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Mauro Fabrizio

Angelo Morro

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Fading Memory Spaces and Approximate Cycles in Linear Viscoelasticity.

MAURO FABRIZIO - ANGELO MORRO (*)

SUMMARY - The theory of linear isothermal viscoelasticity is revisited within the context of materials with fading memory. The compatibility with thermodynamics is investigated by considering approximate cycles in a very general sense. As a result, necessary and sufficient conditions, on the relaxation function, for the validity of the second law are derived which emphasize the role of sinusoidal histories.

1. Introduction.

When dealing with mathematical problems in linear viscoelasticity, any information on the relaxation function makes the developments handier and leads to more detailed conclusions. This is the essential motivation for a renewed interest into the thermodynamic restrictions on the relaxation function (see, e.g., [1, 2]) and references therein. It is the purpose of this paper to develop a general thermodynamic analysis of linear viscoelasticity and provide a necessary and sufficient set of conditions, on the relaxation function, for compatibility with thermodynamics.

In linear viscoelasticity, when the reference state is stress-free, the

(*) Indirizzo degli AA.: M. FABRIZIO: Dipartimento di Matematica, Piazza di Porta S. Donato 5, 40127 Bologna (Italy); A. MORRO: DIBE - Università, Viale Causa 13, 16145 Genova (Italy).

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relevant constitutive equation takes the form

$$T(t) = G(0)E(t) + \int_0^\infty G'(s)E(t-s)\,ds \tag{1.1}$$

where T is the Cauchy stress tensor, E is the infinitesimal strain tensor, and G (with $G' = dG/ds$) is a function mapping R^+ into the space of linear transformations of symmetric tensors into symmetric tensors. The function G, called *relaxation function*, characterizes the material under consideration. Constitutive equations of the form (1.1) were introduced by Boltzmann [3], were studied at length by Volterra [4], and have given rise to a large and growing literature (1).

The principle of fading memory employed by Coleman and Noll [9] furnishes a general hypothesis of smoothness for non-linear simple materials with memory that yields (1.1) as a complete first-order approximation for infinitesimal deformations from a stress-free equilibrium state. In his development of the thermodynamics of (non-linear) materials obeying the principle of fading memory, Coleman [10, 11] showed that, because of the second law, $G(0)$ and $G(\infty)$ (with $G(\infty) = \lim_{s\to\infty} G(s)$) must be symmetric and such that $G(0) - G(\infty)$ is positive semidefinite, namely

$$G(0) = G^T(0)\,, \tag{1.2a}$$

$$G(\infty) = G^T(\infty)\,, \tag{1.2b}$$

and

$$E\cdot[G(0) - G(\infty)]E \geqslant 0 \tag{1.2c}$$

for all symmetric tensors E (2). Later, Day [12] showed that the second law also implies that, for all $s \in R^+$, $G(0) - G(s)$ is positive semidefinite, namely

$$E\cdot[G(0) - G(s)]E \geqslant 0 \tag{1.3}$$

for all symmetric tensors E.

(1) See, e.g., Graffi [5], Bland [6], Gurtin & Sternberg [7], and Leitman & Fisher [8].

(2) Here G^T is the transpose of G, namely $E_1\cdot G^T E_2 = E_2\cdot G E_1$ for all symmetric tensors E_1, E_2, while $E_1\cdot E_2 = \text{trace}\,(E_1, E_2^T)$.

Long before the modern theory of thermodynamics of materials with fading memory was formulated, Graffi [13], while investigating one-dimensional deformations and using a principle of positive work for cycles, obtained a result which, in tensor form, can be stated as follows. If the fourth-order tensor $\boldsymbol{G}(s)$ is symmetric for all s then the second law requires that the sine-transform $\hat{\boldsymbol{G}}_s(\omega)$, defined by

$$(1.4) \qquad \hat{\boldsymbol{G}}'_s(\omega) = \int_0^\infty \boldsymbol{G}'(s) \sin(\omega s)\, ds\,,$$

be negative semidefinite, i.e.

