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Approximate and Relaxed Solutions of Differential Inclusions.

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1. Introduction.

Let $F: \mathbb{R}^{n+1} \to 2^{\mathbb{R}^n}$ be a bounded multifunction. A comparison between the problems

$$\dot{x} \in F(t, x) \tag{1}$$

and

$$\dot{x} \in \overline{\text{co}}\, F(t, x) \tag{2}$$

has been carried out in many papers. In particular, Ważewski [12] proved that, for a continuous F, every solution of (2) is a uniform limit of functions $y_k(\cdot)$ such that

$$d\big(\dot{y}_k(t), \overline{\text{ext}}\,\overline{\text{co}}\, F(t, y_k(t))\big) \to 0 \qquad \text{for a.e. } t\,. \tag{3}$$

(according to Ważewski's definitions: every trajectory of $\overline{\text{co}}\, F$ is a quasitrajectory of $\overline{\text{ext}}\,\overline{\text{co}}\, F$, where $\overline{\text{ext}}\,\overline{\text{co}}\, F(t, x)$ indicates the closure of the extremal points of the closed convex hull of $F(t, x)$). A result in the same direction was proved later by Filippov [5]: he showed that if F is Lipschitzean (with respect to the Hausdorff distance) and compact valued, then the set S_F of solutions to (1) is dense in the set S of solutions to (2), for the uniform convergence topology. Filippov's theorem was generalized by Pianigiani [9], Tolstonogov and

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Finogenko [11], Ornelas [8], and, from other viewpoints, by Bressan [2] and Cellina [4], always under the assumption of continuity (or lower semicontinuity) for F. There is, however, a counterexample due to Pliś [10], showing that Filippov's theorem is false when F is only continuous. Moreover, it is well known that if F is upper semicontinuous solutions to (1) may not exist. Therefore, when F is less than continuous one can still pay attention to the problem of investigating the relationships between the solutions of (2), i.e. the « relaxed » solutions of (1), and the approximate solutions of (1).

In this paper we prove an analogue of Ważewski's result, without requiring any continuity assumption on F (indeed F must only be bounded). Our approach relies on a different notion of quasitrajectory and on a relaxed equation more general than (2). The present result can also be regarded as a multivalued generalization of a theorem by Hájek [7, Corollaries 5.6, 5.7], concerning discontinuous differential equations.

2. Notations and basic definitions.

Let X, Y be subsets of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. We define $d(x, Y) = \inf\{|x-y| : y \in Y\}$, the open ε-neighbourhood of X as $B(X, \varepsilon) = \{y \in \mathbb{R}^n : d(y, X) < \varepsilon\}$ and the separation between X and Y as $h^*(X, Y) = \sup\{d(x, Y) : x \in X\}$; the Hausdorff distance between X and Y is $h(X, Y) = \max\{h^*(X, Y), h^*(Y, X)\}$. The closed convex hull of X is indicated by $\overline{\text{co}}\, X$. If X is convex, we define ext X as the set of all the extreme points of X, i.e. the set of all the points $x \in X$ such that no nondegenerate segment in X exists which contains x in its relative interior; its closure is indicated by $\overline{\text{ext}}\, X$. The set theoretic difference and the symmetric difference between X and Y are denoted, respectively, by $X \setminus Y$ and $X \Delta Y$, while $2^{\mathbb{R}^n}$ means the family of all nonempty subsets of $\mathbb{R}^n$.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\Gamma : \Omega \to 2^{\mathbb{R}^n}$ be a multifunction. We say Γ to be bounded if there exists $M > 0$ such that $\Gamma(x) \subseteq B(O, M)$ for every $x \in \Omega$. The following continuity concept is mainly considered:

Γ is Hausdorff-upper semicontinuous (h-u.s.c.) in Ω iff

$$\forall x_0 \in \Omega, \forall \varepsilon > 0 \ \exists \delta > 0 \ \textit{such that } x \in B(x_0, \delta) \textit{ implies}$$
$$h^*(F(x), F(x_0)) < \varepsilon .$$

The graph of Γ, graph $\{\Gamma\}$, is the set $\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n : y \in \Gamma(x)\}$. We recall that a h-u.s.c. multifunction with closed values has closed graph; conversely, a bounded map with closed graph is h-u.s.c.

