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Note on integral representation of holomorphic functions in several complex variables

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Note on Integral Representation of Holomorphic Functions in Several Complex Variables.

CHEN SHU-JIN (*)

1. Support function.

Integral representation of holomorphic functions of a single variable, i.e. Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)\, d\zeta}{\zeta - z},$$

has two remarkable properties. Firstly, the Cauchy kernel is holomorphic in $z \in D$ for fixed $\zeta \in \partial D$ and $\zeta - z \neq 0$, if $\zeta \neq z$; the Cauchy kernel $1/2\pi i(\zeta - z)$ does not depend on the shape of D, so it is universal, that is true for any domain D with a sufficiently nice boundary ∂D. Secondly, the value of holomorphic functions in domain D is defined by its whole boundary value. The situation in several variables is quite different.

Let D be a bounded domain with piecewise smooth boundary in $\mathbf{C}^n$. $\Phi(z, \zeta)$ is called a support function on D, if $\Phi(z, \zeta) \neq 0$ for any $\zeta \in \partial D$ and $z \in D$. Especially: 1) if it is a continuously differentiable funcation on D, then $\Phi(z, \zeta)$ is called a continuously differentiable support function, denoted by $\Phi(z, \zeta) \in B$; 2) if it is a holomorphic function of $z \in D$ for any fixed $\zeta \in \partial D$, and is continuously differentiable at $\zeta \in \partial D$ for any fixed $z \in D$, then $\Phi(z, \zeta)$ is called a

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holomorphic support function in z, denoted by $\Phi(z, \zeta) \in C$; 3) if it is a holomorphic function on D, then $\Phi(z, \zeta)$ is called a holomorphic support function, denoted by $\Phi(z, \zeta) \in A$. Evidently a function $\zeta - z$ is a holomorphic support function on any bounded domain D in $\mathbf{C}^1$.

In the establishing of integral representation of holomorphic functions in several complex variables, the key point is to find support functions with good properties and wide use. Naturally, one wishes to find support functions in $\mathbf{C}^n$ $(n \geqslant 2)$ with first property as in space $\mathbf{C}^1$, i.e. seek a holomorphic support functions with a universal property. But it is false in the several complex variables case. In fact, J. J. Kohn and L. Nirenberg [1] give an example of a pseudoconvex domain, which don't possess any holomorphic support function in z. On the other hand, the values of functions $f(z)$ of several complex variables in the domain D is sometimes determined by their values on partial boundary faces.

Integral representations of several complex variables may be divided into three kinds. 1) There is a continuous differentiable support function with a universal property, such that the integral representation of holomorphic functions with integral kernel is defined by these support functions, such as Cauchy-Fantappiè formula, Bochner-Martinelli formula. In this case the support function as well as the integral kernel does not depend on the shape of D. Although the integral representation has a universal property, the integral kernel is not holomorphic in z. 2) The domains with special boundary construction, such as convex domains [2], [3], strictly pseudoconvex domain [4], [5], [6], [7], classical domains [8], may use their boundary character to construct the holomorphic support functions in z. 3) On polyhedron domains in holomorphic domains of space $\mathbf{C}^n$ or Stein manifolds, there exists holomorphic support functions, for example integral representations of polycylindrical domains, analytic polyhedrons, or generalized polyhedron in [9], [10]. Moreover, values of holomorphic functions in these domains are defined by their partial boundary values.

2. Integral representation of holomorphic functions in several complex variables.

Now we obtain a general integral formula of holomorphic functions in several complex variables, such that it contains the above three kinds of integral representations.

Given a B-harmonic mapping on the domain of holomorphy Ω in space $\mathbf{C}^n$ or Stein manifold:

$$U^\alpha:\ \Omega \to \mathbf{R}^{m_\alpha}, \qquad \alpha = 1, 2, \dots, N, \qquad \sum_{\alpha=1}^{N} m_\alpha \geqslant n .$$

Let D_α be a bounded domain in space $\mathbf{R}^{m_\alpha}$ with piecewise C^1-boundary, denote $\delta_\alpha = \{z \in \Omega:\ u(z) \in D_\alpha\}$, $r_\alpha = \{z \in \Omega:\ u^\alpha(z) \in \partial D_\alpha\}$. The intersection $\bigcap_{\alpha=1}^{N} \delta_\alpha$ consists of a series of domains, let $\mathfrak{D}$ be one of the domains and $\overline{\mathfrak{D}} \subset \Omega$. Let $X^\alpha(z)$ be holomorphic functions and $\operatorname{Re} X^\alpha = U^\alpha$ or $\operatorname{Im} X^\alpha = U^\alpha$ on $\overline{\mathfrak{D}}$, then $F_\alpha(\zeta, z) \equiv X^\alpha(\zeta) - X^\alpha(z) \in A$. By Hefer theorem, there are holomorphic functions $\Phi_{\alpha_s}(\zeta, z)$ on domain of holomorphy $\Omega_\xi \times \Omega_z$ such that

$$F_\alpha(\xi, z) \equiv X^\alpha(\xi) - X^\alpha(z) = \sum_{S=1}^{n} (\xi_s - z_s) \Phi_{\alpha_s}(\zeta, z) . \tag{1}$$

