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Solutions of minimal period of a wave equation via a generalization of a Hofer’s theorem

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Solutions of Minimal Period of a Wave Equation
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A. Salvatore (*)

0. Introduction.

Consider the following semilinear wave equation

\begin{equation}
    u_{tt} - u_{xx} + g(u, t, x) = 0 \quad t \in \mathbb{R}, \ x \in [0, \pi]
\end{equation}

under boundary and periodicity conditions

\begin{equation}
    \begin{cases}
        u(t, 0) = u(t, \pi) = 0 \\
        u(t, x) = u(t + T, x) 
    \end{cases} \quad t \in \mathbb{R}, \ x \in [0, \pi]
\end{equation}

where $T$ is a rational multiple of $\pi$. The problem of the existence of solutions of (0.1)-(0.2) has been studied by many authors (cf. e.g. the review article of Brezis [7]), but very little is known on the minimality on their period. Solutions of (0.1)-(0.2) with minimal period $T$ have been found in [17], when the nonlinear term $g(u, t, x)$ is sublinear in $u$ and the period $T$ satisfies a condition of «ammissibility». Arguing differently, in [16] we have proved the existence of solutions with minimal period in the autonomous case, when the nonlinear term

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g(u) is either sublinear or superlinear in u. Now we shall consider the nonautonomous superlinear case; more precisely, we shall assume

\((G_1)\) \quad g(u, t, x) \in C^1(\mathbb{R} \times \mathbb{R} \times [0, \pi]) , \quad g(u, \cdot, x) \text{ is } T\text{-periodic ,} \quad \frac{\partial g}{\partial u}(u, t, x) > 0 \quad \forall u, t, x ; \quad g(0, t, x) = 0 \quad \forall t, x .

\((G_2)\) \quad \text{there exist } \beta > 2 \text{ and some positive constants } c_1, c_2, R \text{ such that}

i) \quad g(u, t, x) \leq c_1|u|^\beta - 1 \quad |u| > R ,

ii) \quad \frac{\partial g(u, t, x)}{\partial u} \geq c_2|g(u, t, x)|^{(\beta - 2)/(\beta - 1)} \quad |u| > R ;

\((G_3)\) \quad \text{there exists a positive constant } c_3 \text{ such that}

\[ \frac{1}{2} u g(u, t, x) - G(u, t, x) \geq c_3|u|^\beta \quad |u| > R \]

where \( G(u, t, x) = \int_0^u g(s, t, x) \, ds . \)

\((G_4)\) \quad g(u, t, \bar{x} - x) = g(u, t, x) \quad \forall u, t, x .

**Remark 0.3.** By assumptions \((G_1), (G_2)\) it follows that \( G(\cdot, t, x) \) is strictly convex, superquadratic at infinity and quadratic at zero.

Before stating our result, we have to introduce the Legendre transform \( H(\cdot, t, x) \) defined on \( \mathbb{R} \) by

\[ H(v, t, x) = \sup_{u \in \mathbb{R}} \{ uv - G(u, t, x) \} . \]

Since \( G(u, t, x) \) satisfies \((G_1)-(G_2)\), by classical arguments in Convex Analysis, obtain that \( (\partial G/\partial u)(u, t, x) = g(u, t, x) \) is a global homeomorphism, whose inverse \( h(v, t, x) \) is the derivative of \( H(v, t, x) \) respect to \( v \), i.e.

\[ u(t, x) = h(v, t, x) \quad \text{iff } v(t, x) = g(u, t, x) . \]

Moreover let \( \alpha = \beta/(\beta - 1) \). Then by \((G_1)-(G_2)\) and \((0.4)\) it follows that \( H \) satisfies the following properties:

\((G_4^\star)\) \quad H(\cdot, t, x) \in C^2(\mathbb{R}, \mathbb{R}) , \quad H(0, t, x) = \frac{\partial H}{\partial v}(0, t, x) = 0 \quad \forall t, x ;

\[ H(v, t, x) \text{ is convex in } v \text{ and } T\text{-periodic in } t ; \]
(G₄) there exist some positive constants cᵣ, cₛ and R* s.t.

i) \( h(v, t, x) \geq cᵣ^* |v|^{α-1} |v| > R* \),

ii) \( \frac{∂h(v, t, x)}{∂v} < cₛ^* |v|^{α-2} |v| > R* \).