$$(1.5) \qquad \boldsymbol{E}\cdot\hat{\boldsymbol{G}}'_s(\omega)\boldsymbol{E} \leqslant 0$$

for every $\omega \in R^+$ and all symmetric tensors $\boldsymbol{E}$. In this paper we show that the validity of the condition

$$(1.6) \qquad \boldsymbol{E}_1\cdot[\boldsymbol{G}^T(0) - \boldsymbol{G}(0)]\boldsymbol{E}_2 \int_0^\infty [\boldsymbol{E}_1\cdot\boldsymbol{G}'(s)\boldsymbol{E}_1 + \boldsymbol{E}_2\cdot\boldsymbol{G}'(s)\boldsymbol{E}_2] \sin(\omega s)\, ds - \\ - \int_0^\infty \boldsymbol{E}_1\cdot[\boldsymbol{G}'(s) - \boldsymbol{G}'^T(s)]\boldsymbol{E}_2 \cos(\omega s)\, ds \geqslant 0\,,$$

for all symmetric tensors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, is necessary and sufficient for the validity of the second law of thermodynamics. When the material symmetry implies that $\boldsymbol{G}(s) = \boldsymbol{G}^T(s)$ for all s (3), the relation (1.6) is equivalent to (1.2a) and (1.5): in such a case (1.2a) and (1.5) are also sufficient for compatibility of the relaxation function with thermodynamics. This emphasizes the importance of the inequality (1.5) for $\hat{\boldsymbol{G}}'_s$ (4).

To get the aforementioned result we develop an approach which can be summarized as follows. In section 2 we define a topology for the state space Σ; such a topology is similar to that introduced by Coleman and Mizel [14] and used by Coleman and Owen [15]. On

(3) This occurs, for example, when the material is isotropic.

(4) Its opposite, $-\hat{\boldsymbol{G}}'_s$, is usually called the *loss modulus*.

the basis of this topology we introduce the concept of approximate cycle. Moreover, in section 3, we show that the set of periodic histories is dense in Σ, relative to the topology at hand. In section 4 we derive the work inequality induced by the second law of thermodynamics on approximate cycles (in linear isothermal viscoelasticity). Finally, in section 5, we prove that the relation (1.6) is necessary and sufficient for the validity of the second law (work inequality).

The results of this paper generalize those exhibited in [16] in that they are based on the more general notion of approximation cycle as stated in [15, 17].

2. Fading memory space and approximate cycles.

As usual, by a history we mean a function defined on R^+ and by a past history a function defined on R^{++}. Given any function φ on R and a time $t \in R$, we define the history φ^t of φ up to time t as

$$\varphi^t(s) = \varphi(t - s) , \qquad s \in R^+ ;$$

the restriction of φ^t to R^{++} provides the past history ${}_r\varphi^t$ of φ up to time t.

Since we are dealing with linear viscoelastic materials we consider histories of the infinitesimal-strain tensor $\boldsymbol{E}$; their values are in the set $\mathfrak{I}_{sym}$ of symmetric second-order tensors. The relaxation function $\boldsymbol{G}$ is assumed to be a C^1 map from R^+ into the set $[\mathfrak{I}_{sym}]$ of endomorphysms of $\mathfrak{I}_{sym}$. The equilibrium elastic modulus $\boldsymbol{G}(\infty)$ is taken to be positive definite, namely $\boldsymbol{E}\cdot\boldsymbol{G}(\infty)\boldsymbol{E} > 0$ for each nonzero $\boldsymbol{E} \in \mathfrak{I}_{sym}$.

The state σ (at time t) of a linear viscoelastic material is given by the history $\boldsymbol{E}^t$, namely the pair $(\boldsymbol{E}(t), {}_r\boldsymbol{E}^t)$ of the present value $\boldsymbol{E}(t)$ and the past history ${}_r\boldsymbol{E}^t$. The set of states σ is denoted by Σ. To accomplish our procedure we need a topology for Σ. A standard way of specifying the topology is through a norm involving an influence function h.

