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On the Iterative Process $x_{n+1} = f(x_n, x_{n-1})$.

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SUMMARY - In this paper we consider the iterative process $x_{n+1} = f(x_n, x_{n-1})$, $n \in \mathbb{N}$, where f is a continuous function from $[0, 1]^2$ in $[0, 1]$ and we prove that the condition of the non existence of a pair (x, y) of distinct points of $[0, 1]$ such that $f(x, y) = y$ and $f(y, x) = x$, obviously necessary for the global convergence, is sufficient if f is decreasing with respect to both variables and whatever the point (x_1, x_0) of $[0, 1]^2$ be, the following implications:

$$\max \{x_0, x_1\} < x_2 \Rightarrow \min \{x_0, x_1\} < \min \{x_3, x_4\};$$

$$x_2 < \min \{x_0, x_1\} \Rightarrow \max \{x_3, x_4\} < \max \{x_0, x_1\},$$

are true.

0. Introduction.

Let S be a topological space and f be a continuous function from S^2 in S , after fixing a point (x_1, x_0) of S^2 , let us consider the question of the convergence of the sequence $(x_n)_{n \in \mathbb{N}}$, where:

$$(I) \quad x_{n+1} = f(x_n, x_{n-1}) \quad \forall n \geq 1.$$

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If f depends only on the first variable, i.e.:

$$f(x, y) = \varphi(x), \quad (\varphi \in \mathcal{C}(S, S))$$

the above question becomes the well known convergence of the sequence $(t_n)_{n \in \mathcal{N}}$, obtained by fixing a point t_1 of S and by putting for each $n \in \mathcal{N}$:

$$(II) \quad t_{n+1} = \varphi(t_n).$$

For the « global convergence of the method (II) », that is for the convergence of the sequence:

$$(\varphi^n(t))_{n \in \mathcal{N}}$$

for each point t of S , the condition of non existence of periodic points of φ is necessary. Some papers relate to the question of the sufficiency for the global convergence of the above mentioned condition or furthermore of the condition of non existence of periodic points of φ of period two (see, e.g. [1, 2, 4÷8, 10, 11, 12, 15, 16, 17] ⁽¹⁾).

In relation to this question the equivalence between the global convergence of the method (II) and every one of the following conditions α) and b) has been proved in [17], where S is a totally ordered set, compact, complete and dense in itself:

a) there exist no periodic points of φ of period two (that is, there exist no periodic points x and y of S such that $\varphi(x) = y$ and $\varphi(y) = x$);

b) there exist no points x and y of S such that:

$$\varphi(y) \leq x < y \leq \varphi(x).$$

Let us now return to method (I) and suppose, as in [9], that $S = [0, 1]$. It is spontaneous to ask ourselves if the following conditions—which are, respectively, a natural translation of the conditions α) and b)—are equivalent:

⁽¹⁾ See, also, the papers [13] and [14], in which S is a totally ordered set, complete and dense in itself, and the convergence of the sequence $(\varphi^n(x_0))_{n \in \mathcal{N}}$ is required for any point x_0 of S , such that the sequence itself is bounded.

a') *there exist no distinct points x and y of $[0, 1]$ such that:*

$$f(x, y) = y \quad \text{and} \quad f(y, x) = x;$$

b') *there exist no points x and y of $[0, 1]$ such that:*

$$f(y, x) \leq x < y \leq f(x, y).$$

For the equivalence between a') and b') it is needed the only other hypothesis that f is decreasing with respect to the second variable (see (2.2)).

G. Di Lena has proved in [9] that the condition a') implies the global convergence of the method (I)—i.e., the convergence of the sequence $(x_n)_{n \in \mathbb{N}}$ for each point (x_1, x_0) of $[0, 1]^2$ —if for each point (x_1, x_0) of $[0, 1]^2$ the implications:

$$\gamma_1) \quad \max \{x_0, x_1\} < x_2 \Rightarrow \min \{x_0, x_1\} < x_n \quad \forall n > 1;$$

$$\gamma_2) \quad x_2 < \min \{x_0, x_1\} \Rightarrow x_n < \max \{x_0, x_1\} \quad \forall n > 1,$$

are true.

