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Solutions of minimal period of a wave equation via a generalization of a Hofer's theorem

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Solutions of Minimal Period of a Wave Equation via a Generalization of a Hofer's Theorem.

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0. Introduction.

Consider the following semilinear wave equation

$$u_{tt} - u_{xx} + g(u, t, x) = 0 \qquad t \in \mathbb{R},\ x \in [0, \pi] \tag{0.1}$$

under boundary and periodicity conditions

$$\begin{cases} u(t, 0) = u(t, \pi) = 0 \\ u(t, x) = u(t + T, x) \end{cases} \qquad t \in \mathbb{R},\ x \in [0, \pi] \tag{0.2}$$

where T is a rational multiple of π. The problem of the existence of solutions of (0.1)-(0.2) has been studied by many authors (cf. e.g. the review article of Brezis [7]), but very little is known on the minimality on their period. Solutions of (0.1)-(0.2) with minimal period T have been found in [17], when the nonlinear term $g(u, t, x)$ is sublinear in u and the period T satisfies a condition of « ammissibility ». Arguing differently, in [16] we have proved the existence of solutions with minimal period in the autonomous case, when the nonlinear term

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$g(u)$ is either sublinear or superlinear in u. Now we shall consider the nonautonomous superlinear case; more precisely, we shall assume

(G_1) $g(u, t, x) \in C^1(\mathbf{R}\times\mathbf{R}\times[0,\pi])$, $g(u, \cdot, x)$ is T-periodic,

$$\frac{\partial g}{\partial u}(u, t, x) > 0 \quad \forall u, t, x; \qquad g(0, t, x) = 0 \quad \forall t, x.$$

(G_2) there exist $\beta > 2$ and some positive constants c_1, c_2, R such that

i) $g(u, t, x) \leqslant c_1|u|^{\beta-1} \qquad |u| > R$,

ii) $\dfrac{\partial g(u, t, x)}{\partial u} \geqslant c_2|g(u, t, x)|^{(\beta-2)/(\beta-1)} \qquad |u| > R$;

(G_3) there exists a positive constant c_3 such that

$$\tfrac{1}{2}u\,g(u, t, x) - G(u, t, x) \geqslant c_3|u|^\beta \qquad |u| > R$$

where $G(u, t, x) = \int_0^u g(s, t, x)\,ds$.

(G_4) $g(u, t, \pi - x) = g(u, t, x) \qquad \forall u, t, x$.

REMARK 0.3. By assumptions (G_1), (G_2) it follows that $G(\cdot, t, x)$ is strictly convex, superquadratic at infinity and quadratic at zero.

Before stating our result, we have to introduce the Legendre transform $H(\cdot, t, x)$ defined on $\mathbf{R}$ by

$$H(v, t, x) = \sup_{u\in\mathbf{R}}\{uv - G(u, t, x)\}. \tag{0.4}$$

Since $G(u, t, x)$ satisfies (G_1)-(G_2), by classical arguments in Convex Analysis, obtain that $(\partial G/\partial u)(u, t, x) = g(u, t, x)$ is a global homeomorphism, whose inverse $h(v, t, x)$ is the derivative of $H(v, t, x)$ respect to v, i.e.

$$u(t, x) = h(v, t, x) \quad \text{iff } v(t, x) = g(u, t, x).$$

Moreover let $\alpha = \beta/(\beta - 1)$. Then by (G_1)-(G_2) and (0.4) it follows that H satisfies the following properties:

(G_1^*) $H(\cdot, t, x) \in C^2(\mathbf{R}, \mathbf{R})$, $\quad H(0, t, x) = \dfrac{\partial H}{\partial v}(0, t, x) = 0 \quad \forall t, x$;

$H(v, t, x)$ is convex in v and T-periodic in t;

(G_3^*) there exist some positive constants c_1^*, c_2^* and R^* s.t.

i) $h(v, t, x) \geqslant c_1^* |v|^{\alpha-1} \qquad |v| > R^*$,

ii) $\dfrac{\partial h(v, t, x)}{\partial v} \leqslant c_1^* |v|^{\alpha-2} \qquad |v| > R^*$.