CONSTITUTIVE HYPOTHESIS. *The relaxation function* $\boldsymbol{G}$ *is such that there is a scalar monotone-decreasing function* h, *mapping* R^+ *into* R^+, *with*

$$(2.1) \qquad \int_0^\infty |\boldsymbol{G}'(s)|^2 h^{-1}(s)\, ds < \infty$$

and, for any $\tau \in R^+$,

$$h(s+\tau) < \frac{\gamma}{(1+\tau)^{2+\varkappa}}\, h(s)\,, \qquad \gamma, \varkappa \in R^+\,. \tag{2.2}$$

Let H be the Banach space formed from the set of past histories ${}_r\boldsymbol{E}^t$ defined as

$$H = \left\{ {}_r\boldsymbol{E}^t,\ R^{++} \to \mathfrak{J}_{\text{sym}},\ \int_0^\infty |{}_r\boldsymbol{E}^t|^2\, h(s)\, ds < \infty \right\};$$

the norm of a past history ${}_r\boldsymbol{E}^t$ is then defined by

$$\|{}_r\boldsymbol{E}^t\|^2 = \int_0^\infty |{}_r\boldsymbol{E}^t(s)|^2\, h(s)\, ds\,.$$

The Banach space H becomes a Hilbert space as soon as we introduce the inner product of two past histories ${}_r\boldsymbol{E}_1^{t_1}$, ${}_r\boldsymbol{E}_2^{t_2}$ as

$$({}_r\boldsymbol{E}_1^{t_1}, {}_r\boldsymbol{E}_2^{t_2})_r = \int_0^\infty {}_r\boldsymbol{E}_1^{t_1}(s)\cdot {}_r\boldsymbol{E}_2^{t_2}(s)\, h(s)\, ds\,.$$

The analogous inner product for histories, namely,

$$(\boldsymbol{E}_1^{t_1}, \boldsymbol{E}_2^{t_2}) = \boldsymbol{E}_1(t_1)\cdot \boldsymbol{E}_2(t_2) + \int_0^\infty \boldsymbol{E}_1^{t_1}(s)\cdot \boldsymbol{E}_2^{t_2}(s)\, h(s)\, ds\,, \tag{2.3}$$

ascribes to the set Σ of states σ with finite norm,

$$\|\sigma\| = \left(|\boldsymbol{E}(t)|^2 + \int_0^\infty |\boldsymbol{E}^t(s)|^2\, h(s)\, ds \right)^{\frac{1}{2}}, \tag{2.4}$$

the structure of a Hilbert space called fading memory space of (total) histories.

The definition of Σ leads at once to the following statements [14].

(i) If $\boldsymbol{E}^t$ is in Σ then, for each $\tau > 0$, the static continuation of $\boldsymbol{E}^t$, defined by

$$\boldsymbol{E}^{t+\tau}(s) = \begin{cases} \boldsymbol{E}(t)\,, & s \in [0, \tau]\,, \\ \boldsymbol{E}^t(s-\tau)\,, & s \in (\tau, \infty)\,, \end{cases}$$

is in Σ.

(ii) The distance between the static continuation $\boldsymbol{E}^{t+\tau}$ of $\boldsymbol{E}^t$ and the constant history $\boldsymbol{E}^t(s) = \boldsymbol{E}(t)$ tends to zero, i.e.,

$$\lim_{\tau\to\infty} \|\boldsymbol{E}^{t+\tau} - \boldsymbol{E}(t)\| = 0\,.$$

The assumption (2.1) implies the continuity of the functional $\hat{\boldsymbol{T}}$, given by (1.1), with respect to σ in the sense of the norm (2.4).

A function f is said piecewise continuous on $[0, d_f)$ if $\lim_{\xi\to t^+} f(\xi) = f(t)$ and $\lim f(\xi)$ exists for each $t \in [0, d_f)$ and $\lim_{\xi\to t^-} f(\xi) = f(t)$ at all but a finite number of points in $[0, d_f)$.

Definition 1. A deformation process, of duration $d_p > 0$, is a piecewise continuous function $P\colon [0, d_p) \to \mathfrak{J}_{\text{sym}}$ defined by

$$P(t) = \dot{\boldsymbol{E}}_P(t)\,, \qquad t \in [0, d_p)\,.$$

The collection of deformation processes P is denoted by Π, the restriction of P to $[0, t) \subset [0, d_p)$ by P_t.