Let $\tilde{N}_{pq} = \tilde{N}_{pq}(\zeta, z) \in C$, $\tilde{N}_r = (\tilde{N}'_r, \dots, \tilde{N}^m_r)^t$, here t denotes transpose. For positive integer r, let

$$\Delta^{(q-1)} = \{(\tau_1, \dots, \tau_r):\ \tau_1 + \dots + \tau_r = 1,\ \tau_r \geqslant 0\} \qquad q = 1, \dots, r .$$

LEMMA 1. Let D be a bounded domain in $\mathbf{C}^n$ with piecewise smooth boundary, and there are holomorphic support functions on z

$$\tilde{M}_r = \sum_{p=1}^{m} (\zeta_p - z_p) \tilde{N}^p_r = \sum_{q=1}^{r} \tau_q \sum_{p=1}^{m} (\zeta_p - z_p) \tilde{N}_{qp} \in C . \tag{2}$$

The boundary of the domain D consists of a chain of slit space, and this chain can be written as: 1) $\partial D = S^{(1)} \supset S^{(2)} \supset \dots S^{(r-1)} \supset S^{(r)}$, where $\partial S^{(\theta)} = S^{(\theta+1)}$, $\theta = 1, 2, \dots, r-1$; or 2) $\partial D = S^{(1)} \supset \dots \supset S^{(\eta)} \supset S^{(r)}$, where $S^{(r)}$ be slit of $S^{(\eta)}$, and dimension of $S^{(r)}$ may be greater than dimension of $S^{(\eta)}$ at least one. Let $S^{(r)}$ be a $(2m - r)$ dimensional boundary chain, i.e. there is a $(2m - r + 1)$ dimensional chain $\tilde{C}_1$, such that $\partial \tilde{C}_1 = S^{(r)}$, and when $\zeta \in D$,

$$\operatorname{rank} \frac{\partial(\tilde{N}'_r, \dots, \tilde{N}^m_r)}{\partial(\xi_1, \dots, \xi_m)} \leqslant m - r . \tag{3}$$

Then for a holomorphic function $f(z)$ in the closed domain D, when $z \in D$, we have

$$(4) \qquad f(z) = \frac{\varepsilon}{(2\pi i)^m} \int\limits_{S^{(r)} \times \tilde{\Delta}^{(r-1)}} \frac{f(\zeta)}{\tilde{M}_r^m} \det{}_{(m)} (\tilde{N}_r, \partial_{\bar{\zeta}\tau} \tilde{N}_r, \dots, \partial_{\bar{\zeta}\tau} \tilde{N}_r) \wedge \omega(\zeta) ,$$

where $S^{(r)}$ is a $(2m-r)$ dimensional slit, $\varepsilon = \pm 1$ denote orientation of boundary surface.

PROOF. 1) For convenience, let $\partial \tilde{\Delta}^{(\theta)} = \tilde{\Delta}^{(\theta-1)}$, where $\tilde{\Delta}^{(\theta-1)}$ is the total of $(\theta-1)$ dimensional simplex

$$\{(\tau_{j_1}, \dots, \tau_{j_\theta}) : \tau_{j_1} + \dots + \tau_{j_\theta} = 1, \tau_{j_1}, \dots, \tau_{j_\theta} \geqslant 0\}$$

$1 \leqslant j_1, \dots, j_\theta \leqslant r$. On $\tilde{\Delta}^{(\theta-1)}$ $(\theta = 1, \dots, r)$, let

$$\tilde{Q}_p(\zeta, z, \tau) = \frac{\tilde{N}_\theta^p(\zeta, z, \tau)}{\tilde{M}_\theta} = \sum_{\delta=1}^{\theta} \tau_\delta \frac{\tilde{N}_{p\delta}(\zeta, z)}{\tilde{M}_\theta} , \qquad p = 1, \dots, m;$$

$$\tilde{Q} = (\tilde{Q}_1, \dots, \tilde{Q}_m)^t = \left(\frac{\tilde{N}_\theta'}{\tilde{M}_\theta}, \dots, \frac{\tilde{N}_\theta^m}{\tilde{M}_\theta}\right)^t = \left(\sum_{\delta=1}^{\theta} \tau_{j_\delta} \tilde{Q}_1^\delta(\zeta, z), \dots, \sum_{\delta=1}^{\theta} \tau_{j_\delta} \tilde{Q}_m^\delta(\zeta, z)\right)^t ,$$

where $\tilde{Q}_p^\delta(\zeta, z) = \tilde{N}_{p\delta}(\zeta, z)/\tilde{M}_\theta$. Since

$$(5) \qquad \det{}_{(m)} (\tilde{Q}_1, \partial_{\bar{\zeta}\tau} \tilde{Q}, \dots, \partial_{\bar{\zeta}\tau} \tilde{Q}) = \frac{1}{\tilde{M}_\theta^m} \det{}_{(m)} (\tilde{N}_\theta, \partial_{\bar{\zeta}\tau} \tilde{N}_\theta, \dots, \partial_{\bar{\zeta}\tau} \tilde{N}_\theta) ,$$

we have

$$(6) \qquad \frac{\varepsilon}{(2\pi i)^m} \int\limits_{S^{(\theta)} \times \tilde{\Delta}^{(\theta-1)}} \frac{f(\zeta)}{\tilde{M}_\theta^m} \det{}_{(m)} (\tilde{N}_\theta, \partial_{\bar{\zeta}\tau} \tilde{N}_\theta, \dots, \partial_{\bar{\zeta}\tau} \tilde{N}_\theta) \wedge \omega(\zeta) =$$

$$= \frac{\varepsilon}{(2\pi i)^m} \int\limits_{S^{(\theta)} \times \tilde{\Delta}^{(\theta-1)}} f(\zeta) \det{}_{(m)} (\tilde{Q}, \partial_{\bar{\zeta}\tau} \tilde{Q}, \dots, \partial_{\bar{\zeta}\tau} \tilde{Q}) \wedge \omega(\zeta) .$$