We are now ready to state the following

THEOREM 0.5. Assume (G_1)-(G_4) and

(G_5) there exists $\mu \in]0, 1[$ such that for any $v \in \mathbb{R}$ it results

$$\frac{\partial h(v, t, x)}{\partial v} v^2 \leqslant \mu h(v, t, x) v .$$

Then there exists $\bar{T} > 0$ s.t. for any T, $0 < T < \bar{T}$, $T/2\pi = q/p$, p and q odd, problem (0.1)-(0.2) either has a periodic solution having T as minimal period, or has a periodic solution which is an accumulation point of periodic solutions.

REMARK 0.6. In terms of G, assumptions (G_5) can be stated

$$\frac{\partial g(u, t, x)}{\partial u} u^2 \geqslant \frac{1}{\mu} u g(u, t, x) .$$

The same hypotesis has been introduced in [1] for the study of periodic solutions with prescribed minimal period of a superquadratic Hamiltonian system.

This paper is organized as follows: we shall consider the dual functional restricted to a suitable subspace. Then we relate, as in [6], the Morse index to the minimal period of its critical points. Moreover we shall give a generalization of a Hofer's theorem (cf. [11], [12]). An easy consequence will be that there exists a critical point of the dual functional, obtained by Ambrosetti-Rabinowitz mountain pass theorem, which gives either a solution of (0.1)-(0.2) with minimal period T or a solution which is an accumulation point of periodic solutions.

1. Dual formulation and an useful lemma.

Let be $\Omega = [0, T] \times [0, \pi]$ and consider the linear operator

$$Au = u_{tt} - u_{xx}$$

acting on the function $u \in L^\beta = L^\beta(\Omega)$ and satisfying conditions (0.2) with $T = 2\pi(q/p)$, $p, q \in N$. It is known that the kernel $N(A)$ of A is the closed subspace of L^β given by

$$N(A) = \left\{u(t, x) = h(t + x) - h(t - x),\ h \in L^\beta_{\mathrm{loc}},\ h\ \frac{2\pi}{p}\text{ - periodic}\right\}.$$

Moreover for $\alpha = \beta/(\beta - 1)$, consider the Banach space

$$F = L^\alpha \cap R(A) = \left\{f \in L^\alpha: \iint_\Omega f(t, x)\Phi(t, x)\,dt\,dx = 0,\ \forall \Phi \in N(A)\right\}$$

(equipped with the usual $\|\ \|_\alpha$ norm). Denote by $\langle\cdot,\cdot\rangle_\beta$ the pairing between L^β and L^α.

Then A, as an operator from F into F, has a continuous inverse K. An explicit formula for Kf (cf. [8] and [13]), permits to prove that there exists $c_T > 0$ such that

$$|Kf|_{C^{0,s}} \leqslant c_T \|f\|_\alpha \qquad \text{with } s = 1 - \frac{1}{\alpha} \tag{1.1}$$

and

$$\iint_\Omega (Kf)\,g = \iint_\Omega f(Kg) \qquad \forall f, g \in F, \tag{1.2}$$

then K is a compact selfadjoint operator in the space

$$\left\{f \in L^2: \iint_\Omega f\Phi = 0\ \ \forall \Phi \in N(A) \cap L^2\right\}.$$

Moreover in this space the spectrum $\sigma(K)$ of K is given by

$$\sigma(K) = \\ = \left\{\mu_{kj} = \frac{1}{k^2 - ((2\pi/T)j)^2},\ k \neq \frac{2\pi}{T}j,\ k = 1, 2, \ldots;\ j = 0, \pm 1, \pm 2, \ldots\right\}$$

and the corresponding eigenfunctions are

$$\psi_{kj} = \sin kx \exp\left[ij\,\frac{2\pi}{T}\,t\right]. \tag{1.3}$$

Moreover it is known that by (G_1^*), (G_2^*) the term $\iint_\Omega H(v,t,x)$ is C^1 on L^α. By the duality principle (cf. e.g. [7]), the solutions of (0.1)-(0.2) correspond to the critical points of the functional

$$\begin{cases} f^*(v) = \dfrac{1}{2}\displaystyle\iint_\Omega (Kv, v) + \iint_\Omega H(v) \\ \text{subject to the constraint } v \in R(A)\,. \end{cases}$$

More precisely, $\bar{u}(t,x)$ is a solution of (0.1)-(0.2) iff $\bar{v}(t,x) = g(\bar{u},t,x)$ is a critical point of f^* on F.

