smooth curve $Y \subset \mathbb{P}^N$ is said to be canonical if $\mathcal{O}_Y(1) \cong K_Y$. A curve $Z \subset \mathbb{P}^N$ is said to have maximal rank if the restrictions maps $r_{Z,N}(k): H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(Z, \mathcal{O}_Z(k))$ have maximal rank for all integers $k$. In [4] it was proved that for every $N > 3$, $g > N$, a general non-degenerate canonical curve with genus $g$ in $\mathbb{P}^N$ has maximal rank.

Here we prove the following results (over $\mathbb{C}$).

**Theorem 1.** For all integers $N > 3$, $g > N$, a general trigonal (resp. bielliptic) non-degenerate canonical curve of genus $g$ in $\mathbb{P}^N$ has maximal rank.

**Theorem 2.** For all integers $N, d, g$, with $g > N > 3$, $d > 2g$, a general embedding of degree $d$ in $\mathbb{P}^N$ of a general hyperelliptic curve of genus $g$ has maximal rank.

The proofs of theorem 1 and theorem 2 is modulo a smoothing result given in § 1, almost the same that the proofs in [4]; in particular in § 3 we omit the details which can be found in [4] or [3]. For § 1 we use the theory of admissible coverings ([7]) which is very useful to obtain results about general k-gonal curves by degeneration techniques. The inductive method of § 2, § 3, (the Horace's method) was introduced in [8].

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In § 4 we show that, by the theory of admissible covers, the results proved in [6] about the minimal free resolution of general canonical curves are true (with the same proof) for general $k$-gonal curves for suitable $k$.

0. Notations.

Let $V$ be a variety (over $\mathbb{C}$) and $S$ a closed subscheme of $V$; $\mathcal{I}_{S,V}$ is the ideal sheaf of $S$ in $V$ and $\mathcal{N}_{S,V}$ its normal sheaf (or normal bundle). Assume that we have fixed an embedding of $V$ into $\mathbb{P}^k$, so that $\mathcal{O}_V(t)$ and $\mathcal{O}_S(t)$ are defined. Then $r_{S,V}(t): H^0(V, \mathcal{O}_V(t)) \to H^0(S, \mathcal{O}_S(t))$ is the restriction map. If $V = \mathbb{P}^k$, we write often $r_{V,k}(t)$, $N_{S,k}$ instead of $r_{S,V}(t)$, $N_{S,V}$.

A curve $T \subset \mathbb{P}^n$ is called a bamboo of degree $d$ if it is reduced, connected, with at most nodes as singularities, $\deg(T) = d$, its irreducible components are lines, and each line in $T$ intersects at most two other irreducible components of $T$; equivalently, we may order the lines $L_1, \ldots, L_d$ of $T$ so that $L_i \cap L_j \neq \emptyset$ iff and only if $|i - j| < 2$. A connected, reduced curve $X \subset \mathbb{P}^n$, $\deg(X) = 2d$, with only ordinary nodes as singularities, is called a chain of $d$ conics if its irreducible components $C_1, \ldots, C_d$ are conics, $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| < 2$, card $(C_i \cap C_{i+1}) = 2$ if $0 < i < d$. Sometimes we will allow the reducibility of some of the conics $C_i$ in a chain of conics; if $C_i$ is reducible, we assume that every line of $C_i$ intersects the adjacent conics. A line (resp. a conic) of a bamboo (resp. chain of conics) $T$ is called final if it intersects at most another irreducible component of $T$.

We will write $\binom{(a; b)}{c}$ for the binomial coefficient; thus $\binom{(a; b)}{c} := \binom{a!}{(a-b)! b!}$. A triple of integers $(d, g; N)$ with $d > g + N$, $N > 2$, $g > 0$ has critical value $k$ if $k$ is the first integer $t > 0$ such that $td + 1 - g < (N + t; N)$. We define integers $r(k, g, N)$, $q(k, g, N)$ by the following relations:

$$kr(k, g, N) + 1 - g - q(k, g, N) = ((N + k; N)),$$

$0 < q(k, N) < k$

A smooth curve $T$ of degree $d$ and genus $g$ in $\mathbb{P}^n$ with $r(k - 1, g, N) < d < r(k, g, N)$ and $h^1(T, \mathcal{O}_T(1)) < 2 < k$, has critical value $k$ (i.e. $(d, g; N)$ has critical value $k$); $T$ has maximal rank if and only if $r_{T,k}(k - 1)$ is injective and $r_{T,k}(k)$ is surjective (Castelnuovo-Mumford’s Lemma).
We define integers \( c(k, N), e(k, N) \) by the following relations:

\[
(2k - 1)e(k, N) + 2 + e(k, N) = ((N + k; N)) , \quad 0 \leq e(k, N) \leq 2k - 2.
\]