On the other hand, since $\sum_{p=1}^{m} (\zeta_p - z_p) \tilde{Q}_p(\zeta, z, \tau) \equiv 1$ on $\Delta^{(\theta)}$, it follows $\sum_{p=1}^{m} (\zeta_p - z_p) \partial_{\bar{\zeta}\tau} \tilde{Q}_p(\zeta, z, \tau) = 0$. It is easily seen that

$$(7) \quad d[\det{}_{(m)} (\tilde{Q}, \partial_{\bar{\zeta}\tau} \tilde{Q}, \dots, \partial_{\bar{\zeta}\tau} \tilde{Q}) \wedge \omega(\zeta)] = \det{}_{(m)} (\partial_{\bar{\zeta}\tau} \tilde{Q}, \dots, \partial_{\bar{\zeta}\tau} \tilde{Q}) \wedge \omega(\zeta) = 0 ,$$

$$(8) \quad d[f(\zeta) \det{}_{(m)} (\tilde{Q}, \partial_{\bar{\zeta}\tau} \tilde{Q}, \dots, \partial_{\bar{\zeta}\tau} \tilde{Q}) \wedge \omega(\zeta)] = 0 .$$

And what is more

$$(9) \qquad \partial(S^{(\theta)}\times\tilde{\Delta}^{(\theta)}) = S^{(\theta+1)}\times\tilde{\Delta}^{(\theta)} + \varepsilon_1 S^{(\theta)}\times\tilde{\Delta}^{(\theta-1)}, \qquad \theta = 1, \dots, r-1\,.$$

From (6), (8) and (9), using Stokes theorem repeatedly we may obtain

$$\frac{\varepsilon}{(2\pi i)^m}\int_{S_1^{(r-1)}\times\tilde{\Delta}^{(r-1)}} \frac{f(\zeta)}{\tilde{M}_r^m}\det{}_{(m)}(\tilde{N}_r, \partial_{\bar{\zeta}\tau}\tilde{N}_r, \dots, \partial_{\bar{\zeta}\tau}\tilde{N}_r)\wedge\omega(\zeta) =$$

$$= \frac{\varepsilon_0}{(2\pi i)^m}\int_{\partial D}\frac{f(\zeta)}{\tilde{M}_1^m}\det{}_{(m)}(\tilde{N}_1, \partial_{\bar{\zeta}}\tilde{N}_1, \dots, \partial_{\bar{\zeta}}\tilde{N}_1)\wedge\omega(\zeta)\,,$$

where ε_0, $\varepsilon = \pm 1$ denote orientation of surface. Then formula (3) follows from Lemma 3 in [3].

2) Let

$$\tilde{C} = \partial(\tilde{C}_1\times\tilde{\Delta}^{(r-1)}) =$$

$$= \partial\tilde{C}_1\times\tilde{\Delta}^{(r-1)} + \varepsilon_2\tilde{C}_1\times\partial\tilde{\Delta}^{(r-1)} = S^{(r)}\times\tilde{\Delta}^{(r-1)} + \varepsilon_2\tilde{C}_1\times\partial\tilde{\Delta}^{(r-1)}$$

where $\varepsilon_2 = \pm 1$, then C is a $(2n-1)$ dimensional cycle.
On the $(r-1)$ dimensional simplex $\tilde{\Delta}^{(r-1)}$, we can write

$$\det{}_{(m)}(\tilde{Q}, \partial_{\bar{\zeta}\tau}\tilde{Q}, \dots, \partial_{\bar{\zeta}\tau}\tilde{Q}) = X_0(\zeta, \tau) + \dots + X_{r-1}(\zeta, \tau)\,,$$

where $X_\alpha(\zeta, \tau)$ are differential forms, the degree of $d\tau$ and $d\zeta_j$ is r and $m-r-1$ respectively. Thus

$$\int_{\tilde{C}_1\times\partial\tilde{\Delta}^{(r-1)}} f(\zeta)\det{}_{(m)}(\tilde{Q}, \partial_{\bar{\zeta}\tau}\tilde{Q}, \dots, \partial_{\bar{\zeta}\tau}\tilde{Q})\wedge\omega(\zeta) = 0\,.$$

Therefore we obtain

$$(10) \qquad \frac{\varepsilon}{(2\pi i)^m}\int_{S^{(r)}\times\tilde{\Delta}^{(r+1)}} f(\zeta)\det{}_{(m)}(\tilde{Q}, \partial_{\bar{\zeta}\tau}\tilde{Q}, \dots, \partial_{\bar{\zeta}\tau}\tilde{Q})\wedge\omega(\zeta) =$$

$$= \frac{\varepsilon}{(2\pi i)^m}\int_{C} f(\zeta)\det{}_{(m)}(\tilde{Q}, \partial_{\bar{\zeta}\tau}\tilde{Q}, \dots, \partial_{\bar{\zeta}\tau}\tilde{Q})\wedge\omega(\zeta)\,.$$

Applying (5) to left of (10), applying Cauchy-Fantappiè formula to right of (10), we obtain (4).