We can now make precise the meaning of state transformation function $\hat{\varrho}\colon \Sigma\times\Pi \to \Sigma$ providing the final state $\sigma^f = (\boldsymbol{E}(t+d_p), {}_r\boldsymbol{E}^{t+d_P})$ in terms of the initial state $\sigma^i = (\boldsymbol{E}(t), {}_r\boldsymbol{E}^t)$ and of the deformation process $P\colon [0, d_p) \to \mathfrak{J}_{\text{sym}}$, according to Definition 1. Specifically

$$\sigma^f = \hat{\varrho}(\sigma^i, P) = (\boldsymbol{E}(t+d_p), {}_r\boldsymbol{E}^{t+d_P}) \tag{2.5}$$

with

$$\boldsymbol{E}(t+d_p) = \int_t^{t+d_P} \dot{\boldsymbol{E}}_p(u)\,du + \boldsymbol{E}(t)\,,$$

$${}_r\boldsymbol{E}^{t+d_P}(s) = \begin{cases} \displaystyle\int_t^{t+d_P} \dot{\boldsymbol{E}}_P(u)\,du + \boldsymbol{E}(t)\,, & s \in [0, d_P]\,, \\ \boldsymbol{E}^t(s-d_P)\,, & s \in (d_P, \infty)\,. \end{cases} \tag{2.6}$$

For any $P \in \Pi$ and $t \in [0, d_p)$, the mapping $\hat{\varrho}(\cdot, P_t)\colon \Sigma \to \Sigma$, characterized by (2.6), is uniformly continuous for any t [15].

A pair $(\sigma, P) \in \Sigma \times \Pi$ such that $\hat{\varrho}(\sigma, P) = \sigma$ is called a *cycle*. For materials with fading memory cycles are rare; a broader class of processes, to which our investigation applies, is that of *approximate cycles*. To make the meaning precise let $\mathcal{O}_\varepsilon(\sigma)$ stand for the ε-neighbourhood of σ consisting of the set of elements $\bar{\sigma}$ such that $\|\bar{\sigma} - \sigma\| < \varepsilon$. A pair (σ, P) is called an *ε-approximate cycle* if $\hat{\varrho}(\sigma, P) \in \mathcal{O}_\varepsilon(\sigma)$.

3. Periodic histories.

For any history $\boldsymbol{E}^t$ in Σ we define the periodic history $\boldsymbol{E}^t_\tau$ of period τ as

$$\boldsymbol{E}^t_\tau(s) = \boldsymbol{E}^t(s - k\tau)\,, \qquad k\tau \leqslant s < (k+1)\tau\,,\ k = 0, 1, 2, \dots . \tag{3.1}$$

The following theorem ascribes to the set $\mathcal{T}$ of all periodic histories (in Σ) a prominent role.

THEOREM 1. The set $\mathcal{T}$ of all periodic histories belonging to Σ is dense in Σ.

PROOF. For any history $\boldsymbol{E}^t$ in Σ let $\boldsymbol{E}_{\tau_n}$ be the associated periodic history according to the definition (3.1). For the sake of definiteness we let $\tau_n = n\alpha$, $\alpha \in R^{++}$. As n runs from 1 to any natural number, the set $\boldsymbol{E}^t_{\tau_n}$ constitutes a sequence of elements of Σ. Since the present value of $\boldsymbol{E}^t$ is equal to that of $\boldsymbol{E}^t_{\tau_n}$, to prove the theorem we have simply to show that

$$\lim_{n\to\infty} \int_0^\infty |\boldsymbol{E}^t(s) - \boldsymbol{E}^t_{\tau_n}(s)|^2 h(s)\, ds = 0\,. \tag{3.2}$$