We are now ready to state the following

**THEOREM 0.5.** Assume (G₁)-(G₄) and

\( G₅ \) there exists \( p ∈ ]0, 1[ \) such that for any \( v ∈ \mathbb{R} \) it results

\[ \frac{∂h(v, t, x)}{∂v} v^2 < μ h(v, t, x) v. \]

Then there exists \( \bar{T} > 0 \) s.t. for any \( T, 0 < T < \bar{T}, T/2π = q/p, p \)

and \( q \) odd, problem (0.1)-(0.2) either has a periodic solution having \( T \)

as minimal period, or has a periodic solution which is an accumula-

tion point of periodic solutions.

**REMARK 0.6.** In terms of \( G \), assumptions (G₅) can be stated

\[ \frac{∂g(u, t, x)}{∂u} u^2 > \frac{1}{μ} u g(u, t, x). \]

The same hypothesis has been introduced in [1] for the study of pe-

riodic solutions with prescribed minimal period of a superquadratic

Hamiltonian system.

This paper is organized as follows: we shall consider the dual

functional restricted to a suitable subspace. Then we relate, as in [6],

the Morse index to the minimal period of its critical points. More-

over we shall give a generalization of a Hofer’s theorem (cf. [11],

[12]). An easy consequence will be that there exists a critical point

of the dual functional, obtained by Ambrosetti-Rabinowitz mountain

pass theorem, which gives either a solution of (0.1)-(0.2) with mini-

mal period \( T \) or a solution which is an accumulation point of periodic

solutions.

1. Dual formulation and an useful lemma.

Let be \( Ω = [0, T] \times [0, π] \) and consider the linear operator

\[ Au = u_{tt} - u_{xx} \]
acting on the function \( u \in L^\beta = L^\beta(\Omega) \) and satisfying conditions (0.2) with \( T = 2\pi(q/p) \), \( p, q \in N \). It is known that the kernel \( N(A) \) of \( A \) is the closed subspace of \( L^\beta \) given by

\[
N(A) = \left\{ u(t, x) = h(t + x) - h(t - x), \ h \in L^\beta_{\text{loc}}, \ h \frac{2\pi}{p} \text{- periodic} \right\}.
\]

Moreover for \( \alpha = \beta/(\beta - 1) \), consider the Banach space

\[
F = L^\alpha \cap R(A) = \left\{ f \in L^\alpha : \frac{1}{\Omega} \int f(t, x)\Phi(t, x) \, dt \, dx = 0, \ \forall \Phi \in N(A) \right\}
\]

(equipped with the usual \( \| \cdot \|_\alpha \) norm). Denote by \( \langle \cdot, \cdot \rangle_\beta \) the pairing between \( L^\beta \) and \( L^\alpha \).

Then \( A \), as an operator from \( F \) into \( F \), has a continuous inverse \( K \). An explicit formula for \( Kf \) (cf. [8] and [13]), permits to prove that there exists \( c_r > 0 \) such that

\[
|Kf|_{c_r,s} \leq c_r \| f \|_{\alpha} \quad \text{with} \quad s = 1 - \frac{1}{\alpha}
\]

and

\[
\frac{1}{\Omega} \int f(Kg) = \frac{1}{\Omega} \int f(Kg) \quad \forall f, g \in F,
\]

then \( K \) is a compact selfadjoint operator in the space

\[
\left\{ f \in L^2 : \frac{1}{\Omega} \int f\Phi = 0 \ \forall \Phi \in N(A) \cap L^2 \right\}.
\]

Moreover in this space the spectrum \( \sigma(K) \) of \( K \) is given by

\[
\sigma(K) = \left\{ \mu_k = \frac{1}{k^2 - ((2\pi/T)j)^2}, \ k \neq \frac{2\pi}{T} j, \ k = 1, 2, \ldots; \ j = 0, \pm 1, \pm 2, \ldots \right\}
\]

and the corresponding eigenfunctions are

\[
\psi_{k,j} = \sin kx \exp \left[ \frac{i}{j} \frac{2\pi}{T} t \right].
\]
Moreover it is known that by \((G_1^\ast), (G_2^\ast)\) the term \(\int_H(v, t, x)\) is \(C^1\) on \(L^\infty\). By the duality principle (cf. e.g. [7]), the solutions of (0.1)-(0.2) correspond to the critical points of the functional

\[
\begin{aligned}
f^\ast(v) &= \frac{1}{2} \int_Q (K v, v) + \int_Q H(v) \\
\text{subject to the constraint } &v \in R(A).
\end{aligned}
\]

More precisely, \(\bar{u}(t, x)\) is a solution of (0.1)-(0.2) iff \(\bar{v}(t, x) = g(\bar{u}, t, x)\) is a critical point of \(f^\ast\) on \(F\).