Moreover, by symmetry assumption (G_4), it follows that we can look for solutions of (0.1)-(0.2) which belong to a suitable subspace. In fact, consider the following subspace of $L^\alpha \cap R(A)$

$$E = \{u \in L^\alpha \cap R(A) \colon u(t, \pi - x) = u(t,x) \text{ for any } (t,x) \in \Omega\}\,.$$

Let be $T = 2\pi q/p$, p and q odd. By combining the Coron's idea (cf. [9]) and the dual formulation, we have that the critical points of the functional f^* restricted to E are the classical solutions of problem (0.1)-(0.2). In the sequel we still denote by f^* the restriction $f^*_{|E}$.

Assume for a moment that $\bar{v}$ is a critical point of f^* on E and $\bar{u}(t,x) = h(\bar{v},t,x)$ is nontrivial, i.e. $\bar{u}(t,x) \neq 0$ on a set of positive measure. Let $m(\bar{v})$ the Morse index of $\bar{v}$; then $m(\bar{v})$ coincides with the index of $f^{*\prime\prime}(\bar{v})$ in $L^2 \cap E$.

The following lemma permits to give a lower bound to the Morse index (cf. [6]).

Lemma 1.4. Suppose that (G_5) holds. Let $\bar{v}(t,x)$ be a nontrivial critical point of f^* with minimal period T/l. Then

$$l \leqslant m(\bar{v})\,.$$

Proof. We shall argue as in lemma 2.3 of [6].

Let $T_0 = 0 < T_1 < \ldots < T_{l-1} < T_l = T$ s.t.

$$\bar{v}(T_i, x) = \bar{v}(0,x) \qquad i = 0, 1, \ldots, l \qquad \forall x \in [0,\pi]\,.$$

Set

$$\Omega_i = [T_{i-1}, T_i] \times [0,\pi] \qquad i = 1, \ldots, l$$

and

$$\alpha_i(t,x) = \begin{cases} \bar{v}(t,x) & \text{if } (t,x)\in\Omega_i \\ 0 & \text{if } (t,x)\notin\Omega_i \end{cases} \qquad i=1,\dots,l\,.$$

Obviously α_i $(i=1,\dots,l)$ are linearly indipendent in E. Let V_l denote the vector space of E spanned by $\{\alpha_i\}$.

We will prove that $f^{*''}(\bar{v})$ is negative definite on V_l. Let

$$v\in V_l\setminus\{0\}\,,\qquad v=\sum_{i=1}^{l} c_i\alpha_i\,,\qquad c_i\in\mathbf{R}\,.$$

Then

$$\begin{aligned}\langle f^{*''}(\bar{v})v, v\rangle &= \iint_{\Omega}\left[(Kv,v)+\frac{\partial h(\bar{v},t,x)}{\partial v}\,v^2\right] = \\ &= \sum_{i=1}^{l} c_i^2\iint_{\Omega_i}\left[(K\alpha_i,\alpha_i)+\frac{\partial h(\bar{v},t,x)}{\partial v}\,\alpha_i^2\right] = \\ &= \sum_{i=1}^{l} c_i^2\iint_{\Omega_i}\left[(K\bar{v},\bar{v})+\frac{\partial h(\bar{v},t,x)}{\partial v}\,\bar{v}^2\right] \leqslant \\ &\leqslant \sum_{i=1}^{l} c_i^2\iint_{\Omega_i}[(K\bar{v},\bar{v})+\mu h(\bar{v},t,x)\,\bar{v}] < 0\,.\end{aligned}$$

The last inequality follows from the fact that $h(\bar{v},t,x)\,\bar{v}$ is positive and $\bar{v}$ is a (T/l)-periodic solution of $f^{*'}(v)=0$.

REMARK 1.5. Let us observe that we have restricted the functional f^* to E because if we take $\bar{v}\in R(A)$, generally α_i does not belong to $R(A)$ and therefore the proof of lemma 1.4 is not true.

2. Proof of theorem 0.5.

First of all, we note that f^* satisfies the assumptions of the mountain pass theorem (cf. [2], [5]).

In fact by (G_1^*), (G_2^*) and for T small enough, it results:

i) there are constants $r>0$ and $\varrho>0$ such that $f^*(v)\geqslant\varrho$ for every $v\in E$ with $\|v\|_\alpha = r$;

ii) $f^*(0)=0$ and $f^*(e_0)<\varrho$ for some $e_0\in E$ with $\|e_0\|_\alpha > r$.