Because $\boldsymbol{E}^t(s) - \boldsymbol{E}^t_{\tau_n}(s) = \boldsymbol{0}$ as $s \in [0, \tau_n)$, we have

$$\int_0^\infty |\boldsymbol{E}^t(s) - \boldsymbol{E}^t_{\tau_n}(s)|^2 h(s)\, ds = \int_{\tau n}^\infty |\boldsymbol{E}^t(s) - \boldsymbol{E}_{\tau_n}(s)|^2 h(s)\, ds \leqslant$$

$$\leqslant 2\int_{\tau_n}^\infty [|\boldsymbol{E}^t(s)|^2 + |\boldsymbol{E}(s)|^2] h(s)\, ds\,.$$

By hypothesis

$$\int_0^\infty |\boldsymbol{E}^t(s)|^2 h(s)\, ds < \infty$$

and hence

$$\lim_{n\to\infty} \int_{\tau_n}^\infty |\boldsymbol{E}^t(s)|^2 h(s)\, ds = 0\ .$$

As to the integral concerning $\boldsymbol{E}^t_{\tau_n}$ we observe that

$$\int_{\tau_n}^\infty |\boldsymbol{E}^t_{\tau_n}(s)|^2 h(s)\, ds = \sum_{r=1}^\infty \int_{r\tau_n}^{(r+1)\tau_n} |\boldsymbol{E}^t(s - r\tau_n)|^2 h(s)\, ds\ .$$

Because of (2.2)

$$\sum_{r=1}^\infty \int_{r\tau_n}^{(r+1)\tau_n} |\boldsymbol{E}^t(s - r\tau_n)|^2 h(s)\, ds < \gamma |\boldsymbol{E}|^2_{\max} \left(\sum_{r=1}^\infty \frac{1}{r^{2+\varkappa}}\right) \frac{1}{\tau_n^{1+\varkappa}}\ .$$

Then, taking the limit as n tends to infinity yields

$$\lim_{n\to\infty} \int_{\tau_n}^\infty |\boldsymbol{E}^t_{\tau_n}(s)|^2 h(s)\, ds = 0\ .$$

This proves that (3.2) holds and hence that $\mathcal{T}$ is dense in Σ. □

4. Second law of thermodynamics and work inequality.

Following Coleman and Owen [15] we take the action having the Clausius property (at any material point of the body) as the statement of the second law of thermodynamics. To make this assertion operative we observe that, as usual in the approximation of small displacement gradients, the rate-of-strain tensor can be replaced by the time derivative $\dot{\boldsymbol{E}}$. Then it follows that the power of the stress is $\boldsymbol{T}\cdot\dot{\boldsymbol{E}}$.

Hence the Clausius property at a state σ is expressed by saying that for each $\varepsilon > 0$, there exists $\nu_\varepsilon > 0$ such that, for each ν_ε-approximate cycle (σ, P),

$$\int_0^{d_P} \left\{ \frac{1}{\theta(t)} [\dot{\check{e}}(\sigma_t) - \check{\boldsymbol{T}}(\sigma_t) \cdot \dot{\boldsymbol{E}}(t)] + \frac{1}{\varrho_0 \theta^2(t)} (\boldsymbol{q} \cdot \boldsymbol{g})(t) \right\} dt < \varepsilon \tag{4.1}$$

where $\sigma_t = \hat{\varrho}(\sigma, P_t)$; here $\boldsymbol{q}$ is the heat flux, θ the temperature, and ϱ_0 the (constant) mass density while $\boldsymbol{g} = \partial\theta/\partial \boldsymbol{x}$. When θ is constant, and $\boldsymbol{g} = \boldsymbol{0}$, the contribution to the internal energy e is simply $[\check{e}(\sigma_{d_P}) - \check{e}(\sigma)]/\theta$. Then, in view of the continuity of the functional $\check{e}$ we can write the second law of thermodynamics in the following form.

WORK INEQUALITY. *For each $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that for each η_ε-approximate cycle (σ, P),*

$$\int_0^{d_P} \check{\boldsymbol{T}}(\sigma_t) \cdot \dot{\boldsymbol{E}}(t)\, dt > -\varepsilon\,. \tag{4.2}$$

The work inequality (4.2) may be phrased by saying that the work done in an approximate cycle is approximately positive.