LEMMA 2. If $\mathfrak{D}$ is a bounded domain in the space $\mathbf{C}^n$, its boundary $\partial\mathfrak{D}$ consists of chain of slit space. If this chain can be written as:

$$1)\quad \partial\mathfrak{D} = \sigma^{(1)} \supset \sigma^{(2)} \supset \ldots \supset \sigma^{(\beta)} = \sigma_1^{(\beta-1)} \supset \sigma_2^{(\beta-1)} \supset \ldots \supset \sigma_{k+1}^{(\beta-1)},$$

where $\sigma_\nu^{(\beta-1)} = \bigcup\limits_{j_1,\ldots,j_\nu} \sigma_{j_1,\ldots,j_\nu}^{(\beta-1)}$, $\sigma_{j_1,\ldots,j_\nu}^{(\beta-1)}$ is of real dimension $2n-\beta-\nu+1$; or

$$2)\quad \partial\mathfrak{D} = \sigma^{(1)} \supset \ldots \supset \sigma^{(\eta)} \supset \sigma^{(\beta)} = \sigma_1^{(\beta-1)} \supset \sigma_2^{(\beta-1)} \supset \ldots \supset \sigma_{k+1}^{(\beta-1)},$$

where $\sigma^{(\beta)}$ is a $(2n-\beta)$ dimensional boundary chain, i.e. there is $(2n-\beta+1)$ dimensional chain C_1, such that $\partial C_1 = \sigma^{(\beta)}$. Assume that there are holomorphic support functions

$$F_{j_1}(\xi, z) = \sum_{p=1}^{n} (\xi_p - z_p)\varphi_{j_1 p}(\xi, z) \in A$$

on $\sigma_1^{(\beta-1)}$, denote $h_{j_1 p} = \varphi_{j_1 p}(\xi, z)/F_{j_1}(\xi, z)$. If there are continuously differentiable support functions

$$M_\beta = \sum_{p=1}^{n} (\xi_p - z_p) N_\beta^p = \sum_{q=1}^{\beta} \mu_q \sum_{p=1}^{n} (\xi_p - z_p) N_{pq} \in B,$$

where $N_\beta = (N_\beta', \ldots, N_\beta^n)^t$, which satisfy condition (3). Then for the holomorphic function $f(z)$ in a closed domain $\overline{\mathfrak{D}}$, when $z \in \mathfrak{D}$, we have

$$(11)\qquad f(z) = \frac{\varepsilon}{(2\pi i)^n} \sum_{j_1<\ldots<j_{k+1}} \sum_{\varrho=1}^{k+1} (-1)^{k+1-\varrho} \int\limits_{\Delta_{[\varrho]}^{(k-1)} \times \sigma_{j_1\ldots j_{k+1}}^{(\beta-1)} \times \Delta^{(\beta-1)}} \frac{f(\xi)}{M_\beta^{n-k}} \cdot$$

$$\cdot \det{}_{(n)} (H_{[j_\varrho]}^{(k)}, \overbrace{\partial_\lambda H_{[j_\varrho]}^{(k)}, \ldots, \partial_\lambda H_{[j_\varrho]}^{(k)}}^{k-1} N_\beta, \partial_{\bar\xi_\mu} N_\beta, \ldots, \partial_{\bar\xi_\mu} N_\beta) \wedge \omega(\xi) +$$

$$+ \frac{1}{(2\pi i)^n} \sum_{\alpha=1}^{k} \sum_{j_1<\ldots<j_k} \varepsilon_\alpha^0 \int\limits_{\Delta^{(\alpha-1)} \times \sigma_{j_1\ldots j_\alpha}^{(\beta-1)} \times \partial\Delta^{(\beta-1)}} \frac{f(\xi)}{M_{\beta-1}^{n-k}} \cdot$$

$$\cdot \det{}_{(n)} (H_j^{(\alpha)}, \overbrace{\partial_\lambda H_j^{(\alpha)}, \ldots, \partial_\lambda H_j^{(\alpha)}}^{\alpha-1}, N_{\beta-1}, \partial_{\bar\xi_\mu} N_{\beta-1}, \ldots, \partial_{\bar\xi_\mu} N_{\beta-1}) \wedge \omega(\xi).$$

where

$$\Lambda_{[\varrho]}^{(k-1)} = \{(\lambda_1, \dots, [\lambda_\varrho], \dots, \lambda_{k+1}) : \lambda_1 + \dots + [\lambda_\varrho] + \dots + \lambda_{k+1} = 1,$$
$$\lambda_1 \geqslant 0, \dots, [\lambda_\varrho] \geqslant 0, \dots, \lambda_{k+1} \geqslant 0\} ;$$