A regularization of a bounded (possibly non-measurable) multifunction F can be constructed as follows:

DEFINITION 1. *Let $F: \Omega \to 2^{\mathbf{R}^n}$ be a bounded multifunction. The h-u.s.c., convex-valued regularization of F is the map*

$$G: \Omega \to 2^{\mathbf{R}^n}, \qquad G(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}}\, \{u : u \in F(y),\, |y - x| < \varepsilon\}\,.$$

The map G can be seen as the smallest multifunction Γ with convex values and closed graph such that $F(x) \subseteq \Gamma(x)$ for every $x \in \Omega$. Notice that G is bounded by the same constant as F.

We now introduce an analogue of *Ważewski's* concept of quasitrajectory, which is more suitable for u.s.c. maps.

DEFINITION 2. *Let $\Gamma: \Omega \to 2^{\mathbf{R}^n}$ be a bounded multivalued map and $I \subseteq \mathbf{R}$ a compact interval. An absolutely continuous function $x: I \to \Omega$ such that $\dot{x}(t) \in \Gamma(x(t))$ for a.e. $t \in I$ (i.e. a solution of the differential inclusion $\dot{x} \in \Gamma(x)$) is said a* trajectory *of Γ; a function $y: I \to \Omega$ is called a* quasitrajectory *of Γ if there exists a sequence of measurable functions $\xi_k: I \to \mathbf{R}^n$ such that $\xi_k \to 0$ uniformly on I and a sequence $(y_k)_{k \geq 1}$ of solutions of*

$$\dot{u}_k(t) \in \Gamma\big(u_k(t) + \xi_k(t)\big)\,, \tag{4}$$

defined on I, which converges to y uniformly on I.

The above definition of quasitrajectory is entirely analogous to the concept of Hermes solution of the control system $\dot{x} = f(t, x, u)$, $u \in \mathcal{U}(t, x)$ given in Hájek [7, Definition 2.3]. The difference between Definition 2 and Ważewski's definition of quasitrajectory consists in the type of perturbation of the field: in (3) there is an « outer » perturbation, while in (4) an « inner » one.

Finally, we say that an absolutely continuous function $u: I \to \mathbf{R}^n$ is a *quasipolygonal* if its derivative $\dot{u}$ is a simple function with respect to the σ-algebra $\mathcal{L}$ of Lebesgue measurable subsets of the interval I.

3. Main result.

Let t_0, $T \in \mathbb{R}$, $T > 0$: in what follows, I indicates the interval $[t_0, t_0 + T]$. The announced result is

THEOREM 1. *Let $F: \Omega \to 2^{\mathbb{R}^n}$ be a bounded multifunction, let G be its h-u.s.c., convex-valued regularization, and let $x_0 \in \Omega$. Then $x: I \to \Omega$ is a trajectory of G if and only if it is a quasitrajectory of F (according to Definition 2). More precisely, for every solution x of $\dot{x} \in G(x)$, $x(t_0) = x_0$ and for every $\varepsilon > 0$ there exist a quasipolygonal function $y: I \to \Omega$ and a function $\xi: I \to \mathbb{R}^n$ such that $y(t_0) = x_0$, $|x(t) - y(t)| < \varepsilon$, $|\xi(t)| < \varepsilon$ for every $t \in I$ and*

$$\dot{y}(t) \in F(y(t) + \xi(t)) \qquad \textit{for a.e. } t \in I .$$

Moreover, the same holds with $\operatorname{ext} \overline{\operatorname{co}} F$ *in place of F.*

The proof of Theorem 1 is a refinement of the argument presented in [1, Theorem 2.4.2] to demonstrate Ważewski's theorem. We begin by stating a lemma contained in a paper of Cellina [3, Theorem 1], which is itself of interest, because it provides a kind of uniform upper semicontinuity for a map defined on a compact space.

PROPOSITION 1 (Cellina). *Let (X, d_X), (Y, d_Y) be two metric spaces, with X compact, and $\Gamma: X \to 2^Y$ be a h-u.s.c. multivalued map. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\forall x \in X \;\; \exists x' \in B(x, \delta) \colon \; \Gamma(B(x, \delta)) \subseteq B(\Gamma(x'), \varepsilon) .$$

PROOF. Fix $\varepsilon > 0$ and, for each $x \in X$, define the function $\varrho(\cdot): X \to \mathbb{R}$ as

$$\varrho(x) = \sup \{\delta > 0 \colon \exists x' \in B(x, \delta) \colon \Gamma(B(x, \delta)) \subseteq B(\Gamma(x'), \varepsilon)\} . \tag{5}$$

To prove our thesis we are going to show that $\varrho(x)$ is positive and bounded away from zero on X.