Moreover, by symmetry assumption \((G_4)\), it follows that we can look for solutions of (0.1)-(0.2) which belong to a suitable subspace. In fact, consider the following subspace of \(L^\infty \cap R(A)\)

\[ E = \{ u \in L^\infty \cap R(A) : u(t, \pi - x) = u(t, x) \text{ for any } (t, x) \in \Omega \}. \]

Let be \(T = 2\pi q/p, p\) and \(q\) odd. By combining the Coron's idea (cf. [9]) and the dual formulation, we have that the critical points of the functional \(f^\ast\) restricted to \(E\) are the classical solutions of problem (0.1)-(0.2). In the sequel we still denote by \(f^\ast\) the restriction \(f^\ast_E\).

Assume for a moment that \(v\) is a critical point of \(f^\ast\) on \(E\) and \(\bar{u}(t, x) = g(\bar{v}, t, x)\) is nontrivial, i.e., \(\bar{u}(t, x) \neq 0\) on a set of positive measure. Let \(m(\bar{v})\) the Morse index of \(\bar{v}\); then \(m(\bar{v})\) coincides with the index of \(f^\ast_E(\bar{v})\) in \(L^2 \cap E\).

The following lemma permits to give a lower bound to the Morse index (cf. [6]).

**Lemma 1.4.** Suppose that \((G_4)\) holds. Let \(\bar{v}(t, x)\) be a nontrivial critical point of \(f^\ast\) with minimal period \(T/l\). Then

\[ l \leq m(\bar{v}). \]

**Proof.** We shall argue as in lemma 2.3 of [6].

Let \(T_0 = 0 < T_1 < \ldots < T_{l-1} < T_l = T\) s.t.

\[ \bar{v}(t_i, x) = \bar{v}(0, x) \quad i = 0, 1, \ldots, l \quad \forall x \in [0, \pi]. \]

Set

\[ \Omega_i = [T_{i-1}, T_i] \times [0, \pi] \quad i = 1, \ldots, l \]
and

\[ \alpha_i(t, x) = \begin{cases} \bar{v}(t, x) & \text{if } (t, x) \in \Omega_i \\ 0 & \text{if } (t, x) \notin \Omega_i \end{cases}, \quad i = 1, \ldots, l. \]

Obviously \( \alpha_i \) \((i = 1, \ldots, l)\) are linearly independent in \( E \). Let \( V_i \) denote the vector space of \( E \) spanned by \( \{\alpha_i\} \).

We will prove that \( f^{**}(\bar{v}) \) is negative definite on \( V_i \). Let

\[ v \in V_i \setminus \{0\}, \quad v = \sum_{i=1}^{l} c_i \alpha_i, \quad c_i \in \mathbb{R}. \]

Then

\[ \langle f^{**}(\bar{v}) v, v \rangle = \iint_{\Omega} \left[ (Kv, v) + \frac{\partial h(\bar{v}, t, x)}{\partial v} v^2 \right] = \]

\[ = \sum_{i=1}^{l} c_i^2 \iint_{\Omega_i} \left[ (K\alpha_i, \alpha_i) + \frac{\partial h(\bar{v}, t, x)}{\partial v} \alpha_i^2 \right] = \]

\[ = \sum_{i=1}^{l} c_i^2 \iint_{\Omega_i} \left[ (K\bar{v}, \bar{v}) + \frac{\partial h(\bar{v}, t, x)}{\partial v} \bar{v}^2 \right] < \]

\[ < \sum_{i=1}^{l} c_i^2 \iint_{\Omega_i} \left[ (K\bar{v}, \bar{v}) + \mu h(\bar{v}, t, x) \bar{v} \right] < 0. \]

The last inequality follows from the fact that \( h(\bar{v}, t, x) \bar{v} \) is positive and \( \bar{v} \) is a \((T/l)\)-periodic solution of \( f^{**}(v) = 0 \).

**Remark 1.5.** Let us observe that we have restricted the functional \( f^* \) to \( E \) because if we generally \( \bar{v} \notin R(A) \), generally \( \alpha_i \) does not belong to \( R(A) \) and therefore the proof of lemma 1.4 is not true.