$$\Lambda^{(\alpha-1)} = \{(\lambda_1, \lambda_2, \dots, \lambda_\alpha) | \lambda_1 + \lambda_2 + \dots + \lambda_\alpha = 1, \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \dots, \lambda_\alpha \geqslant 0\},$$

$$H_{[j_\varrho]}^{(k)} = (H_{[j_\varrho]1}^{(k)}, H_{[j_\varrho]2}^{(k)}, \dots, H_{[j_\varrho]n}^{(k)}),$$

$$H_{[j_\varrho]p}^{(k)} = h_{j_1 p}\lambda_1 + \dots + [h_{j_\varrho p}, \lambda_p] + \dots + h_{j_{k+1}p}\lambda_{k+1},$$

$$H_j^{(\alpha)} = (H_{j1}^{(\alpha)}, H_{j2}^{(\alpha)}, \dots, H_{jn}^{(\alpha)})$$

$$H_{jp}^{(\alpha)} = h_{j_1 p}\lambda_1 + h_{j_2 p}\lambda_2 + \dots + h_{j_\alpha p}\lambda_\alpha,$$

$$h_{j_\iota p} = h_{j_\iota p}(\xi, z) = \varphi_{j_\iota p}(\xi, z)/F_{j_\iota}(\xi, z).$$

PROOF. Since

$$(12)\quad \det_{(n)}(H_{[j_\varrho]}^{(k)}, \partial_\lambda H_{[j_\varrho]}^{(k)}, \dots, \partial_\lambda H_{[j_\varrho]}^{(k)}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q) \wedge \omega(\xi) =$$
$$= (k-1)! \sum_{r=1\,(r\neq\varrho)}^{k+1} (-1)^{r-1}\lambda_r\, d\lambda_{[r]} \cdot$$
$$\cdot \det_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_{k+1}}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q) \wedge \omega(\xi),$$

$$(13)\quad \det_{(n)}(H_j^{(\alpha)}, \overbrace{\partial_\lambda H_j^{(\alpha)}, \dots, \partial_\lambda H_j^{(\alpha)}}^{a-1}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q) \wedge \omega(\xi) =$$
$$= (a-1)! \sum_{\nu=1}^{a} (-1)^{\nu-1}\lambda_\nu\, d\lambda_{[\nu]} \det_{(n)}(h_{j_1}, \dots, h_{j_\alpha}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q) \wedge \omega(\xi),$$

$$(14)\quad \therefore \frac{\varepsilon}{(2\pi i)^n} \sum_{j_1<\dots<j_{k+1}} \sum_{\varrho=1}^{k+1} (-1)^{k+1-\varrho} \int_{\Lambda_{[\varrho]}^{(k-1)} \times \sigma_{j_1\cdots j_{k+1}}^{(\beta-1)} \times \Delta^{(\beta-1)}} f(\xi) \cdot$$
$$\cdot \det_{(n)}(H_{[j_\varrho]}^{(k)}, \overbrace{\partial_\lambda H_{[j_\varrho]}^{(k)}, \dots, \partial_\lambda H_{[j_\varrho]}^{(k)}}^{(k-1)}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\xi_\mu} Q) \wedge \omega(\xi) +$$
$$+ \frac{1}{(2\pi i)^n} \sum_{\alpha=1}^{k} \sum_{j_1<\dots<j_\alpha} \varepsilon_\alpha^0 \int_{\Lambda^{(\alpha-1)} \times \sigma_{j_1\cdots j_\alpha}^{(\beta-1)} \times \partial\Delta^{(\beta-1)}} f(\xi) \cdot$$
$$\cdot \det_{(n)}(H_j^{(\alpha)}, \overbrace{\partial_\lambda H_j^{(\alpha)}, \dots, \partial_\lambda H_j^{(\alpha)}}^{\alpha-1}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q) \wedge \omega(\xi) =$$
$$= \frac{\varepsilon}{(2\pi i)^n} \sum_{j_1<\dots<j_{k+1}} \int_{\sigma_{j_1\cdots j_{k+1}}^{(\beta-1)} \times \Delta^{(\beta-1)}} f(\xi) \sum_{\varrho=1}^{k+1} (-1)^{k+1-\varrho} \cdot$$
$$\cdot \det_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_{k+1}}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q) \wedge \omega(\xi) + \frac{1}{(2\pi i)^n} \cdot$$
$$\cdot \sum_{\alpha=1}^{k} \varepsilon_\alpha^0 \sum_{j_1<\dots<j_\alpha} \int_{\sigma_{j_1\cdots j_\alpha}^{(\beta-1)} \times \partial\Delta^{(\beta-1)}} f(\xi) \det_{(n)}(h_{j_1}, \dots, h_{j_\alpha}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q) \wedge \omega(\xi).$$

By a similar computation as in proving (4), we may obtain

$$(15)\qquad \frac{1}{M_\beta^{n-k}}\det{}_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_{k+1}}, N_\beta, \partial_{\bar\xi_\mu} N_\beta, \dots, \partial_{\bar\xi_\mu} N_\beta) = \\ = \det{}_{(n)}(h_{j_1}, \dots, [h_{j_\nu}], \dots, h_{j_{k+1}}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q)\,.$$