Moreover by (G_3) it follows (cf. [5]) that f^* satisfies the following condition (which is a weakened version of the Palais-Smale condition):

(*C*) If $\{v_n\} \in E$, $f^*(v_n)$ is bounded and $\|f^{*\prime}(v_n)\|_\beta \|v_n\|_\alpha \to 0$, then there exists a subsequence v_{n_k} convergent in E.

Then we can find that f^* has a critical point in E.

Unfortunately, we cannot conclure, as in [11], that there exists a critical point $\bar{v}$ such that $m(\bar{v}) \leqslant 1$, because E is not a Hilbert space. In the following we shall adapte the arguments contained in [11] to our situation.

Let us still denote by f^* the restriction of f^* on $L^2 \cap E$. Since f^* does not satisfy condition (*C*) or (*PS*) condition on L^2, we shall introduce the following compacteness condition.

Let $c \in \mathbf{R}$. We say that f^* satisfies condition $(\overline{PS})_c$ provided:

$(\overline{PS})_c$ If $\{v_n\} \in L^2 \cap E$, $f^*(v_n) \to c$ and $\|f^{*\prime}(v_n)\|_2 \|v_n\|_2 \to 0$, then there exists a subsequence v_{n_k} convergent to v in L^α. Moreover $\langle f^{*\prime}(v), v\rangle_\beta = 0$.

LEMMA 2.1. The functional f^* satisfies $(\overline{PS})_c$ condition.

PROOF. Let $\{v_n\} \in L^2 \cap E$ s.t.

$$f^*(v_n) \to c\,, \tag{2.2}$$

$$\|f^{*\prime}(v_n)\|_2 \|v_n\|_2 \to 0\,. \tag{2.3}$$

Obviously (2.3) implies that

$$\langle f^{*\prime}(v_n), v_n\rangle_\beta \to 0\,. \tag{2.4}$$

Since f^* verifies condition (*C*) on L^α (cf. [5]), there exists a subsequence, still denoted by $\{v_n\}$, which convergeas to v in L^α, i.e.

$$v_n \to v \quad \text{in } L^\alpha\,. \tag{2.5}$$

Then

$$f^*(v_n) \to f^*(v) = c\,, \tag{2.6}$$

$$f^{*\prime}(v_n) \to f^{*\prime}(v) \quad \text{in } L^\beta\,, \tag{2.7}$$

and therefore $\langle f^{*\prime}(v), v\rangle_\beta = 0$.

An obvious consequence of $(\overline{PS})_c$, is the following result:

COROLLARY 2.8. $\forall c \in \mathbb{R}\ \exists \sigma, M, \gamma > 0$ s.t.

$$\forall v \in f^{*-1}([c-\sigma, c+\sigma])\ , \qquad \|v\|_\alpha \geqslant M\colon \|f^{*\prime}(v)\|_2 \|v\|_2 \geqslant \gamma\ .$$

Condition $(\overline{PS})_c$ is a weakening of the condition (C), introduced in [3], and the condition $(PS)_c$, introduced in [8]. Arguing as in [8], $(\overline{PS})_c$ implies that f^* has a critical value in L^2, but this is known already. To prove the existence of a critical point $\bar{v}$ with $m(\bar{v}) \leqslant 1$, it needs the standard deformation lemma (cf. [14]), but we think that $(\overline{PS})_c$ does not suffice to prove it. Then the following lemma will be useful.

LEMMA 2.9. Let $\{v_n\} \in L^2 \cap E$ a bounded sequence in L^α s.t.

$$f^{*\prime}(v_n) \rightharpoonup 0 \quad \text{in } L^2 \tag{2.10}$$

Then there exists a subsequence $\{v_{n_k}\}$ s.t. $f^{*\prime}(v_{n_k}) \rightharpoonup 0$ in L^β.

PROOF. Since $\{v_n\}$ is bounded in L^α, there exists a subsequence, still denoted by $\{v_n\}$, s.t. $v_n \rightharpoonup v$ in L^α. Then

$$Kv_n \to Kv \quad \text{in } L^\beta . \tag{2.11}$$

Moreover by (G_2^*) there exist some constants M_i s.t.

$$\begin{aligned}\|h(v_n, t, x)\|_\beta^\beta &= \iint_\Omega |h(v_n, t, x)|^\beta = \iint_{\Omega_n} |h(v_n, t, x)|^\beta + \\ &+ \iint_{\Omega\setminus\Omega_n} |h(v_n, t, x)|^\beta \leqslant M_1 \iint_{\Omega_n} |v_n|^{\beta(\alpha-1)} + M_2 \leqslant M_1 \|v_n\|_\alpha^\alpha + M_2 \leqslant M_3\ ,\end{aligned} \tag{2.12}$$

where $\Omega_n = \{(t, x) \in \Omega\colon |v_n(t, x)| \geqslant R^*\}$.