By the h-u.s.c. of Γ, for every $x \in X$ there exists $\eta(x) > 0$ such that $\Gamma\big(B(x, \eta(x))\big) \subseteq B(\Gamma(x), \varepsilon)$. Therefore, setting $x' = x$ in (5), we see that $0 < \eta(x) \leqslant \varrho(x)$. Assume now, by contradiction that there exist two sequences $(\zeta_n)_n$ and $(x_n)_n$ such that $\zeta_n \in \mathbb{R}$, $\zeta_n \downarrow 0$, $x_n \in X$ and $\varrho(x_n) < \zeta_n$. By the compactness of X, we can suppose that

$x_n \to x_0 \in X$. Consider the number $\eta_0 = \eta(x_0)$: when $d_X(x_n, x_0) < \eta_0/2$, we have

$$\Gamma(B(x_n, \eta_0/2)) \subseteq \Gamma(B(x_0, \eta_0)) \subseteq B(\Gamma(x_0), \varepsilon) ,$$

and therefore $\varrho(x_n) \geqslant \eta_0/2$, a contradiction. ∎

Proof of Theorem 1. Since the map G is h-u.s.c. with compact convex values, it is well known that the differential inclusion $\dot{u} \in G(u)$, $u(t_0) = x_0$ admits solutions. Let therefore $x: I \to \Omega$ be one such solution, and fix $\varepsilon > 0$. We can suppose $B(x(I), 2\varepsilon) \subseteq \Omega$ and also that F and G are bounded by $M > 1$. The function $\Gamma: I \to 2^{\mathbf{R}^n}$, $\Gamma(t) = G(x(t))$, is h-u.s.c. By Proposition 1 there exists a $\delta < \varepsilon/6M$ such that

$$\forall t \in I \;\; \exists t' \in B(t, \delta): \; |t - s| < \delta \Rightarrow \Gamma(s) \subseteq B(\Gamma(t'), \varepsilon/18T) . \tag{6}$$

Partition I into N intervals $I_i = [t_i, t_{i+1}]$ of length $T/N < \delta$, such that $MT/N < \varepsilon/9$. For $i = 0, \dots, N-1$, choose a point $t'_i \in B(t_i, \delta)$ such that (6) holds for $t = t_i$, $t' = t'_i$ and define $\Phi_i = \Gamma(t'_i) = G(x(t'_i))$.

Fix now $i \in \{0, \dots, N-1\}$ and consider a partition of the set

$$S_i = \bigcup_{t \in I_i} \Gamma(t)$$

made of a finite number of Borel subsets S_{ij} having diameter not larger than $\varepsilon/18T$; choose moreover a subset $J_i \subseteq I_i$ such that $\operatorname{meas}(J_i) = \operatorname{meas}(I_i)$ and $\dot{x}(t)$ exists for each $t \in J_i$. Set $H_{ij} = \{t \in J_i: \dot{x}(t) \in S_{ij}\}$ and $\chi_{ij}(\cdot) = \chi_{H_{ij}}(\cdot)$, and let z_{ij} be some point in S_{ij}. Since $z_{ij} \in S_i$, by (6) and by our choice of the interval I_i,

$$d(z_{ij}, \Phi_i) < \varepsilon/18T . \tag{7}$$

Define the map $z: I_i \to \mathbf{R}^n$ as $z(t) = \sum_j z_{ij}\chi_j(t)$ if $t \in J_i$ for some i, and $z(t) = 0$ if $t \notin \bigcup J_i$: z is a simple function such that $|z(t) - \dot{x}(t)| \leqslant \varepsilon/18T$ for every $t \in \bigcup J_i$. The derivative of the quasitrajectory we are looking for will be obtained from this first approximation of $\dot{x}$.