By Stokes theorem, and substituting (15) into (14), it follows from Lemma 1, that

$$(16)\quad \left\{\begin{aligned}
&\sum_{\varrho=1}^{k+1}\sum_{j_1,\dots,[j_\varrho],\dots,j_{k+1}}\Bigg[\sum_{j_\varrho}\int_{\sigma^{(\beta-1)}_{j_1\dots j_{k+1}}\Delta^{(\beta-1)}}(-1)^{k+1-\varrho}\frac{f(\xi)}{M_\beta^{n-k}}\cdot\\
&\cdot\det{}_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_{k+1}}, N_\beta, \partial_{\bar\xi_\mu} N_\beta, \dots, \partial_{\bar\xi_\mu} N_\beta)\wedge\omega(\xi) +\\
&+\varepsilon_1^0\int_{\sigma^{(\beta-1)}_{j_1\dots[j_\varrho]\dots j_{k+1}}\times\partial\Delta^{(\beta-1)}}\frac{f(\xi)}{M_{\beta-1}^{n-k}}\cdot\\
&\cdot\det{}_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_{k+1}}, N_{\beta-1}, \partial_{\bar\xi_\mu} N_{\beta-1}, \dots, \partial_{\bar\xi_\mu} N_{\beta-1})\wedge\\
&\wedge\omega(\xi)\Bigg] = (k+1)\sum_{j_1,\dots,j_k}\int_{\sigma^{(\beta-1)}_{j_1\dots j_k}\times\Delta^{(\beta-1)}}\frac{f(\xi)}{M_\beta^{n-k+1}}\sum_{\varrho=1}^{k}(-1)^{k-\varrho}\cdot\\
&\qquad\cdot\det{}_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_k}, N_\beta, \partial_{\bar\xi_\mu} N_\beta, \dots, \partial_{\bar\xi_\mu} N_\beta)\wedge\omega(\xi)\,.\\
&\because\ \sum_{j_1,\dots,j_{k+1}}\int_{\sigma^{(\beta-1)}_{j_1\dots j_{k+1}}\times\Delta^{(\beta-1)}} f(\xi)\sum_{\varrho=1}^{k+1}(-1)^{k+1-\varrho}\cdot\\
&\cdot\det{}_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_{k+1}}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q)\wedge\omega(\xi) =\\
&=\sum_{\varrho-1}^{k+1}\sum_{j_1,\dots,[j_\varrho],\dots,j_{k+1}}\sum_{j_\varrho}\int_{\sigma^{(\beta-1)}_{j_1\dots j_{k+1}}\times\Delta^{(\beta-1)}}(-1)^{k+1-\varrho}f(\xi)\det{}_{(n)}\cdot\\
&\qquad\cdot(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_{k+1}}, Q, \partial_{\bar\xi_\mu} Q, \dots, \partial_{\bar\xi_\mu} Q)\wedge\omega(\xi)\,,\\
&\therefore\ \frac{\varepsilon}{(2\pi i)^n}\sum_{j_1,\dots,j_{k+1}}\int_{\sigma^{(\beta-1)}_{j_1\dots j_{k+1}}\times\Delta^{(\beta-1)}}\frac{f(\xi)}{M_\beta^{n-k}}\sum_{\varrho=1}^{k+1}(-1)^{k+1-\varrho}\cdot\\
&\cdot\det{}_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_{k+1}}, N_\beta, \partial_{\bar\xi_\mu} N_\beta, \dots, \partial_{\bar\xi_\mu} N_\beta)\wedge\omega(\xi) +\\
&+\frac{(k+1)}{(2\pi i)^n}\varepsilon_1^0\sum_{j_1,\dots,j_k}\int_{\sigma^{(\beta-1)}_{j_1\dots j_k}\times\partial\Delta^{(\beta-1)}}\frac{f(\xi)}{M_{\beta-1}^{n-k}}\cdot\\
&\cdot\det{}_{(n)}(h_{j_1}, \dots, h_{j_k}, N_{\beta-1}, \partial_{\bar\xi_\mu} N_{\beta-1}, \dots, \partial_{\bar\xi_\mu} N_{\beta-1})\wedge\omega(\xi) =\\
&=\frac{k+1}{(2\pi i)^n}\sum_{j_1,\dots,j_k}\int_{\sigma^{(\beta-1)}_{j_1\dots j_k}\times\Delta^{(\beta-1)}}\frac{f(\xi)}{M_\beta^{n-k+1}}\sum_{\varrho=1}^{k}(-1)^{k-\varrho}\cdot\\
&\qquad\cdot\det{}_{(n)}(h_{j_1}, \dots, [h_{j_\varrho}], \dots, h_{j_k}, N_\beta, \partial_{\bar\xi_\mu} N_\beta, \dots, \partial_{\bar\xi_\mu} N_\beta)\wedge\omega(\xi)\,,
\end{aligned}\right.$$