Upon substitution of the constitutive relation (1.1) it follows from (4.2) that

$$\int_0^{d_P} \dot{\boldsymbol{E}}(t) \cdot \boldsymbol{G}(0) \boldsymbol{E}(t)\, dt + \int_0^{d_P} \dot{\boldsymbol{E}}(t) \cdot \int_0^{\infty} \boldsymbol{G}'(s) \boldsymbol{E}^t(s)\, ds\, dt > -\varepsilon\,. \tag{4.3}$$

In the next section we derive the restrictions placed by the work inequality on the relaxation function $\boldsymbol{G}$. In this regard we observe that, in the case of cycles, the inequality (4.3) is replaced by [15]

$$\int_0^{d_P} \dot{\boldsymbol{E}}(t) \cdot \boldsymbol{G}(0) \boldsymbol{E}(t)\, dt + \int_0^{d_P} \dot{\boldsymbol{E}}(t) \cdot \int_0^{\infty} \boldsymbol{G}'(s) \boldsymbol{E}^t(s)\, ds\, dt \geqslant 0\,. \tag{4.4}$$

For later convenience we prove now that the work performed by the stress is a continuous functional on Σ; the work $L(\sigma, P)$ associated with the process P starting from the state σ is expressed by

$$L(\sigma, P) = \int_0^{d_P} \check{\boldsymbol{T}}(\sigma_t)\cdot \dot{\boldsymbol{E}}(t)\, dt\,.$$

LEMMA 1. For every process P the functional

$$L(\cdot, P)\colon\ \Sigma \to R$$

is continuous on Σ.

PROOF. In view of the uniform continuity of $\hat{\varrho}(\cdot, P_t)$ on Σ it follows that for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that

$$\|\hat{\varrho}(\sigma_1, P_t) - \hat{\varrho}(\sigma_2, P_t)\| < \varepsilon\,, \qquad t \in [0, d_P)\,,$$

whenever

$$\|\sigma_1 - \sigma_2\| < \eta_\varepsilon\,. \tag{4.5}$$

On the other hand,

$$|L(\sigma_1, P) - L(\sigma_2, P)| \leqslant \int_0^{d_P} |\check{\boldsymbol{T}}(\hat{\varrho}(\sigma_1, P_t)) - \check{\boldsymbol{T}}(\hat{\varrho}(\sigma_2, P_t))|\,|P(t)|\, dt\,.$$

Then, since the functional $\check{\boldsymbol{T}}(\sigma_t)$ is linear and bounded while the function P is bounded on $[0, d_P)$, there exists a number M_P such that

$$|L(\sigma_1, P) - L(\sigma_2, P)| < M_P\,\varepsilon\,,$$

whenever (4.5) holds, which proves the continuity of $L(\cdot, P)$. □

5. Thermodunamic restrictions on the relaxation function.

Besides being important for easy correlation with experiments [6], as we shall see in a moment sinusoidally time variation proves to be especially fruitful in connection with the derivation of thermodynamic restrictions on the relaxation function [1]. This motivates the recourse to oscillatory strain-tensor evolution of the form

$$\tilde{\boldsymbol{E}}(t) = \boldsymbol{E}_1 \cos(\omega t) + \boldsymbol{E}_2 \sin(\omega t)\,, \qquad \omega \in R^{++}\,. \tag{5.1}$$

The corresponding process $\tilde{P} \in \Pi$ may be expressed as

$$\tilde{P}(t) = -\omega \boldsymbol{E}_1 \sin(\omega t) + \omega \boldsymbol{E}_2 \cos(\omega t), \qquad t \in [0, d_{\tilde{P}}), \tag{5.2}$$

with $d_{\tilde{P}} = 2\pi m/\omega$, m being any integer. Accordingly, it follows at once that any current state

$$\tilde{\sigma}_t = (\tilde{\boldsymbol{E}}(t), {}_r\tilde{\boldsymbol{E}}^t) \tag{5.3}$$

and the process $\tilde{P}$ constitute a cycle in that

$$\hat{\varrho}(\tilde{\sigma}_t, \tilde{P}) = \tilde{\sigma}_t .$$

We are now in a position to derive some necessary conditions for the validity of the second law.