A chain \( F \) of \( c(k, N) \) conics in \( \mathbb{P}^N \) has critical value \( k \); it has maximal rank if and only if \( r_{F,N}(k - 1) \) is injective and \( h^0(\mathbb{P}^N, J_{F,N}(k)) = e(k, N) \).

Define integers \( y(k, N), k > 0, N > 2, \) in the following way. Set \( y(1, N) := c(1, N) \) and assume defined \( y(k - 1, N) \). Set \( y(k, N) = y(k - 1, N) + (c(k, N) - c(k - 1, N)) \) with \( e = -1 \) if \( e(k, N) < e(k - 1, N) \), \( e = 0 \) otherwise. Hence \( c(k, N) > y(k, N) > c(k, N) - k \).

Define integers \( x(k, N), j(k, N) \) by the following relations:

\[
(2k - 1)x(k, N) + j(k, N) = ((N + k; N)) , \quad 0 \leq j(k, N) \leq 2k - 2.
\]

A canonical curve \( C \) of degree \( d \) in \( \mathbb{P}^n \) has critical value \( k \) if and only if \( 2x(k - 1, N) + 2 < d < 2x(k, N) \). Set \( y'(k, N) := y(k, N) - \left\lfloor \frac{(N + 5)}{3} \right\rfloor - 1 \).

A finite subset \( S \subset \mathbb{P}^k \) is said to be in Linear general position if every subset \( W \) of \( S \) spans a linear space of dimension \( \min (k, \text{card } (W) - 1) \).

A bielliptic curve is a smooth, connected, complete curve with a degree two morphism onto an elliptic curve.

Let \( Y = A \cup B \subset \mathbb{P}^3 \), \( A \) chain of conics, \( B \) bamboo, \( A \) intersecting \( B \) at a unique point, \( P \), and quasi-transversally, \( P \) belonging to a final line of \( B \) and a final conic of \( A \). An irreducible component of \( Y \) is called free if it intersects only another irreducible component of \( Y \).

1. A smooth, connected curve \( E \subset \mathbb{P}^n \) is called canonical if \( O_X(1) \cong K_X \). Let \( C(g, n) \) be the closure in \( \text{Hilb}(\mathbb{P}^n) \) of the set of smooth canonical curves of genus \( g \) in \( \mathbb{P}^n \), and \( C(g, n, k) \) (resp. \( C(g, n; \text{biel}) \)) the closure in \( \text{Hilb}(\mathbb{P}^n) \) of the set of smooth canonical curves of genus \( g \) in \( \mathbb{P}^n \), which are \( k \)-gonal (resp. bielliptic) as abstract curves.

**Proposition 1.1.** Fix a smooth canonical trigonal curve \( C \subset \mathbb{P}^n \), \( \deg (C) = 2g - 2, 2 \) points \( A, B \) in the same fiber of a \( g^1_4 \) on \( C \), and a chain \( D \) of \( r \) conics in \( \mathbb{P}^n \), \( D \) intersecting \( C \) quasi-transversally and exactly at \( A \) and \( B \), \( A \) and \( B \) belonging to a final conic of \( D \). Then \( C \cup D \in C(g + r, n, 3) \).