2. **Proof of theorem 0.5.**

First of all, we note that \( f^* \) satisfies the assumptions of the mountain pass theorem (cf. [2], [5]).

In fact by \((G_1^*), \ (G_2^*)\) and for \( T \) small enough, it results:

i) there are constants \( r > 0 \) and \( \alpha > 0 \) such that \( f^*(v) > \alpha \) for every \( v \in E \) with \( \|v\|_\alpha = r \);

ii) \( f^*(0) = 0 \) and \( f^*(v_0) < \alpha \) for some \( v_0 \in E \) with \( \|v_0\|_\alpha > r \).
Moreover by \( (G_3) \) it follows (cf. [5]) that \( f^* \) satisfies the following condition (which is a weakened version of the Palais-Smale condition):

\[(C) \text{ If } \{v_n\} \subset E, \ f^*(v_n) \text{ is bounded and } \|f^*(v_n)\|_\beta \|v_n\|_\alpha \to 0, \text{ then there exists a subsequence } v_{n_k} \text{ convergent in } E.\]

Then we can find that \( f^* \) has a critical point in \( E \).

Unfortunately, we cannot conclude, as in [11], that there exists a critical point \( \bar{v} \) such that \( m(\bar{v}) < 1 \), because \( E \) is not a Hilbert space. In the following we shall adapt the arguments contained in [11] to our situation.

Let us still denote by \( f^* \) the restriction of \( f^* \) on \( L^2 \cap E \). Since \( f^* \) does not satisfy condition \( (C) \) or \( (PS) \) condition on \( L^2 \), we shall introduce the following compacteness condition.

Let \( c \in \mathbb{R} \). We say that \( f^* \) satisfies condition provided:

\[(PS)_c \text{ If } \{v_n\} \in L^2 \cap E, \ f^*(v_n) \to c \text{ and } \|f^*(v_n)\|_2 \|v_n\|_2 \to 0, \text{ then there exists a subsequence } v_{n_k} \text{ convergent to } v \text{ in } L^2. \text{ Moreover } \langle f^*(v), v \rangle_\beta = 0.\]

**Lemma 2.1.** The functional \( f^* \) satisfies \( (PS)_c \) condition.

**Proof.** Let \( \{v_n\} \in L^2 \cap E \) s.t.

\[(2.2) \quad f^*(v_n) \to c, \quad (2.3) \quad \|f^*(v_n)\|_2 \|v_n\|_2 \to 0.\]

Obviously \( (2.3) \) implies that

\[(2.4) \quad \langle f^*(v_n), v_n \rangle_\beta \to 0.\]

Since \( f^* \) verifies condition \( (C) \) on \( L^2 \) (cf. [5]), there exists a subsequence, still denoted by \( \{v_n\} \), which converges to \( v \) in \( L^2 \), i.e.

\[(2.5) \quad v_n \to v \quad \text{in } L^2.\]

Then

\[(2.6) \quad f^*(v_n) \to f^*(v) = c, \quad (2.7) \quad f^*(v_n) \to f^*(v) \quad \text{in } L^\beta,\]

and therefore \( \langle f^*(v), v \rangle_\beta = 0.\)
An obvious consequence of \((PF)\), is the following result:

**Corollary 2.8.** \(\forall c \in \mathbb{R} \exists \sigma, M, \gamma > 0\) s.t.

\[
\forall v \in f^*([c - \sigma, c + \sigma]), \quad \|v\|_p \geq M: \|f^*(v)\|_2 \leq 2 > \gamma.
\]

Condition \((PF)\) is a weakening of the condition \((C)\), introduced in [3], and the condition \((PS)\), introduced in [8]. Arguing as in [8], \((PF)\) implies that \(f^*\) has a critical value in \(L^2\), but this is known already. To prove the existence of a critical point \(\bar{v}\) with \(m(\bar{v}) < 1\), it needs the standard deformation lemma (cf. [14]), but we think that \((PF)\) does not suffice to prove it. Then the following lemma will be useful.

**Lemma 2.9.** Let \(\{v_n\} \in L^2 \cap E\) a bounded sequence in \(L^2\) s.t.

\[
(2.10) \quad f^{*\prime}(v_n) \rightarrow 0 \quad \text{in} \quad L^2.
\]

Then there exists a subsequence \(\{v_n\}\) s.t. \(f^{*\prime}(v_n) \rightarrow 0\) in \(L^2\).