Hence

$$(17)\qquad \frac{\varepsilon}{(2\pi i)^n}\sum_{j_1<\dots<j_{k+1}}\int_{\sigma^{(\beta-1)}_{j_1\dots j_{k+1}}\times\Delta^{(\beta-1)}}\frac{f(\xi)}{M_\beta^{n-k}}\sum_{\varrho=1}^{k}(-1)^{k+1-\varrho}\cdot$$

$$\cdot\det{}_{(n)}(h_{j_1},\dots,[h_{j_\varrho}],\dots,h_{j_{k+1}},N_\beta,\partial_{\bar\xi_\mu}N_\beta,\dots,\partial_{\bar\xi_\mu}N_\beta)\wedge\omega(\xi)+$$

$$+\frac{\varepsilon_1^0}{(2\pi i)^n}\sum_{j_1<\dots<j_k}\int_{\sigma^{(\beta-1)}_{j_1\dots j_k}\times\partial\Delta^{(\beta-1)}}\frac{f(\xi)}{M_{\beta-1}^{n-k}}\cdot$$

$$\cdot\det{}_{(n)}(h_{j_1},\dots,h_{j_k},N_{\beta-1},\partial_{\bar\xi_\mu}N_{\beta-1},\dots,\partial_{\bar\xi_\mu}N_{\beta-1})\wedge\omega(\xi)=$$

$$=\frac{\varepsilon}{(2\pi i)^n}\sum_{j_1<\dots<j_k}\int_{\sigma^{(\beta-1)}_{j_1\dots j_k}\times\Delta^{(\beta-1)}}\frac{f(\xi)}{M_\beta^{n-k+1}}\sum_{\varrho=1}^{k}(-1)^{k-\varrho}\cdot$$

$$\cdot\det{}_{(n)}(h_{j_1},\dots,[h_{j_\varrho}],\dots,h_{j_k},N_\beta,\partial_{\bar\xi_\mu}N_\beta,\dots,\partial_{\bar\xi_\mu}N_\beta)\wedge\omega(\xi)\,.$$

Using (17) repeatedly, we obtain

$$(18)\qquad \frac{\varepsilon}{(2\pi i)^n}\sum_{j_1<\dots<j_{k+1}}\int_{\sigma^{(\beta-1)}_{j_1\dots j_{k+1}}\times\Delta^{(\beta-1)}}\frac{f(\xi)}{M_\beta^{n-k}}\sum_{\varrho=1}^{k}(-1)^{k+1-\varrho}\cdot$$

$$\cdot\det{}_{(n)}(h_{j_1},\dots,[h_{j_\varrho}],\dots,h_{j_{k+1}},N_\beta,\partial_{\bar\xi_\mu}N_\beta,\dots,\partial_{\bar\xi_\mu}N_\beta)\wedge\omega(\xi)+$$

$$+\frac{1}{(2\pi i)^n}\sum_{\alpha=1}^{k}\sum_{j_1<\dots<j_\alpha}\varepsilon_\alpha^0\int_{\sigma^{(\beta-1)}_{j_1\dots j_\alpha}\times\partial\Delta^{(\beta-1)}}\frac{f(\xi)}{M_{\beta-1}^{n}}\cdot$$

$$\cdot\det{}_{(n)}(h_{j_1},\dots,h_{j_\alpha},N_{\beta-1},\partial_{\bar\xi_\mu}N_{\beta-1},\dots,\partial_{\bar\xi_\mu}N_{\beta-1})\wedge\omega(\xi)=$$

$$=\frac{\varepsilon}{(2\pi i)^n}\int_{\sigma_1^{(\beta-1)}\times\Delta^{(\beta-1)}}\frac{f(\xi)}{M_\beta^n}\det{}_{(n)}(N_\beta,\partial_{\bar\xi_\mu}N_\beta,\dots,\partial_{\bar\xi_\mu}N_\beta)\wedge\omega(\xi)\,,$$

where ε, $\varepsilon_\alpha^0=\pm1$ denote the orientation of boundary surface and the lower dimensional boundary surface respectively. By (16), we may write (14) again as

$$(19)\qquad \frac{\varepsilon}{(2\pi i)^n}\sum_{j_1<\dots<j_{k+1}}\sum_{\varrho=1}^{k+1}(-1)^{k+1-\varrho}\int_{\Delta^{(k-1)}_{[\varrho]}\times\sigma^{(\beta-1)}_{j_1\dots j_{k+1}}\times\Delta^{(\beta-1)}}\frac{f(\xi)}{M_\beta^{n-k}}\cdot$$

$$\cdot \det_{(n)}(H^{(k)}_{[j_\varrho]}, \overbrace{\partial_\lambda H^{(k)}_{[j_\varrho]}, \ldots, \partial_\beta H^{(k)}_{[j_\varrho]}}^{(k-1)}, N_\beta, \partial_{\bar\xi_\mu} N_\beta, \ldots, \partial_{\bar\xi_\mu} N_\beta) \wedge \omega(\xi) +$$

$$+ \frac{1}{(2\pi i)^n} \sum_{\alpha=1}^{k} \sum_{j_1<\ldots<j_\alpha} \varepsilon^0_\alpha \int\limits_{\Delta^{(\alpha-1)} \times \sigma^{(\beta-1)}_{j_1\ldots j_\alpha} \times \partial\Delta^{(\beta-1)}} \frac{f(\xi)}{M^{n-\alpha}_{\beta-1}} \cdot$$

$$\cdot \det_{(n)}(H^{(\alpha)}_j, \overbrace{\partial_\lambda H^{(\alpha)}_j, \ldots, \partial_\lambda H^{(\alpha)}_j}^{\alpha-1}, N_{\beta-1}, \partial_{\bar\xi_\mu} N_{\beta-1}, \ldots, \partial_{\bar\xi_\mu} N_{\beta-1}) \wedge \omega(\xi) =$$