**Proof.** By definition \( C(g + r, n, 3) \) is closed in \( \text{Hilb}(\mathbb{P}^n) \). Hence we may assume \( C, A, D \) general. We know that \( C \cup D \in C(g, n) \) ([5], [4], § 2).
First assume $r = 1$. Taking a projection, we may assume $n = g$, $C \cup D$ spanning $\mathbb{P}^n$, $C$ spanning a hyperplane $M$, and $h^1(C, N_{C,M}) = 0$. We know that $h^1(C \cup D, N_{C \cup D,M}) = 0$ ([4], proof of 2.1), hence $C \cup D$ is a smooth point of $\text{Hilb} (\mathbb{P}^n)$. Since $C \cup D$ is semi-stable, we have a morphism $h$ from a neighborhood of $C \cup D$ in $\text{Hilb} (\mathbb{P}^n)$ to the moduli scheme $\overline{M}_{g+1}$ of stable curves of genus $g+1$ such that $h(C \cup D)$ is the curve $C'$ obtained from $C$ pinching together the points $A, B$. By the generality of $C, A$, we may assume $\text{Aut} (C') = \{1\}$, i.e. $C'$ is a smooth point of $\overline{M}_{g+1}$. To obtain 1.1 for $r = 1$, it is sufficient to check that $h$ is flat, hence open, at $C \cup D$. By the smoothness of $\text{Hilb} (\mathbb{P}^n)$ and $\overline{M}_{g+1}$ at the corresponding points, it is sufficient to check that the fiber $h^{-1}(C')$ has the right dimension $n^2 + 2n = \dim (\text{Aut} (\mathbb{P}^n))$ in a neighborhood of $C \cup D$. A priori near $C \cup Dh^{-1}(C')$ contains either curves abstractly isomorphic to $C'$ (i.e. irreducible canonical stable curves) or curves isomorphic to $C \cup D$. The first type of curves has dimension $n^2 + 2n$. Since Pic $(C \cup D)$ has a 1-dimensional non-compact factor, we see easily that, up to projective transformations, there is exactly a one dimensional family of curves $C'' \cup D' \cong C \cup D$. However for any 2 triples $\{E_i\}, \{F_i\}, i = 1, 2, 3$, of distinct points of $D''$, there is $m \in \text{Aut} (\mathbb{P}^n)$ with $m(E_i) = F_i$ for every $i$. Hence the stabilizer of any $C'' \cup D''$ in $\text{Aut} (\mathbb{P}^n)$ is one-dimensional, concluding the proof of the case $r = 1$. By induction on $r$, if $r > 1$ it is sufficient to prove the following claim stronger than the case $r = 1$ just proven.

**Claim.** Assume $r = 1$ and fix 2 general points $E, F$ of $D$; then there is a flat family $X \to T, T$ smooth irreducible affine curve, $X \subset T \times \mathbb{P}^n$, with $X_0 = C \cup D, X_t$ smooth, canonical and trigonal for $t \in T, t \neq 0$, and a family $m_t$ of 3-coverings, $m_t: X_t \to \mathbb{P}^1$ such that $m_0|C$ is the given $g_1^1$, $m_0|D$ sends $A, B$ to one point of $\mathbb{P}^1$ and $E, F$ to another point of $\mathbb{P}^1$.

By the theory of admissible coverings ([7]) there is a morphism $b: T \to \overline{M}_{g+1}$, with $b(0) = C', b(t)$ a smooth 3-gonal curve for $t \neq 0$. By the first part of the proof we may assume $X_t = b(t)$. By [7], proof of th. 5(a), we may assume the existence of a family $m_t: X_t \to \mathbb{P}^1, t \in T \setminus \{0\}$, which, as $t$ goes to 0, tends to an admissible covering $m_0$ with $m_0(A) = m_0(B), m_0(E) = m_0(F)$. Taking a suitable fiber product, we obtain the claim. ■

**Proposition 1.2.** Fix a canonical bielliptic curve $C \subset \mathbb{P}^n$, deg $(C) = 2g - 2$, 2 points $A, B$ on $C$ with the same image under the 2 to 1 map of $C$ to an elliptic curve, and a chain $D$ of $r$ conics in $\mathbb{P}^n, D$ inter-
secting $C$ quasi-transversally and exactly at $A$ and $B$, $A$ and $B$ belonging to a final conic of $D$. Then $C \cup D \in C(g + r, n; \text{biel})$.