**Proof.** Since \(\{v_n\}\) is bounded in \(L^2\), there exists a subsequence, still denoted by \(\{v_n\}\), s.t. \(v_n \rightarrow v\) in \(L^2\). Then

\[
(2.11) \quad K v_n \rightarrow K v \quad \text{in} \quad L^2.
\]

Moreover by \((G^*_2)\) there exist some constants \(M_i\) s.t.

\[
(2.12) \quad \|h(v_n, t, x)\|_p^\beta = \int_{\Omega} |h(v_n, t, x)|^\beta = \int_{\Omega} |h(v_n, t, x)|^\beta + \int_{\Omega} |h(v_n, t, x)|^\beta < M_1 \int_{\Omega} \sup_{\Omega} |v_n|^{\beta(z-1)} + M_2 < M_1 \|v_n\|_p^\alpha + M_2 < M_3,
\]

where \(\Omega_n = \{(t, x) \in \Omega : |v_n(t, x)| > R^*\}\).

By (2.11) and (2.12) we have that \(f^{*\prime}(v_n)\) is bounded in \(L^2\), then the conclusion follows by (2.10).

Give now some definitions. Let \(\delta > 0, c \in \mathbb{R}\). We set

\[
K_c = \{v \in E : f^*(v) = c \ \text{and} \ f^{*\prime}(v) = 0\} ;
\]

\[
A_c = \{v \in E : f^*(v) < c\} ,
\]

\[
\hat{A}_c = \{v \in E : f^*(v) < c\} ,
\]

\[
M^*_\delta = \{v \in E : \text{dist}_n(v, K_c) < \delta\} ,
\]
$D(f^*) = \{ \sigma : [0, 1] \times L^2 \cap E \rightarrow L^2 \cap E, \sigma \text{ continuous } |\sigma(0, \cdot)| = I d_{L^2 \cap E},
\sigma(t) \rightarrow f^*(\sigma(t, u)) \text{ is nonincreasing for all } u \in L^2 \cap E \}$.

We shall prove the following deformation lemma (cf. [3] and [14])

**Lemma 2.13.** Given $c \in R$ s.t. $K_c \neq \emptyset$ and $V, W$ open neighbourhoods of $K_c$ in $L^2$, with $V = M^c_\delta \cap L^2$, $W = M^c_{\delta b} \cap L^2$, there exist $\eta \in D(f^*)$ and constants $\varepsilon > \varepsilon > 0$ satisfying the following properties

\[ i) \quad \eta(1, A_{c+\varepsilon} \setminus V) \subset A_{c-\varepsilon}; \]
\[ ii) \quad \eta(t, u) = u \quad (t, u) \in [0, 1] \times (A_{c+\varepsilon} \setminus A_{c-\varepsilon}); \]
\[ iii) \quad \eta([0, 1] \times (\bar{V} \cap A_c)) \subset W, \]

where $\bar{V} = \{ v \in L^2 \cap E \mid \text{dist}_a(v, K_c) < \delta \}$.

**Proof.** Let $c \in R$ s.t. $K_c \neq \emptyset$. Since $f^*$ satisfies condition (C) on $L^2$, $K_c$ is compact in $L^2$. We shall prove that there exist $\varepsilon, b, b_1 > 0$ s.t.

\[
(a) \quad \| f^*'(v) \|_2 \geq b, \quad \forall v \in (A_{c+\varepsilon} \setminus A_{c-\varepsilon}) \cap (M^c_\delta \setminus M^c_{\delta b}) \cap L^2,
(b) \quad \| f^*'(v) \|_2 \geq b_1, \quad \forall v \in (A_{c+\varepsilon} \setminus A_{c-\varepsilon}) \cap (B^c_\delta \setminus M^c_{\delta b}) \cap L^2.
\]

If (2.14)(a) does not hold, then there exists

\[
\{v_n\} \in L^2 \cap (M^c_\delta \setminus M^c_{\delta b}) \text{ s.t.}
\]

\[
f^*(v_n) \rightarrow c \quad \text{and} \quad f^*'(v_n) \rightarrow 0 \quad \text{in } L^2.
\]

Since $M^c_\delta$ is bounded in $L^2$, by lemma 2.9 it follows that

\[
f^*(v_n) \rightarrow 0 \quad \text{in } L^\delta.
\]

But this is not true because condition (C) on $L^2$ implies (cf. [3], theorem 1.3) that there exists $\bar{b}, \varepsilon_i > 0$ s.t.

\[
\| f^*'(v_n) \|_2 \geq \bar{b}, \quad \forall v \in (A_{c+\varepsilon_i} \setminus A_{c-\varepsilon_i}) \cap (M^c_\delta \setminus M^c_{\delta b}) \cap L^2.
\]

The same proof holds for (2.14)-(b).