By (7) and by the definition of $G(x)$, for each i, j there exist finitely many points x_{ijk}, y_{ijk} and coefficients α_{ijk} such that

$$|x_{ijk} - x(t'_i)| < \varepsilon/3 , \qquad y_{ijk} \in F(x_{ijk}) , \tag{8}$$

$$\alpha_{ijk} \geqslant 0 , \qquad \sum_k \alpha_{ijk} = 1 , \qquad \Big|z_{ij} - \sum_k \alpha_{ijk} y_{ijk}\Big| < \frac{\varepsilon}{18T} . \tag{9}$$

The function $y|I_i$ will be constructed by assigning the vectors y_{ijk} as derivatives on suitable subsets of I_i. To this purpose, select for each j, by Liapunov's Convexity Theorem [6, Proposition 1.1], a family $(A_{ij}(\alpha))_{\alpha\in[0,1]}$ of Lebesgue measurable subsets of H_{ij} such that

i) $A_{ij}(\alpha) \subseteq A_{ij}(\beta)$ if $\alpha \leqslant \beta$,

ii) $\operatorname{meas}(A_{ij}(\alpha)) = \alpha \cdot \operatorname{meas}(H_{ij})$ $(\alpha \in [0, 1])$,

and set for each k

$$p_0 = 0\,, \quad p_k = \sum_{l=1}^{k} \alpha_{ijl} \quad \text{and} \quad \chi_{ijk} = \chi_{A_{ij}(p_k)\setminus A_{ij}(p_{k-1})}\,.$$

Define the simple function $\varrho\colon I \to \mathbf{R}^n$ as

$$\varrho(t) = \begin{cases} \sum_k y_{ijk}\,\chi_{ijk}(t) & \text{for } t \in H_{ij}\,, \\ 0 & \text{for } t \in I \setminus \bigcup_{ij} H_{ij}\,, \end{cases}$$

and set

$$y(t) = x_0 + \int_{t_0}^{t} \varrho(s)\,ds\,. \tag{10}$$

Define also the function $\xi\colon I \to \mathbf{R}^n$ as

$$\xi(t) = \begin{cases} \sum_k \big(x_{ijk} - y(t)\big) \cdot \chi_{ijk}(t) & \text{for } t \in H_{ij}\,, \\ 0 & \text{for } t \in I \setminus \bigcup_{ij} H_{ij}\,, \end{cases} \tag{11}$$

We claim that the function y defined by (10) is the desired approximation. To see this, notice first that $\dot y(t) \in F(\Omega)$ a.e., and therefore y is Lipschitzean with the same constant M as x. Fix $t \in I$. For some i, $t \in I_i$ and we have

$$|y(t) - x(t)| \leqslant |y(t) - y(t_i)| + |y(t_i) - x(t_i)| + |x(t_i) - x(t)|\,.$$

By our choice of N, the first and the last term of the right-hand side are smaller than $\varepsilon/9$. To estimate the second term, remark that, on each I_i,

$$\int_{I_i} z(s)\,ds = \sum_j \text{meas}\,(H_{ij})\,z_{ij}$$

and

$$\int_{I_i} \varrho(s)\,ds = \int_{I_i} \sum_j \Big(\sum_k y_{ijk}\cdot \chi_{ijk}(s)\Big)\,ds = \sum_j \Big(\sum_k \alpha_{ijk}\cdot \text{meas}\,(H_{ij})\cdot y_{ijk}\Big)\,.$$

Thus, by the preceding remark and (9),

$$(12)\qquad \Big|\int_{I_i} \varrho(s)\,ds - \int_{I_i} z(s)\,ds\Big| = \Big|\sum_j \text{meas}\,(H_{ij})\cdot\Big(z_{ij} - \sum_k \alpha_{ijk} y_{ijk}\Big)\Big| \leqslant$$
$$\leqslant \sum_j \text{meas}\,(H_{ij})\,\Big|z_{ij} - \sum_k \alpha_{ijk} y_{ijk}\Big| \leqslant \text{meas}\,(I_i)\cdot\frac{\varepsilon}{18T}\,.$$