**Proof.** The proof of 1.1 works with two minor twists. We use the notations introduced in the proof of 1.1. Since $\text{Aut}(C)$ and $\text{Aut}(C')$ is not trivial, $\overline{M}_{g+1}$ is not smooth at $C'$. Instead of $\overline{M}_{g+1}$, we may however use (over $C$) the Kuranishi local deformation space of $C'$ (or a suitable rigidification of $\overline{M}_{g+1}$). Instead of admissible coverings of $\mathbb{P}^1$, we have to use admissible 2-coverings of curves of arithmetic genus 1. Since these coverings are cyclic, there is no need here of a general theory. Take a 2-covering $c: C \to Z$, $Z$ elliptic curve, with $c(A) = c(B)$ (hence $c$ not ramified at $A, B$) and a 2-covering $d: D \to \mathbb{P}^1$ with $d(A) = d(B)$ (hence unramified at $A$ and $B$). Take as $Z \cup \mathbb{P}^1$ the glueing of $Z$ and $\mathbb{P}^1$ along $c(A)$ and $d(E)$. Then $c, d$ induce a 2-covering $u: C \cup D \to Z \cup \mathbb{P}^1$. Let $a_1, \ldots, a_{2g}$ be the ramification points of $u$, with $a_i \in Z$ if and only if $i < 2g - 2$. Take any flat family $s: W \to T$, with $W_0 = Z \cup \mathbb{P}^1$, $W_t$ smooth elliptic for $t \neq 0$, and, in an etale neighborhood of 0, any $2g$ sections $s_1, \ldots, s_{2g}$ of $s$ with $s_i(0) = a_i$: The divisor $s_1(t) + \ldots + s_{2g}(t)$ on $W_t$ induces a cyclic 2-covering which tends to $u$ when $t$ goes to 0. 

Let $Z(d, g, n; k)$ (resp. $Z(d, g, n; \text{biel})$) be the closure in $\text{Hilb}(\mathbb{P}^n)$ of the set of smooth, connected, $k$-gonal (resp. bielliptic) curves of degree $d$, genus $g$, and with non special hyperplane section. $Z(d, g, n; k)$ and $Z(d, g, n; \text{biel})$ are irreducible (and not empty if $d > g + n$). We have the following result.

**Proposition 1.3.** Fix $C \subset \mathbb{P}^n$, $C \subset Z(d, g, n; k)$ (resp. $Z(d, g, n; \text{biel})$), 2 smooth points $A, B$ on $C$ with the same image under an admissible covering of degree $k$ (resp. a 2-to-1 covering of a curve of genus 1) of $C$, and a chain $D$ of $r$ conics in $\mathbb{P}^n$, $D$ intersecting $C$ quasi-transversally, and exactly at $A, B$, $A$ and $B$ belonging to the same final conic of $D$. Then $C \cup D \in Z(d + 2r, g + r, n, k)$ (resp. $Z(d + 2r, g + r, n, k)$) (resp. $Z(d + 2r, g + r, n, k)$) (resp. $Z(d + 2r, g + r, n; \text{biel})$).

**Proof.** It is much easier than the proof of 1.1, 1.2. Take any flat family of admissible covering, $(W \to T, U \to T, W \to U)$ with $W_0 = C \cup D$, $U_0 = Z \cup \mathbb{P}^1$, $p_0(Z) = 0$ (resp. 1) and $d + 2r$ sections $w_1$ of $T$ with $w_1(0) + \ldots + w_{d+2r}(0)$ an hyperplane section of $W_0$. Embedd $W_t$ using $|w_1(t) + \ldots + w_{d+2r}(t)|$, and, if necessary, project the result in $\mathbb{P}^n$. 

We shall use often the following fact ([9], [2], 3.5). Fix a smooth curve \( C \subset \mathbb{P}^n \), \( \deg(C) = d \), and a smooth, connected curve \( D \), \( \deg(D) = r \), \( D \) intersecting quasi-transversally \( C \) and exactly at a point. Assume \( D \) rational and that the hyperplane section of \( C \) is non-special. Then \( C \cup D \) is a limit of embeddings of degree \( d + r \) of \( C \) into \( \mathbb{P}^n \).

2. In this section we construct curves \( Y = Z \cup T \subset \mathbb{P}^3 \), with \( Z \cap T = \emptyset \), \( Z \) chain of conics, \( T \) bamboo, and with good postulation. This construction will be used in the next section to prove theorems 1, 2 in \( \mathbb{P}^4 \).

**Lemma 2.1.** Fix non-negative integers \( a, b, r, s \), and a smooth quadric \( Q \) in \( \mathbb{P}^3 \). Assume either (i) \( a = b > 2 \), or (ii) \( a > b > 1 \). Then there is \( (Y, Z, T) \) with \( Y = Z \cup T \subset \mathbb{P}^3 \), \( Z \cap T = \emptyset \), \( Z \) chain of \( r \) conics, \( T \) bamboo of degree \( s \), \( \dim(Y \cap Q) = 0 \), and \( h^0(Q, \mathcal{O}_{\mathbb{P}^3}(a, b)) = \max(0, (a + 1)(b + 1) - 4r - 2s) \).