In this paper we prove that, if f is decreasing with respect to both variables, the mentioned theorem by Di Lena still holds if the implications $\gamma_1)$ and $\gamma_2)$ are replaced by the following ones:

$$\Gamma_1) \quad \max \{x_0, x_1\} < x_2 \Rightarrow \min \{x_0, x_1\} < \min \{x_3, x_4\};$$

$$\Gamma_2) \quad x_2 < \min \{x_0, x_1\} \Rightarrow \max \{x_3, x_4\} < \max \{x_0, x_1\}.$$

We obtain this result in n. 3, making use of the following proposition proved in n. 1:

A) *Let: (T, \leq) be a totally ordered set, g be a function from T^2 in T , decreasing with respect to both variables, and such that:*

b'') *there exist no points x and y of T such that:*

$$g(y, x) \leq x < y \leq g(x, y).$$

Then, set for each point (x_1, x_0) of T^2 :

$$x_{n+1} = g(x_n, x_{n-1}) \quad \forall n > 1,$$

whatever the point (x_1, x_0) of T^2 be, the implications $\gamma_1)$ and $\gamma_2)$ hold, provided that, whatever the point (x_1, x_0) of T^2 be, the implications $\Gamma_1)$ and $\Gamma_2)$ are true.

1. Propositions relative to the antitone functions.

In this section we denote by $(T, <)$ a totally ordered set and by g a function from T^2 in T .

Let us suppose that:

$$(x_1 < y_1 \text{ and } x_2 < y_2) \Rightarrow g(y_1, y_2) < g(x_1, x_2),$$

that is, g is decreasing with respect to both variables or, as it is said, g is antitone.

For each point (x_1, x_0) of T^2 we assume:

$$x_{n+1} = g(x_n, x_{n-1}), \quad \forall n \geq 1.$$

That being stated, let us consider the following property to which we will refer in the next propositions:

b'') It does not exist a pair (x, y) of points of T such that:

$$g(y, x) < x < y < g(x, y).$$

We will now prove the following propositions:

(1.1). *The function g satisfies b'') and the condition:*

(c) *whatever the point (x_1, x_0) of T^2 be:*

$$\max \{x_0, x_1\} < x_2 \Rightarrow \min \{x_0, x_1\} < \min \{x_3, x_4\}.$$

Moreover, there exist a point (x'_1, x'_0) of T^2 and an integer $k > 1$ such that:

$$\max \{x'_0, x'_1\} < x'_2 \quad \text{and} \quad x'_k \leq \min \{x'_0, x'_1\}.$$

Then, there exist a point (x_1, x_0) of T^2 and an even number $m > 4$ such that:

$$(1) \quad x_m \leq \min \{x_0, x_1\} < x_3 < x_5 < \dots < x_{m-1} \leq x_{m-2} \leq \dots \leq x_4 \leq x_2.$$

PROOF. Obviously, the set of integers greater than 1 such that:

$$\exists (y_1, y_0) \in T^2: \max \{y_0, y_1\} < y_2 \text{ and } y_k \leq \min \{y_0, y_1\},$$

is nonempty.

Let us denote by m the minimum of the above mentioned set, in accordance with (c) m is greater than 4, and by (x_1, x_0) a point of T^2 such that:

$$(2) \quad \max \{x_0, x_1\} < x_2 \quad \text{and} \quad x_m \leq \min \{x_0, x_1\} < x_h, \\ \forall h \in \{2, \dots, m-1\}.$$

Let us suppose now the following three propositions of which we postpone the demonstration:

$$(3) \quad x_h \leq \max \{x_{h-2}, x_{h-1}\}, \quad \forall h \in \{3, \dots, m\}.$$

$$(4) \quad \text{It does not exist } k \in \{2, \dots, m-2\} \text{ such that } x_{k+1} < x_k < x_{k-1}.$$

$$(5) \quad x_m \leq \min \{x_0, x_1\} < x_3 < x_5 < x_4 < x_2.$$

That being stated, we observe that (1.1) will be proved if we demonstrate that, whatever $k \in \mathbb{N}$ be such that $2k + 1 < m$, it results:

$$(6) \quad x_m \leq \min \{x_0, x_1\} < x_3 < \dots < x_{2k+1} < x_{2k} \leq \dots \leq x_4 < x_2 \quad (2).$$

Now then, reasoning by recurrence, we observe that, according to (5), the (6) is true for $k = 1$ and, if i is such that $2(i + 1) + 1 < m$, supposing that (6) is true for $k = i$ let us prove it for $k = i + 1$.

We therefore known that:

$$(7) \quad x_m \leq \min \{x_0, x_1\} < x_3 < \dots < x_{2i+1} \leq x_{2i} \leq \dots \leq x_4 \leq x_2$$

and we must prove that:

$$(8) \quad x_{2i+1} < x_{2i+3} \leq x_{2i+2} \leq x_{2i}.$$

(2) Indeed, the validity of (6) for every $k \in \mathbb{N}$ such that $2k + 1 < m$, implying that $x_m < x_d$ for each odd d greater than 1 and less than or equal to m , forbids m to be odd.

Since from (7) it results $x_{2i+1} \leq x_{2i}$, from (3) and (4) we have, respectively, that:

$$(9) \quad x_{2i+2} \leq x_{2i} \quad \text{and} \quad x_{2i+1} < x_{2i+2};$$

moreover it results $x_{2i+3} \leq x_{2i+2}$ in accordance with the second of (9) and (3).