By (2.11) and (2.12) we have that $f^{*\prime}(v_n)$ is bounded in L^β, then the conclusion follows by (2.10).

Give now some definitions. Let $\delta > 0$, $c \in \mathbb{R}$. We set

$$K_c = \{v \in E\colon f^*(v) = c \text{ and } f^{*\prime}(v) = 0\}\ ;$$

$$A_c = \{v \in E\colon f^*(v) \leqslant c\}\ ,$$

$$\mathring{A}_c = \{v \in E\colon f^*(v) < c\}\ ,$$

$$M_\delta^\alpha = \{v \in E\colon \operatorname{dist}_\alpha(v, K_c) < \delta\}\ ,$$

$D(f^*) = \{\sigma\colon [0,1]\times L^2 \cap E \to L^2 \cap E,\ \sigma \text{ continuous } |\sigma(0,\cdot) = \mathrm{Id}_{L^2\cap E},$
$t \to f^*(\sigma(t,u))$ is nonincreasing for all $u \in L^2 \cap E\}$.

We shall prove the following deformation lemma (cf. [3] and [14]):

LEMMA 2.13. Given $c \in \mathbb{R}$ s.t. $K_c \neq \emptyset$ and V, W open neighbourhoods of K_c in L^2, with $V = M_\delta^\alpha \cap L^2$, $W = M_{2\delta}^\alpha \cap L^2$, there exist $\eta \in D(f^*)$ and constants $\bar\varepsilon > \varepsilon > 0$ satisfying the following properties

i) $\eta(1, A_{c+\varepsilon}\setminus V) \subset A_{c-\varepsilon}$;

ii) $\eta(t,u) = u \qquad (t,u) \in [0,1]\times(A_{c+\bar\varepsilon}\setminus A_{c-\bar\varepsilon})$;

iii) $\eta([0,1]\times(\bar V \cap \mathring{A}_c)) \subset W$,

where $\bar V = \{v \in L^2 \cap E \mid \mathrm{dist}_\alpha(v, K_c) \leqslant \delta\}$.

PROOF. Let $c \in \mathbb{R}$ s.t. $K_c \neq \emptyset$. Since f^* satisfies condition (C) on L^α, K_c is compact in L^α. We shall prove that there exist $\bar\varepsilon$, b, $b_1 > 0$ s.t.

$$(2.14)\quad \begin{cases} (a) & \|f^{*\prime}(v)\|_2 \geqslant b \qquad \forall v \in (A_{c+\bar\varepsilon}\setminus A_{c-\bar\varepsilon}) \cap (M_\delta^\alpha \setminus M_{\delta/8}^\alpha) \cap L^2, \\ (b) & \|f^{*\prime}(v)\|_2 \geqslant b_1 \qquad \forall v \in (A_{c+\bar\varepsilon}\setminus A_{c-\bar\varepsilon}) \cap (B_M^\alpha \setminus M_{\delta/8}^\alpha) \cap L^2. \end{cases}$$

If (2.14)(a) does not hold, then there exists

$$\{v_n\} \in L^2 \cap (M_\delta^\alpha \setminus M_{\delta/8}^\alpha) \text{ s.t.}$$
$$f^*(v_n) \to c \quad \text{and} \quad f^{*\prime}(v_n) \to 0 \quad \text{in } L^2.$$

Since M_α^δ is bounded in L^α, by lemma 2.9 it follows that

$$f^{*\prime}(v_n) \rightharpoonup 0 \quad \text{in } L^\beta.$$

But this is not true because condition (C) on L^α implies (cf. [3], theorem 1.3) that there exists $\bar b$, $\bar\varepsilon_1 > 0$ s.t.

$$(2.16)\quad \|f^{*\prime}(v_n)\|_\beta \geqslant \bar b > 0 \qquad \forall v \in (A_{c+\bar\varepsilon_1}\setminus A_{c-\bar\varepsilon_1}) \cap (M_\delta^\alpha \setminus M_{\delta/8}^\alpha) \cap L^2.$$

The same proof holds for (2.14)-(b).