**Proof.** From now on, we assume \( s = 0 \), the general case being similar (or use [1], 6.2). If \( s = 0 \), it is sufficient to prove 2.1 when \( (a + 1)(b + 1) - 4 < 4r < (a + 1)(b + 1) + 4 \). By the properness of \( \text{Hilb}(\mathbb{P}^3) \), for any \( u \) and any plane \( H \) there is a scheme \( W \), with \( W_{\text{red}} \subset H \), \( W_{\text{red}} \) with only ordinary double points, \( W \) reduce outside the singular locus of \( W_{\text{red}} \), \( W \) limit of a family of chains of \( u \) conics. Just to fix the notations, we assume \( a + b + 1 = 3 \mod(4) \), the remaining cases being similar. Fix 3 general planes \( M, N, R \). Take limits \( W, X, D \), respectively of chains of \( [(a + b + 1)/4], [(a + b - 1)/4] \), and 2, conics, with \( W_{\text{red}} \subset M \), \( X_{\text{red}} \subset X \), \( D_{\text{red}} \subset R \), \( W \cup X \cup D \) intersecting transversally \( Q \), \( W \cup X \cup D \) limit of a family of chains of conics, \( \text{card}(D \cap Q \cap M) = 2 \), \( \text{card}(D \cap Q \cap N) = 2 \). \( \text{card}(X \cap Q \cap M) = 1 \). Set

\[
M' := M \cap Q, \quad N := N \cap Q, \quad A := (W \cup X \cup D) \cap Q.
\]

Any forms of type \( (a, b) \) vanishing on \( A \), vanishes on \( a + b + 1 \) points on \( M' \), hence on \( M' \).

Any form of type \( (a - 1, b - 1) \) vanishing on the points of \( A \setminus (A \cap M') \), vanishes on \( a + b - 1 \) points of \( N' \), hence on \( N' \). We reduce to an assertion about forms of type \( (a - 2, b - 2) \) (in which we have to consider also the 4 points in \( D \cap (Q \setminus (M' \cup N')) \). We
continue in the same way, never adding curves intersecting \(M' \cup N'\); for the next step we take \(R\) as first working plane. At the end in case (i) we reduce to the case \(a = 3\) or \(4\) (plus a few points), in case (ii) to cases with \(a < 4\). Consider for instance the case \(a = b = 3\) and no point left from the previous construction (the worst case). Take 3 general smooth curves \(L, L', L''\) of type \((1, 1)\) on \(Q\), and let \(H, H', H''\) the planes they span. The following chain \(A \cup A' \cup A'' \cup B\) of conics solves our problem. \(A\) (resp. \(A'\)) is a sufficiently general conic in \(H\) (resp. \(H'\)), \(A''\) is a sufficiently general conic in \(H''\) containing a point of \(L, B\) is a conic containing 2 points of \(L\) and a point of \(L'\), but not intersecting \(L''\).

The aim of this section is the proof of the following lemma.

**Lemma 2.2.** Fix integers \(n, r\), with \(n > 29, 0 \leq a \leq 2n - 2\). There exists \((Y, Z, T)\) with \(Y = Z \cup T \subseteq \mathbb{P}^3, Z \cap T = \emptyset\), \(Z\) chain of conics, \(T\) bamboo of degree \(a\), \(r_{x,3}(n)\) surjective, \(\deg (Y) > r(n, 0, 3) - 9 - n/6\) (hence \(\dim (\text{Ker}(r_{Y,3}(n))) < r(n, 0, 3)/2 + (n^2/6) + 10n + 1\).

**Proof.** Let \(s\) be the maximal integer with \(s \leq n, s \equiv n \mod (4)\), say \(n = s + 4t, \; r(s, 0, 3) - 3 < (r(n, 0, 3) - a)/2 - (n - s)/3\). By [4], 3.1, there is a bamboo \(E \subseteq \mathbb{P}^3, \deg (E) = r(s, 0, 3) - 2, \; \text{with } r_{E,3}(s)\) surjective. If \(a = 0\), set \(F := E, \; T = \emptyset\). If \(a > 0\), take \(F \subseteq E, \deg (F) = \deg (E) - 1, \; F\) union of two disjoint bamboos \(A, T, \deg (T) = a\). Then we fix a smooth quadric \(Q\) and we apply the so-called Horace's method (introduced in [8]) 2\(t\) times. At the odd (resp. even) steps we add in \(Q\) lines of type \((1, 0)\) (resp. \((0, 1)\)). We order the lines \(L_1, ..., L_{\deg (E)}\) of \(E\) in such a way that \(L_i \cap L_j = \emptyset\) if and only if \(|i - j| < 2\). Just to fix the notations we assume \(s = 6k\), \(k\) integer (which, together with \(s = 6k + 5\), is the worst case). We add in \(Q\) the union \(U\) of \(4k + 2 = r(6k + 2, 0, 3) - (6k + 2, 0, 3) - 1\) lines \(A_i, 1 \leq i \leq 4k + 2, \; \text{of type } (1, 0)\) with \(A_i\) intersecting \(L_{a(i-1)}\) for every \(i\). We claim that \(r_{U \cup E,3}(6k + 2)\) is surjective for general \(F\). To prove the claim, it is sufficient to find \(S \subseteq \mathbb{P}^3\),