$$= \frac{\varepsilon}{(2\pi i)^n} \sum_{j_1<\ldots<j_{k+1}} \int\limits_{\sigma^{(\beta-1)}_{j_1\ldots j_{k+1}} \times \Delta^{(\beta-1)}} \frac{f(\xi)}{M^{n-k}_\beta} \sum_{\varrho=1}^{k+1} (-1)^{k+1-\varrho} \cdot$$

$$\cdot \det_{(n)}(h_{j_1}, \ldots, [h_{j_\varrho}], \ldots, h_{j_{k+1}}, N_\beta, \partial_{\bar\xi_\mu} N_\beta, \ldots, \partial_{\bar\xi_\mu} N_\beta) \wedge \omega(\xi) +$$

$$+ \frac{1}{(2\pi i)^n} \sum_{\alpha=1}^{k} \sum_{j_1<\ldots<j_\alpha} \varepsilon^0_\alpha \int\limits_{\sigma^{(\beta-1)}_{j_1\ldots j_\alpha} \times \partial\Delta^{(\beta-1)}} \frac{f(\xi)}{M^{n-\alpha}_{\beta-1}} \cdot$$

$$\cdot \det_{(n)}(h_{j_1}, \ldots, h_{j_\alpha}, N_{\beta-1}, \partial_{\bar\xi_\mu} N_{\beta-1}, \ldots, \partial_{\bar\xi_\mu} N_{\beta-1}) \wedge \omega(\xi) .$$

Then Lemma 2 is obtained by (19), (18) and Lemma 1.

From Lemma 1 and Lemma 2, we have the following main theorem.

THEOREM. Let D be a bounded domain as defined in Lemma 1, $\mathfrak{D}$ a bounded domain as in Lemma 2. Then for the holomorphic function $F(z, w)$ in the closed domain $\bar{D} \times \bar{\mathfrak{D}}$, when $z \in D$, $w \in \mathfrak{D}$, $\zeta \in \partial D$, $\zeta \in \partial\mathfrak{D}$, we have

$$(20) \qquad F(z, w) = \frac{\varepsilon}{(2\pi i)^m} \int\limits_{S^{(r)} \times \tilde\Delta^{(r-1)}} \frac{1}{\tilde M^m_r} \det_{(m)} (\tilde N_r, \partial_{\bar\zeta_\tau} \tilde N_r), \ldots, \partial_{\bar\zeta_\tau} \tilde N_r) \wedge \omega(\zeta) \cdot$$

$$\cdot \left[\frac{\varepsilon_1}{(2\pi i)^n} \sum_{j_1<\ldots<j_{k+1}} \sum_{\varrho=1}^{k+1} (-1)^{k+1-\varrho} \int\limits_{\Delta^{(k-1)}_{[\varrho]} \times \sigma^{(\beta-1)}_{j_1\ldots j_{k+1}} \times \Delta^{(\beta-1)}} \frac{F(\xi, \zeta)}{M^{n-k}_\beta} \cdot \right.$$

$$\cdot \det_{(n)} (H^{(k)}_{[j_\varrho]}, \partial_\lambda H^{(k)}_{[j_\varrho]}, \ldots, \partial_\lambda H^{(k)}_{[j_\varrho]}, N_\beta, \partial_{\bar\xi_\mu} N_\beta, \ldots, \partial_{\bar\xi_\mu} N_\beta) \wedge \omega(\xi) +$$

$$+ \frac{1}{(2\pi i)^n} \sum_{\alpha=1}^{k} \sum_{j_1<\ldots<j_\alpha} \varepsilon^0_\alpha \int\limits_{\Delta^{(\alpha-1)} \times \sigma^{(\beta-1)}_{j_1\ldots j_\alpha} \times \partial\Delta^{(\beta-1)}} \frac{F(\xi, \zeta)}{M^{n-\alpha}_{\beta-1}} \cdot$$

$$\left. \cdot \det_{(n)} (H^{(\alpha)}_j, \overbrace{\partial_\lambda H^{(\alpha)}_j, \ldots, \partial_\lambda H^{(\alpha)}_j}^{\alpha-1}, N_{\beta-1}, \partial_{\bar\xi_\mu} N_{\beta-1}), \ldots, \partial_{\bar\xi_\mu} N_{\beta-1}) \wedge \omega(\xi) \right],$$

COROLLARY 1. When $F(z, w) = f(z)$ on $\bar{D} \times \bar{\mathfrak{D}}$, where D is a convex domain, or a domain with piecewise smooth strictly pseudoconvex boundaries, or classical domains, then from (20) we obtain the integral representation of second kind in § 1.

COROLLARY 2. When $F(z, w) = f(w)$ on $\bar{D} \times \bar{\mathfrak{D}}$, where $\mathfrak{D}$ is a polyhedron defined by B-harmonic mapping (specially holomorphic mapping) as in the begining of this section, then from (20) we obtain integral representation of third kind in § 1.

COROLLARY 3. When $F(z, w) = f(w)$ on $\bar{D} \times \bar{\mathfrak{D}}$, and $\beta = 1$, $k = 0$, then from (20) we obtain integral representation of first kind in § 1.

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