Since (2.14) still holds if $\bar\varepsilon$ is decreased, we can assume

$$(2.17)\qquad \bar\varepsilon < \min\left\{\frac{b\delta}{8}, \sigma\right\},$$

where σ is the constant of corollary (2.8).

Moreover by condition (C) on L^α we have

$$\|f^{*\prime}(v)\|_2 > 0 \qquad \forall v \in (A_{c+\bar\varepsilon} \setminus A_{c-\bar\varepsilon}) \setminus M^\alpha_{\delta/2} \tag{2.18}$$

Now let $0 < \varepsilon < \bar\varepsilon$. As in theorem 1.3 of [3], we can define

$$\chi: L^2 \cap E \to [0, 1] \text{ s.t.}$$

$$\chi(v) = \begin{cases} 0 & \text{if } v \notin f^{*-1}([c-\bar\varepsilon, c+\bar\varepsilon]) \text{ or } v \in M^\alpha_{\delta/8} \\ 1 & \text{if } v \in f^{*-1}([c-\varepsilon, c+\varepsilon]) \setminus M^\alpha_{\delta/4} \end{cases} \tag{2.19}$$

and

$$V: L^2 \cap E \to L^2 \text{ s.t.}$$

$$V(v) = \begin{cases} -\chi(v)\Phi(v) & \text{if } v \in \tilde L^2 = \{v \in L^2 : f^{*\prime}(v) \neq 0\} \\ 0 & \text{otherwise} \end{cases} \tag{2.20}$$

where Φ is the «pseudogradient vector field» associated to f^* (cf. [3] and [14]).

By corollary (2.8) and (2.14)-(*b*) it follows that

$$\|V(v)\|_2 \leqslant K_1 + K_2 \|v\|_2 \tag{2.21}$$

where K_1, K_2 are positive constants independent of $v \in L^2$. Consider now the following initial value problem

$$\frac{d\eta}{dt} = V(\eta), \qquad \eta(0) = x \qquad x \in L^2 \cap E. \tag{2.22}$$

Since V is locally Lipschitz continuous, for any inizial value $x \in L^2 \cap E$ (2.22) possesses a unique solution $\eta(\cdot, x)$ which, by virtue of (2.21), is defined in $\mathbf{R}_+ = \{t \in \mathbf{R}: t \geqslant 0\}$. By (2.19)-(2.20), it is clear that $\eta(t, \cdot)$ satisfies ii), for any $t \in \mathbf{R}_+$. Arguing as in [3], it can been proved that there exists $\bar t$ s.t.

$$\eta(\bar t, A_{c+\varepsilon} \setminus V) \subset A_{c-\varepsilon}.$$

Then making a reparametrisation of the time t, one obtains the desired map η satisfying i).

We give now the proof of property iii). Denoting by $c(\alpha)$ the imbedding constant of L^2 into L^α, then by (2.21)-(2.22), for any $x \in L^2 \cap E$ we have

$$(2.23) \qquad \|\eta(t, x) - \eta(0, x)\|_\alpha \leqslant c(\alpha)\|\eta(t, x) - \eta(0, x)\|_2 = \\ = c(\alpha)\left\| \int_0^t V(\eta(\tau, x))\, d\tau \right\|_2 \leqslant (K_1 + K_2\|x\|_2)\, c(\alpha)\, t\,.$$

By (2.23), it follows that for any $v \in \bar{V}$, $w \in K_c$,

$$\|\eta(t, v) - w\|_\alpha \leqslant (K_1 + K_2\|v\|_2)\, c(\alpha)\, t + \delta\,.$$

Observe now that $\bar{V} \cap \mathring{A}_c$ is bounded in L^2 (for details we refer to the proof of theorem 2.27). Then the conclusion follows for $\bar{t}$ small enough.

We introduce now a variant of a definition given by Hofer in [11].

DEF. 2.24. Let $c \in \mathbf{R}$ and $v_0 \in K_c$. We say that v_0 is of almost mountain pass type (a. mp-type) in L^2 if for all open neighbourhood U of v_0 in $L^\alpha \cap E$ the topological space $U \cap L^2 \cap \mathring{A}_c$ is nonempty and not path-connected in $L^2 \cap E$.

Following [11], we shall state the existence of a critical point of f^* of a. mp-type in L^2.