At each nodal point t_h, we have, by our choice of z and by (12),

$$|y(t_h) - x(t_h)| \leqslant \Big|x(t_h) - x_0 - \int_{t_0}^{t_h} z(s)\,ds\Big| + \Big|\int_{t_0}^{t_h} (z(s) - \varrho(s))\,ds\Big| \leqslant$$
$$\leqslant \frac{\varepsilon}{18} + \sum_{i<h} \text{meas}\,(I_i)\cdot\frac{\varepsilon}{18T} \leqslant \frac{\varepsilon}{9}\,,$$

and hence

$$\sup_{t\in I} |y(t) - x(t)| \leqslant \varepsilon/3\,.$$

Finally, by (8), (10) and (11)

$$\dot y(t) \in F\big(y(t) + \xi(t)\big) \qquad \text{for a.e. } t\in I\,.$$

Moreover, for a.e. $t\in I$ and for some i, j, k,

$$|\xi(t)| \leqslant |x_{ijk} - y(t)| \leqslant |x_{ijk} - x(t_i')| +$$
$$+ |x(t_i') - x(t_i)| + |x(t_i) - x(t)| + |x(t) - y(t)| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon\,,$$

thanks to our choice of δ and of N and to (8), (11).

The proof of the necessity is concluded.

To prove the sufficiency, notice that, by the necessity, the set of quasitrajectories of F is nonempty. Let therefore y be one of them, with sequences $\xi_k, y_k\colon I \to \mathbf{R}^n$ such that ξ_k is measurable and $|\xi_k(t)| \to 0$ uniformly on I, y is absolutely continuous, $\dot{y}_k(t) \in F(y_k(t) + \xi_k(t))$ for a.e. $t \in I$ and $y_k(t) \to y(t)$ uniformly on I. Since F is bounded, by a compactness argument (see Theorem 0.3.4. in [1]) the sequence $\dot{y}_k$ can be supposed to converge weakly in $L^1(I, \mathbf{R}^n)$ to $\dot{y}$. Since $G(y) \supseteq F(y)$ for every $y \in \Omega$, we have

$$d\Big((y_k(t), \dot{y}_k(t)), \text{graph}\,\{G\}\Big) \leqslant d\Big((y_k(t), \dot{y}_k(t)), \text{graph}\,\{F\}\Big) = |\xi_k(t)| \to 0\,.$$

Therefore, the Convergence Theorem 1.4.1 in [1] (see also the First Proof of Theorem 2.1.3 in the same book) can be applied, yielding

$$\big(y(t), \dot{y}(t)\big) \in \text{graph}\,\{G\}\,,$$

i.e. $\dot{y}(t) \in G(y(t))$, and the proof of the sufficiency is concluded.

Finally, remark that the regularization (according to Definition 1) of the function $x \to \overline{\text{co}}\, F(x)$ is the same as the regularization G of $x \to F(x)$. Therefore, since by Krein-Milman's theorem $\overline{\text{co}}\,\text{ext}\,\overline{\text{co}}\, F(x) = \overline{\text{co}}\, F(x)$ for every $x \in \Omega$, by applying the above arguments to the function $\tilde{F}(x) = \text{ext}\,\overline{\text{co}}\, F(x)$, we obtain for $\tilde{F}$ the same results as for F. The proof of Theorem 1 is concluded. ∎

COROLLARY. *Let $F\colon \Omega \to \mathbf{R}^n$ be a bounded h-u.s.c. multifunction with closed values. Then $x\colon I \to \mathbf{R}^n$ is a trajectory of $\overline{\text{co}}\, F$ if and only if it is a quasitrajectory of* $\text{ext}\,\overline{\text{co}}\, F$.

Indeed, the regularization $G(x)$ coincides with the convexification $\overline{\text{co}}\, F(x)$.

REMARKS.

1) The argument of the sufficiency part of Theorem 1 still applies if the property

$$\lim_{k\to\infty} d\Big((x_k(t), \dot{x}_k(t)), \text{graph}\,\{F\}\Big) \to 0\,,$$

which is more general than (4), holds. This approximation is usually said « in the sense of graph »: it contains both an inner and an outer perturbation of F.

Therefore the following statement holds:

If F is (locally) bounded, then every uniform limit of approximate solutions in the sense of graph of

$$\dot{x} \in F(x)\,, \qquad x(t_0) = x_0$$

is a solution of the relaxed problem

$$\dot{x} \in G(x)\,, \qquad x(t_0) = x_0\,.$$

2) Theorem 1 holds also in the nonautonomous case, provided the regularization G is made also with respect to time, following Definition 2.4 in [7].

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