\[
\text{card } (S) = q(6k + 2, 0, 3) + (6k + 2) (r(6k + 2, 0, 3) - \deg (F \cup U)),
\]

with \(r_{U \cup E,3}(6k + 2)\) injective. Take \(S = S' \cup S''\), with

\[
\text{card } (S') = q(6k + 2, 0, 3) + (6k)(r(6k + 2, 0, 3) - \deg (F)),
\]
Consider the following assertion \( T(n, a, b) \), defined for all integers \( n, a, b \): There is \( (Y, Z, T) \) with \( Y = Z \cup T \subset \mathbb{P}^s \), \( Z \cap T = \emptyset \), \( Z \) chain of a conics, \( T \) bamboo of degree \( b \), and with \( r_{x, a}(n) \) surjective.

In the following section, for the proof of theorems 1, 2 we will need the assertion \( T(n, a, b) \) for the values of \( n, a, b \), listed in 2.3.

Lemma 2.3. – The assertion \( T(n, a, b) \) is true if \( (n, a, b) \) has one of the following values: \( (2, 1, 0), (2, 0, 2), (3, 2, 0), (3, 2, 1), (4, 3, 0), (4, 2, 2), (5, 4, 0), (5, 3, 2), (6, 4, 2), (6, 3, 4), (7, 2, 8), (7, 5, 3), (8, 3, 10), (8, 5, 5), (9, 9, 2), (9, 7, 4), (10, 8, 5), (10, 5, 12), (11, 9, 7), (11, 5, 14), (12, 11, 6), (12, 7, 15), (13, 14, 4), (14, 14, 8), (15, 16, 9), (16, 18, 9), (17, 22, 6), (18, 16, 24), (18, 22, 11), (19, 18, 24), (19, 25, 11), (20, 27, 12), (21, 20, 33), (21, 32, 8), (22, 33, 12), (22, 25, 29), (23, 35, 15), (23, 27, 30),
Sketch of proof. The cases with \( n < 24 \) can be done using several times the Horace’s construction applied not to quadrics but to planes: it is easier; if \( n < 10 \), we do not need any nilpotent, if \( n > 9 \) we use nilpotents as in [8]; only the cases with \( n > 21 \) are more difficult; however they can be handle also using quadrics as in the proof of 2.2. If \( n > 23 \), the proof of 2.2 works verbatim, and gives indeed stronger results; for \((24, 39, 13)\) start taking in the proof of 2.2 \( s = 12 \); for the remaining \((n, a, b)\) start from \( s = n - 8 \). ■

3. In this section we show how to modify the proofs in [4], to prove theorems 1, 2. The proof of the case "\( P^4 \)" given in [4], § 8, cannot be adapted, but the results proven here in § 2 are sufficient to prove this case and the inductive assertions of [4] needed for the proofs in \( P^N, N > 5 \). We will use the numbers \( y(k, N), ... \), introduced in § 0.

Consider the following assertions:

\[ Y(k, N), k > 0, n > 3: \] there exists a chain \( Y \) of \( y(k, N) \) smooth conics in \( P^N \) with \( r_{r,N}(k) \) surjective; if either \( k > 6 \) or \( k > 2, N > 4 \), or \( N > 6 \), there is such a \( Y \) which is contained in an integral hypersurface of degree \( k \).

\[ Z(k, N), k > 0, N > 3: \] there exists a chain \( Y \) of \( e(k, N) + k - 1 \) smooth conics in \( P^N \) with \( r_{r,N}(k) \) injective.