Since (2.14) still holds if $\varepsilon$ is decreased, we can assume

\[
\varepsilon < \min \left\{ \frac{b\delta}{8}, \sigma \right\},
\]

where $\sigma$ is the constant of corollary (2.8).
Moreover by condition (C) on $L^a$ we have

$$\|f^*(v)\|_2 > 0 \quad \forall v \in (A_{c+\varepsilon} \setminus A_{c-\varepsilon}) \setminus M_{\delta/2}^a$$

Now let $0 < \varepsilon < \delta$. As in theorem 1.3 of [3], we can define

$$\chi: L^2 \cap E \to [0, 1] \text{ s.t.}$$

$$\chi(v) = \begin{cases} 
0 & \text{if } v \notin f^{-1}([c - \varepsilon, c + \varepsilon]) \text{ or } v \in M_{\delta/8}^a \\
1 & \text{if } v \in f^{-1}([c - \varepsilon, c + \varepsilon]) \setminus M_{\delta/4}^a
\end{cases}$$

and

$$V: L^2 \cap E \to L^2 \text{ s.t.}$$

$$V(v) = \begin{cases} 
-\chi(v) \Phi(v) & \text{if } v \in \bar{L}^a = \{v \in L^a: f^*(v) \neq 0\} \\
0 & \text{otherwise}
\end{cases}$$

where $\Phi$ is the « pseudogradient vector field » associated to $f^*$ (cf. [3] and [14]).

By corollary (2.8) and (2.14)-(b) it follows that

$$\|V(v)\|_2 < K_1 + K_2 \|v\|_2$$

where $K_1$, $K_2$ are positive constants independent of $v \in L^a$. Consider now the following initial value problem

$$\frac{d\eta}{dt} = V(\eta), \quad \eta(0) = x \quad x \in L^2 \cap E.$$  

Since $V$ is locally Lipschitz continuous, for any iniżial value $x \in L^2 \cap E$ (2.22) possesses a unique solution $\eta(\cdot, x)$ which, by virtue of (2.21), is defined in $R_+ = \{t \in R: t > 0\}$. By (2.19)-(2.20), it is clear that $\eta(t, \cdot)$ satisfies ii), for any $t \in R_+$. Arguing as in [3], it can been proved that there exists $\hat{t}$ s.t.

$$\eta(\hat{t}, A_{c+\varepsilon} \setminus V) \subset A_{c-\varepsilon}.$$ 

Then making a reparametrisation of the time $t$, one obtains the desired map $\eta$ satisfying i).
We give now the proof of property iii). Denoting by $o(a)$ the embedding constant of $L^s$ into $L^a$, then by (2.21)-(2.22), for any $x \in L^s \cap E$ we have

$$\|\eta(t, x) - \eta(0, x)\|_a \leq o(a)\|\eta(t, x) - \eta(0, x)\|_a$$

$$= o(a)\left\|\int_0^t V(\eta(\tau, x))\, d\tau\right\|_2 \leq (K_1 + K_2\|x\|_2) o(a) t .$$

By (2.23), it follows that for any $v \in \bar{V}$, $w \in K_c$,

$$\|\eta(t, v) - w\|_a \leq (K_1 + K_2\|v\|_2) o(a) t + \delta .$$

Observe now that $\bar{V} \cap A_c$ is bounded in $L^a$ (for details we refer to the proof of theorem 2.27). Then the conclusion follows for $t$ small enough.

We introduce now a variant of a definition given by Hofer in [11].

**DEF. 2.24.** Let $c \in \mathbb{R}$ and $v_0 \in K_c$. We say that $v_0$ is of almost mountain pass type (a. mp-type) in $L^2$ if for all open neighbourhood $U$ of $v_0$ in $L^a \cap \mathbb{R}$ the topological space $U \cap L^s \cap A_c$ is nonempty and not path-connected in $L^s \cap E$.

Following [11], we shall state the existence of a critical point of $\tilde{f}^*$ of a. mp-type in $L^2$.