THEOREM 2.25. Assume that there exist $\varrho, r > 0$ and $e_0 \in L^2 \cap E$ s.t. $f^*(e_0) < \varrho = \inf_{\|v\|_2 = r} f^*(v)$. Set

$$A = \{a\colon [0, 1] \to L^2 \cap E\colon\ a \text{ continuous},\ a(0) = 0,\ a(1) = e_0\}\,,$$

$$c = \inf_A \max f^*(|a|)\,, \qquad \text{where } |a| = a([0, 1])\,.$$

Then K_c is nonempty. If in addition the critical points in K_c are isolated in L^2, there exists $v_0 \in K_c$ of a. mp-type in L^2.

PROOF. As we have already noted, $K_c \neq \emptyset$ (we recall that the critical point of f^* belong to L^∞). Arguing indirectly, we may assume that K_c contains only a finite number of critical points all being not of a. mp-type in L^2. Let $K_c = \{v_1, \ldots, v_n\}$.

Then we find corresponding open neighbourhoods U_i of v_i in $L^\alpha \cap E$ s.t.

$$U = \bigcup_{i=1}^{n} U_i \supset K_c .$$

Define $\delta > 0$, $\bar{\varepsilon} > 0$, W and V by $\bar{\varepsilon} = \frac{1}{2}(c - d)$, where

$$d = \max\{f^*(0), f^*(e_0)\} ;$$

$$\delta = \tfrac{1}{8}\min\Big\{\mathrm{dist}_\alpha(\partial U \cup \{0, e_0\}, K_c),\ \inf\{\mathrm{dist}_\alpha(v_i, K_c \setminus \{v_i\}) : i = 1, \ldots n\}\Big\},$$

$$W = M^\alpha_{2\delta} \cap L^2 \quad \text{and} \quad V = M^\alpha_\delta \cap L^2 .$$

By lemma 2.13, we find $\varepsilon \in (0, \bar{\varepsilon})$ and $\sigma \in D(f^*)$ satysfying i)-iii). Choose $a \in A$ with $|a| \subset A_{c+\varepsilon}$. Note that

$$W = \Big(\bigcup_{i=1}^{n} W_i\Big) \cap L^2 \quad \text{and} \quad V = \Big(\bigcup_{i=1}^{n} V_i\Big) \cap L^2 ,$$

where W_i and V_i are open 2δ or δ-balls, respectively, around v_i in $L^\alpha \cap E$. Let

$$M = \big\{t \in [0, 1] : a(t) \notin V\big\} \quad \text{and} \quad \Gamma = (U \cap L^2 \cap \mathring{A}_c) \cup \sigma(1, a(M)) .$$

Observe that 0, $e_0 \in \Gamma$. Denote by $\tilde{\Gamma}$ the path-component of Γ in $L^2 \cap E$ containing 0. Arguing as in [11] it is possible to show that $e_0 \in \tilde{\Gamma} \subset \Gamma \subset \mathring{A}_c$, and this contradicts the definition of c.

Remark 2.26. Let us observe that theorem 1 of [11] assures that f^* has a critical point v_0 of mp-type in L^α, but we cannot prove that $m(v_0) \leqslant 1$, since L^α is not a Hilbert space. Moreover f^* does no verify condition (C) on L^2, and therefore we do not know if f^* has a critical point of mp-type in L^2. For this reason we have introduced the definition of critical point of a. mp-type in L^2.

Finally we shall prove the following:

Theorem 2.27. Let v_0 an isolated critical point of f^* of a. mp-type in L^2. Then $m(v_0) \leqslant 1$.

To prove this theorem, we need the following variant of the Morse lemma (cf. [10], [11]).

LEMMA 2.28. Let F be a real Hilbert space, U a nonempty open subset and $\Phi \in C^2(U, \mathbb{R})$ having a gradient of the form identity-compact. Suppose 0 is an isolated critical point of Φ with $\Phi(0) = 0$. Let $F = F^- \oplus F^0 \oplus F^+$ be the canonical decomposition associated to $\Phi''(0)$ via the spectral resolution.

Then there exist an origin-homeomorphism D defined on a 0-neighbourhood into F and an origin-preserving C^1-map β defined on a 0-neighbourhood in F^0 into $F^- \oplus F^+$ s.t.

$$\Phi(Du) = -\tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|z\|^2 + \Phi(\beta(y) + y)$$

for all $u = x + y + z$, $\|u\|$ small.