\[ W(k, a, N, j), k > 2, 0 < a < 2k - 1, N > 3, 1 < j < 2N + 3: \] for every subset \( S \subset P^N \), card \( (S) = j \), \( S \) in linear general position, and every \( A, B \in S, A \neq B \), there is a curve \( Y \subset P^N \) such that:

(a) \( Y \cap S = \{A, B\}, r_{r,c,S,N}(k) \) is surjective and a general hypersurface of degree \( k \) containing \( Y \cup S \) is irreducible;

(b) \( Y = J \cup T \) with \( J \cap T = \emptyset \); \( J \) is a chain of \( y'(k, N) - a - 1 \) conics; \( T = \emptyset \) if \( a = 0 \); if \( a > 0 \), \( T \) is a bamboo of degree \( 2a \).

\[ H(k, N), k > 0, N > 3: \] there exists a curve \( Y = Z \cup T \subset P^N \) such that:

(a) \( Z \) is a canonical trigonal (resp. bielliptic) curve of degree \( 2(x(k, N) - j(k, N)) \) and genus \( x(k, N) - j(k, N) + 1 \);
(b) \( T \) is a bamboo of degree \( 2j(k,N) \) intersecting \( Z \) exactly at a point, say \( P \), and quasi-transversally; \( P \) is a point in a final line of \( T \);

(c) \( r_{Y,N}(k) \) is bijective.

\( W(k,a,N,j) \) and \( H(k,N) \) are slight modifications of the assertions of [4] with the same name. In [4] \( Y(k,N) \) and \( Z(k,N) \) were proved for \( N > 4 \). The same method (Horace’s construction using a hyperplane) gives \( Y(k,4) \), \( Z(k,4) \), using 2.2 if \( k > 30 \) (plus a numerical lemma: \( 2o(k,4) - 2 > r(k,0,3)/2 + (k^2/6) + 10k \) if \( k > 30 \)) whose proof is left to the reader), using 2.3 if \( k < 31 \).

In the same way we get the «new» assertions \( W(k,a,4,j) \), \( H(k,4) \). Then the proofs of the «new» \( W(k,a,N,j) \), \( H(k,N) \), \( N > 4 \), are done by induction as in [4]; the cases with \( N = 4 \) simplify the discussion of the cases with low \( k \) for \( N = 5 \) given in [4], 6.4. Then theorem 1 is proved in \( \mathbb{P}^n \), \( N > 3 \), in the same way the corresponding theorem is proved for \( \mathbb{P}^n \), \( N > 4 \), in [4], end of §7. The same proof works for theorem 0.2, although a simpler one could be done in this case, adding in a hyperplane irreducible hyperelliptic curves.

4. After this paper was typed, we read [6]. It is elementary to show how the results of [6], th. 4, 5, about syzygies of general canonical curves can be adapted to give results about syzygies of general \( k \)-gonal curves for suitable \( k \). We have:

**Proposition 4.1.** Let \( X \) be a non-hyperelliptic \( k \)-gonal genus \( n \) curves. Assume \( K_{p,3}(X) = 0 \) for an integer \( p \) with \( 1 \leq p \leq n - 3 \) \( p < k - 3 \). Then

(a) If \( C \) is a general \( k \)-gonal curve of genus \( n + p + 1 \), then \( K_{p,3}(C) = 0 \).

(b) If \( C \) is a general \( k \)-gonal curve of genus \( m \), where \( m \equiv n \text{ mod } (p + 1) \) and \( m < n \), then \( K_{p,2}(C) = 0 \).

For the proof of 4.1, it is sufficient to take in the proof of [6], th. 4, as divisor \( q_1 + q_2 + \ldots + q_{p+2} \) a divisor contained in a \( g_k \) on \( X \) (strictly contained by the assumption \( k > p + 2 \)) and as smoothing of \( X \cup Y \) an admissible \( k \)-cover. Then from 4.1 we get verbatim the following improved version of [6], th. 5:
PROPOSITION 4.2. Let $C$ be a general $k$-gonal curve of genus $g$.

(a) $K_{2,2}(C) = 0$ if $g > 7$ and either $k > 5$ or $g \equiv 1, 2 \mod (3)$ and $k = 5$.

(b) $K_{3,2}(5) = 0$ if $g > 9$ and either $k > 6$ or $g \equiv 1, 2 \mod (4)$ and $k = 6$.

(c) $K_{1,2}(C) = 0$ if $g \geq 11$, $k > 6$, and $g \equiv 1, 2 \mod (5)$.

REFERENCES


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