Then (8) will be demonstrated as soon as we will have proved that:

$$x_{2i+1} < x_{2i+3}.$$

If, *ad absurdum*, $x_{2i+3} \leq x_{2i+1}$, from (9) and being g antitone we would obtain:

$$g(x_{2i+2}, x_{2i+1}) = x_{2i+3} \leq x_{2i+1} < x_{2i+2} = g(x_{2i+1}, x_{2i}) \leq g(x_{2i+1}, x_{2i+2});$$

consequently, for $x = x_{2i+1}$ and $y = x_{2i+2}$, it would result:

$$g(y, x) \leq x < y \leq g(x, y),$$

in contrast with b'').

Now (6) is completely proved.

We must now conclude the demonstration of this proposition proving (3), (4) and (5), that we have considered to be true, before.

PROOF OF (3). The inequality:

$$x_h \leq \max \{x_{h-1}, x_{h-2}\}$$

easily follows from (2) when $h = m$.

Considering now the case in which $h \in \{3, \dots, m-1\}$, let us start observing that, from the definition of m , it is obvious that:

α) whatever the point (y_1, y_0) of T^2 be, the following implication is true:

$$\max \{y_0, y_1\} < y_2 \Rightarrow \min \{y_0, y_1\} < y_h, \quad \forall h \in \{2, \dots, m-1\}.$$

Now then, if there existed $h \in \{3, \dots, m-1\}$ such that:

$$\max \{x_{h-2}, x_{h-1}\} < x_h,$$

in accordance with α) it would result:

$$\min \{x_{h-2}, x_{h-1}\} < x_m,$$

in contrast with (2).

PROOF OF (4). The existence of $k \in \{2, \dots, m-2\}$ such that:

$$(\beta_1) \quad x_{k+1} \leq x_k \leq x_{k-1},$$

is not possible because, as we are going to prove, (β_1) implies:

$$(\beta_2) \quad x_{k+1} \leq x_m,$$

in contrast with (2).

Well, to prove that (β_1) implies (β_2) it is sufficient to demonstrate, reasoning by recurrence, that:

$$(\beta_3) \quad x_{k+1} \leq x_i \leq x_k, \quad \forall i \in \{k, \dots, m\}.$$

(β_3) is true for $i = k$. Let us prove that, $\forall h \in \{k, \dots, m-1\}$, (β_3) is true for $i = h + 1$ if (β_3) holds $\forall i \in \{k, \dots, h\}$.

Since it results:

$$(\beta_4) \quad \max \{x_{h-1}, x_h\} \leq \min \{x_{k-1}, x_k\},$$

being g antitone, we have:

$$x_{k+1} = g(x_k, x_{k-1}) \leq g(x_h, x_{h-1}) = x_{h+1}.$$

Since, on the other hand, from (3) and (β_4) it follows:

$$x_{h+1} \leq \max \{x_{h-1}, x_h\} \leq x_k,$$

(β_3) holds for $i = h + 1$.

PROOF OF (5). From (2) it results $x_1 < x_2$ and then, from (3), $x_3 \leq x_2$. Consequently, from (3) we have $x_4 \leq x_2$ and from (4) $x_3 < x_4$. So, thanks to (3) again, it results $x_5 \leq x_4$.

In accordance with (2) it results $x_m \leq \min \{x_0, x_1\} < x_3$; then the proof of (5) will be complete when we show that:

$$x_3 < x_5.$$

To this aim we observe that if it were $x_5 \leq x_3$, being g antitone we should have:

$$g(x_4, x_3) = x_5 \leq x_3 < x_4 = g(x_3, x_2) < g(x_3, x_4),$$

and therefore, for $x = x_3$ and $y = x_4$ it would result:

$$g(y, x) \leq x < y \leq g(x, y)$$

in contrast with b'').

(1.2). *The function g satisfies b'') and the following condition:*

(c') *whatever the point (x_1, x_0) of T^2 be:*

$$x_2 < \min \{x_0, x_1\} \Rightarrow \max \{x_3, x_4\} < \max \{x_0, x_1\}.$$

Moreover, there exist a point (x'_1, x'_0) of T^2 and an integer $k > 1$ such that:

$$x'_2 < \min \{x'_0, x'_1\} \quad \text{and} \quad \max \{x'_0, x'_1\} < x'_k.$$

Then, there exist a point (x_1, x_0) of T^2 and an even number $m > 4$ such that:

$$(1') \quad x_2 < x_4 < \dots < x_{m-2} < x_{m-1} < \dots < x_5 < x_3 < \max \{x_0, x_1\} < x_m.$$

PROOF. It is sufficient to apply (1.1), considering the totally ordered set (T, \geq) .