PROOF OF THEOREM 2.27. We may assume $v_0 = 0$. Then $c = f^*(0) = 0$. By lemma 2.28 f^* has the form

$$f^*(v) = -\tfrac{1}{2}\|x\|_2^2 + \tfrac{1}{2}\|z\|_2^2 + \psi(y)\,, \qquad \|v\|_2 \text{ small} \tag{2.28}$$

where $\psi(y) = f^*(\beta(y) + y)$ and 0 is an isolated critical point of ψ.

Let W be a ball of 0 in $L^\alpha \cap E$; then $W \cap L^2$ is a neighbourhood of 0 in $L^2 \cap E$ s.t. $W \cap L^2 = W^- \oplus W^+ \oplus W^0$, where $W^- = E^- \cap W$, $W^+ = E^+ \cap W$, $W^0 = E^0 \cap W$, and $L^2 \cap E = E^- \oplus E^0 \oplus E^+$.

Since $\dim E^- + \dim E^0 < +\infty$, if we choose W small enough, we have that W^- is a δ-ball around 0 in $L^2 \cap E$ and W^0 is a ball in $L^2 \cap E$ with

$$|\psi(y)| \leqslant \frac{\delta^2}{8} \qquad \forall y \in W_0\,. \tag{2.29}$$

Moreover decomposition (2.28) holds on $W^- \oplus W^0$. Obviously we may assume that $K_c \cap (\overline{W} \cap L^2) = \{0\}$. We shall prove that $\dim E^- \leqslant 1$. Namely, if we assume $\dim E^- \geqslant 2$, we will show that $W \cap L^2 \cap \dot{A}_0 = \Gamma$ is path-connected in $L^2 \cap E$, and this contradicts our assumption that 0 is of a. mp-type in L^2. Let $g, g' \in \Gamma$. We shall write $g \sim g'$ iff they are in the same path-component in $L^2 \cap E$. Let $g = x_1 + y_1 + z_1 \in \Gamma$. We shall find

$$g \sim g_1 = x_1 + y_1\,.$$

Namely if we consider the continuous map $h\colon [0, 1] \to L^2 \cap E$, $h(t) = x_1 + y_1 + tz_1$, it is obvious that $|h| \subset W \cap L^2 \cap E$. To prove that

$|h| \subset \mathring{A}_0$, it suffices to choose W small enough such that by Taylor's formula we have

$$f^*(g) = \tfrac{1}{2}[f^{*\prime\prime}(0)g, g] + o(\|g\|_\alpha^2) \qquad \forall g \in W .$$

Then if $g \in W \cap L^2$, there exist some positive constants λ_1, λ_2 s.t.

$$f^*(g) \geqslant \tfrac{1}{2}[\lambda_1\|g_+\|_2^2 - \lambda_2\|g_-\|_2^2] + o(\|g\|_\alpha^2) .$$

Since $g \in W \cap L^2 \cap \mathring{A}_0$ and W is bounded in L^α, it follows that $W^+ \cap \cap L^2 \cap \mathring{A}_0$ is bounded in L^2 and $W \cap L^2 \cap \mathring{A}_0$ is contained in the neighbourhood of 0 in $L^2 \cap E$ on which (2.28) holds.

Then $h(t) \in W \cap L^2 \cap \mathring{A}_0$ and $g \sim g_1$.

Now we can choose $x_2 \in W^- \cap L^2$ with $\|x_2\|_2 > \delta/2$ and

$$\|tx_2 + (1-t)x_1\|_2 \geqslant \|x_1\|_2 \qquad \forall t \in [0, 1] .$$

Since (2.28) holds on $(W^- \oplus W^0) \cap L^2$, it follows that $g_1 \sim g_2 = x_2 + y_1$. Finally by (2.29) $g_2 \sim g_3 = x_2$. Hence we have shown that for every $g \in \Gamma$ there exists $\tilde{g} \in \tilde{\Gamma} = \overline{W}\setminus\{0\}$, with $g \sim \tilde{g}$, provided $E^- \neq \{0\}$. If $\dim E^- \geqslant 2$, the set $\tilde{\Gamma}$ is path-connected, then Γ is path-connected in $L^2 \cap E$, which contradics the fact that 0 is of a. mp-type in L^2. Therefore $\dim E^- \leqslant 1$.

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