**THEOREM 2.25.** Assume that there exist $\rho, r > 0$ and $e_0 \in L^s \cap E$ s.t. $\tilde{f}^*(e_0) < \rho = \inf_{\|v\|_a = r} \tilde{f}^*(v)$. Set

$$A = \{a: [0, 1] \to L^s \cap \mathbb{R}: a \text{ continuous, } a(0) = 0, a(1) = e_0\} ,$$

$$c = \inf_A \max_{a} f^*(|a|) , \quad \text{where } |a| = a([0, 1]) .$$

Then $K_c$ is nonempty. If in addition the critical points in $K_c$ are isolated in $L^a$, there exists $v_0 \in K_c$ of a. mp-type in $L^s$.

**PROOF.** As we have already noted, $K_c \neq \emptyset$ (we recall that the critical point of $f^*$ belong to $L^a$). Arguing indirectly, we may assume that $K_c$ contains only a finite number of critical points all being not of a. mp-type in $L^s$. Let $K_c = \{v_1, ..., v_n\}$. 

Then we find corresponding open neighbourhoods $U_i$ of $v_i$ in $L^2 \cap E$ s.t.

$$U = \bigcup_{i=1}^{n} U_i \subset K_e.$$ 

Define $\delta > 0$, $\bar{\varepsilon} > 0$, $W$ and $V$ by $\bar{\varepsilon} = \frac{1}{2}(c - d)$, where

$$d = \max \{f^*(0), f^*(e_0)\};$$

$$\delta = \frac{1}{2} \min \{ \text{dist}_x(\partial U \cup \{0, e_0\}, K_e), \inf \{ \text{dist}_x(v_i, K_e \setminus \{v_i\}): i = 1, \ldots, n \} \},$$

$$W = M_{2\delta}^x \cap L^2 \quad \text{and} \quad V = M_{\delta}^x \cap L^2.$$ 

By lemma 2.13, we find $\varepsilon \in (0, \bar{\varepsilon})$ and $\sigma \in D(f^*)$ satisfying i)-iii). Choose $a \in A$ with $|a| \subset A_{c+\varepsilon}$. Note that

$$W = \left( \bigcup_{i=1}^{n} W_i \right) \cap L^2 \quad \text{and} \quad V = \left( \bigcup_{i=1}^{n} V_i \right) \cap L^2,$$

where $W_i$ and $V_i$ are open $2\delta$ or $\delta$-balls, respectively, around $v_i$ in $L^2 \cap E$. Let

$$M = \{ t \in [0, 1]: a(t) \notin V \} \quad \text{and} \quad \Gamma = (U \cap L^2 \cap \hat{A}_c) \cup \sigma(1, a(M)).$$

Observe that $0, e_0 \in \Gamma$. Denote by $\hat{\Gamma}$ the path-component of $\Gamma$ in $L^2 \cap E$ containing $0$. Arguing as in [11] it is possible to show that $e_0 \in \hat{\Gamma} \subset \Gamma \subset \hat{A}_c$, and this contradicts the definition of $e$.

**Remark 2.26.** Let us observe that theorem 1 of [11] assures that $f^*$ has a critical point $v_0$ of mp-type in $L^2$, but we cannot prove that $m(v_0) < 1$, since $L^2$ is not a Hilbert space. Moreover $f^*$ does not verify condition $(C)$ on $L^2$, and therefore we do not know if $f^*$ has a critical point of mp-type in $L^2$. For this reason we have introduced the definition of critical point of a mp-type in $L^2$.

Finally we shall prove the following:

**Theorem 2.27.** Let $v_0$ an isolated critical point of $f^*$ of a mp-type in $L^2$. Then $m(v_0) < 1$.

To prove this theorem, we need the following variant of the Morse lemma (cf. [10], [11]).
LEMMA 2.28. Let $F$ be a real Hilbert space, $U$ a nonempty open subset and $\Phi \in C^2(U, \mathbb{R})$ having a gradient of the form identity-compact. Suppose $0 \in \Phi(U)$ is an isolated critical point of $\Phi$ with $\Phi(0) = 0$. Let $F = F^- \oplus F^0 \oplus F^+$ be the canonical decomposition associated to $\Phi''(0)$ via the spectral resolution.

Then there exist an origin-homeomorphism $D$ defined on a $0$-neighbourhood into $F$ and an origin-preserving $C^1$-map $\beta$ defined on a $0$-neighbourhood in $F^0$ into $F^- \oplus F^+$ s.t.

$$\Phi(Du) = -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|z\|^2 + \Phi(\beta(y) + y)$$

for all $u = x + y + z$, $\|u\|$ small.