Making use of the propositions (1.1) and (1.2) let us prove the proposition A) enunciated in the introduction.

Reasoning ab absurdo, let us suppose that the part of the thesis relative to γ_1) is not true.

Then, according to (1.1), it results:

$$(10) \quad x_m < x_3, \quad x_{m-1} \leq x_2;$$

$$(11) \quad x_{m-2} \leq x_4;$$

$$(12) \quad x_m < x_{m-1} \leq x_{m-2},$$

for some point (x_1, x_0) of T^2 and for some $m > 4$.

From (10), being g antitone, it follows:

$$x_4 = g(x_3, x_2) \leq g(x_m, x_{m-1}) = x_{m+1},$$

therefore, from (11), we have:

$$x_{m-2} \leq x_{m+1},$$

and from this, according to (12):

$$x_m < x_{m-1} \leq x_{m-2} \leq x_{m+1}.$$

Consequently, set $y_0 = x_{m-2}$, $y_1 = x_{m-1}$, we have:

$$y_2 < \min \{y_1, y_0\} \quad \text{and} \quad \max \{y_1, y_0\} \leq y_3$$

in contrast with the hypothesis relative to Γ_2).

The part of the thesis relative to γ_1) is then obtained.

As for γ_2) it is sufficient to follow a similar way, making use of (1.2).

2. Remarks on the continuous functions from $[0, 1]^2$ in $[0, 1]$ decreasing with respect to the second variable.

In this section we denote by f a continuous function from $[0, 1]^2$ in $[0, 1]$, decreasing with respect to the second variable.

Said p the (continuous) function from $[0, 1]$ in itself such that:

$$f(x, p(x)) = p(x) \quad \forall x \in [0, 1] \text{ } ^{(3)},$$

we immediately observe that:

(2.1). For each point (x, y) of $[0, 1]^2$ it results:

$$y < f(x, y) \Leftrightarrow y < p(x).$$

⁽³⁾ The existence of the function p can be deduced from well known reasoning regarding the implicit functions, see e.g. [3], pp. 338-339.

PROOF. It is obviously sufficient to demonstrate that:

$$y < f(x, y) \Leftrightarrow y < p(x) .$$

Well, if were:

$$y < f(x, y) , \quad y \geq p(x) ,$$

being f decreasing with respect to the second variable, we should have:

$$y < f(x, y) \leq f(x, p(x)) = p(x) ;$$

similarly it is easy to see that it is not possible to have:

$$y \geq f(x, y) , \quad y < p(x) .$$

From (2.1) and from a theorem by A. Volčič it easily follows that: (2.2). *The following conditions are equivalent:*

- 1) *There exists $(x, y) \in [0, 1]^2$ such that $f(y, x) = x \neq y = f(x, y)$.*
- 2) *There exists $(x, y) \in [0, 1]^2$ such that $f(y, x) \leq x < y \leq f(x, y)$.*
- 3) *There exists a periodic point of p of period two.*

PROOF. The implication 1) \Rightarrow 2) is trivial.

Relatively to the implication 2) \Rightarrow 3) it is sufficient to observe that, since from (2.1) it results:

$$p(y) \leq x < y \leq p(x) ,$$

(3) follows from theorem I of [17].

Finally, 3) \Rightarrow 1) because, if we call x a periodic point of p of period two, and assume $y = p(x)$, it results that $x \neq y$ and:

$$p(y) = x , \quad f(x, y) = y ,$$

and from the first of these equalities it follows:

$$f(y, x) = f(y, p(y)) = p(y) = x .$$

3. Global convergence of the method $x_{n+1} = f(x_n, x_{n-1})$.

Let f be a continuous function from $[0, 1]^2$ in $[0, 1]$.

Let us consider the following iterative process:

$$(1) \quad x_{n+1} = f(x_n, x_{n-1}),$$

which determines a sequence $(x_n)_{n \in \mathbb{N}}$ of points of $[0, 1]$, once a point (x_1, x_0) of $[0, 1]^2$ has been fixed.

The iterative process (1) is said to be *globally convergent* when, for any point (x_1, x_0) of $[0, 1]^2$ the sequence $(x_n)_{n \in \mathbb{N}}$ converges.

Making use of the propositions (2.2) and A) and of theorem by Di Lena mentioned in introduction, it obviously follows the next theorem of global convergence:

Let f be antitone and it does not exist a pair (x, y) of points of $[0, 1]$ such that:

$$f(y, x) = x \neq y = f(x, y).$$

Moreover, whatever the point (x_1, x_0) of $[0, 1]^2$ be let the implications Γ_1) and Γ_2) be true.

Then, the method $x_{n+1} = f(x_n, x_{n-1})$ converges globally.

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