PROOF OF THEOREM 2.27. We may assume $v_0 = 0$. Then $c = f^*(0) = 0$. By lemma 2.28 $f^*$ has the form

$$f^*(v) = -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|z\|^2 + \psi(y), \quad \|v\|_2 \text{ small}$$

where $\psi(y) = f^*(\beta(y) + y)$ and $0$ is an isolated critical point of $\psi$.

Let $W$ be a ball of $0$ in $L^2 \cap E'$; then $W \cap L^2$ is a neighbourhood of $0$ in $L^2 \cap E$ s.t. $W \cap L^2 = W^- \oplus W^+ \oplus W^0$, where $W^- = E^- \cap W$, $W^+ = E^+ \cap W$, $W^0 = E^0 \cap W$, and $L^2 \cap E = E^- \oplus E^0 \oplus E^+$.

Since $\dim E^- + \dim E^0 < +\infty$, if we choose $W$ small enough, we have that $W^-$ is a $\delta$-ball around $0$ in $L^2 \cap E$ and $W^0$ is a ball in $L^2 \cap E$ with

$$|\psi(y)| < \frac{\delta^2}{8} \quad \forall y \in W_0.$$ 

Moreover decomposition (2.28) holds on $W^- \oplus W^0$. Obviously we may assume that $K_\delta \cap (\overline{W} \cap L^2) = \{0\}$. We shall prove that $\dim E^- < 1$. Namely, if we assume $\dim E^- > 2$, we will show that $W \cap L^2 \cap A_0 = \Gamma$ is path-connected in $L^2 \cap E$, and this contradicts our assumption that $0$ is of a mp-type in $L^2$. Let $g, g' \in \Gamma$. We shall write $g \sim g'$ iff they are in the same path-component in $L^2 \cap E$. Let $g = x_1 + y_1 + z_1 \in \Gamma$. We shall find

$$g \sim g_1 = x_1 + y_1.$$ 

Namely if we consider the continuous map $h : [0, 1] \to L^2 \cap E$, $h(t) = x_1 + y_1 + tz_1$, it is obvious that $|h| \subset W \cap L^2 \cap E$. To prove that
\(|h| \subset \mathcal{A}_\phi\), it suffices to choose \(W\) small enough such that by Taylor’s formula we have

\[
f^*(g) = \frac{1}{2} [f^*(0)g, g] + o(\|g\|_2^2) \quad \forall g \in W.
\]

Then if \(g \in W \cap L^2\), there exist some positive constants \(\lambda_1, \lambda_2\) s.t.

\[
f^*(g) > \frac{1}{2} [\lambda_1 \|g_+\|_2^2 - \lambda_2 \|g_-\|_2^2] + o(\|g\|_2^2).
\]

Since \(g \in W \cap L^2 \cap \mathcal{A}_\phi\) and \(W\) is bounded in \(L^2\), it follows that \(W^+ \cap L^2 \cap \mathcal{A}_\phi\) is bounded in \(L^2\) and \(W \cap L^2 \cap \mathcal{A}_\phi\) is contained in the neighbourhood of 0 in \(L^2 \cap \mathcal{E}\) on which (2.28) holds.

Then \(h(t) \in W \cap L^2 \cap \mathcal{A}_\phi\) and \(g \sim g_1\).

Now we can choose \(x_3 \in W^- \cap L^2\) with \(\|x_3\|_2 > \delta/2\) and

\[
\|tx_3 + (1-t)x_2\|_2 > \|x_1\|_2 \quad \forall t \in [0, 1].
\]

Since (2.28) holds on \((W^- \oplus W^0) \cap L^2\), it follows that \(g_1 \sim g_3 = x_2 + y_1\). Finally by (2.29) \(g_3 \sim g = x_2\). Hence we have shown that for every \(g \in \Gamma\) there exists \(\tilde{g} \in \tilde{\Gamma} = \overline{W} \setminus \{0\}\), with \(g \sim \tilde{g}\), provided \(E^- \neq \{0\}\). If \(\dim E^- > 2\), the set \(\tilde{\Gamma}\) is path-connected, then \(\Gamma\) is path-connected in \(L^2 \cap \mathcal{E}\), which contradicts the fact that 0 is of a. mp-type in \(L^2\). Therefore \(\dim E^- < 1\).

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