THEOREM 2. A necessary condition for the validity of the second law is that the inequality

$$\boldsymbol{E}_1 \cdot [\boldsymbol{G}^T(0) - \boldsymbol{G}(0)] \boldsymbol{E}_2 - \int_0^\infty [\boldsymbol{E}_1 \cdot \boldsymbol{G}'(s) \boldsymbol{E}_1 + \boldsymbol{E}_2 \cdot \boldsymbol{G}'(s) \boldsymbol{E}_2] \sin(\omega s)\, ds - \int_0^\infty \boldsymbol{E}_1 \cdot [\boldsymbol{G}'(s) - \boldsymbol{G}'^T(s)] \boldsymbol{E}_2 \cos(\omega s)\, ds \geqslant 0 \tag{5.4}$$

holds for every $\omega \in R^{++}$ and every $\boldsymbol{E}_1, \boldsymbol{E}_2 \in \mathfrak{I}_{\mathrm{sym}}$.

PROOF. Consider the cycle $(\tilde{\sigma}_t, \tilde{P}) \in \Sigma \times \Pi$, with $\tilde{\sigma}_t$, $\tilde{P}$ as given by (5.3), (5.2). Substitution into the inequality (4.4) and integration with respect to $t \in [0, 2\pi/\omega]$ yield

$$\pi \Big\{ \boldsymbol{E}_1 \cdot [\boldsymbol{G}^T(0) - \boldsymbol{G}(0)] \boldsymbol{E}_2 - \int_0^\infty [\boldsymbol{E}_1 \cdot \boldsymbol{G}'(s) \boldsymbol{E}_1 + \boldsymbol{E}_2 \cdot \boldsymbol{G}'(s) \boldsymbol{E}_2] \sin(\omega s)\, ds - \int_0^\infty \boldsymbol{E}_1 \cdot [\boldsymbol{G}'(s) - \boldsymbol{G}'^T(s)] \boldsymbol{E}_2 \cos(\omega s)\, ds \Big\} \geqslant 0$$

whence (5.4). □

As shown below, in the particular cases $\omega \to \infty$, $\omega \to 0$ the inequality (5.4) yields two important results.

COROLLARY 1. Letting $\omega \to \infty$ the integrals in (5.4) vanish. Then the arbitrariness of $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ implies the symmetry of $\boldsymbol{G}(0)$, namely

$$\boldsymbol{G}(0) = \boldsymbol{G}^T(0) \,. \qquad \square \tag{5.5}$$

COROLLARY 2. In the limiting case $\omega \to 0$ the inequality (5.4) reduces to

$$\boldsymbol{E}_1 \cdot [\boldsymbol{G}^T(\infty) - \boldsymbol{G}(\infty)]\boldsymbol{E}_2 \geqslant 0 \,.$$

The arbitrariness of $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ implies that

$$\boldsymbol{G}(\infty) = \boldsymbol{G}^T(\infty) \,. \qquad \square$$

In view of the result (5.5) we can write a significant consequence of the second law on the function $\boldsymbol{G}'(s)$, $s \in R^+$.

COROLLARY 3. A necessary condition for the validity of the second law is that the inequality

$$\int_0^\infty [\boldsymbol{E}_1 \cdot \boldsymbol{G}'(s)\boldsymbol{E}_1 + \boldsymbol{E}_2 \cdot \boldsymbol{G}'(s)\boldsymbol{E}_2] \sin(\omega s)\, ds + \int_0^\infty \boldsymbol{E}_1[\boldsymbol{G}'(s) - \boldsymbol{G}'^T(s)]\boldsymbol{E}_2 \cos(\omega s)\, ds \leqslant 0 \tag{5.6}$$

holds for every $\omega \in R^{++}$ and every $\boldsymbol{E}_1, \boldsymbol{E}_2 \in \mathfrak{J}_{\mathrm{sym}}$. $\square$

Let $\hat{\boldsymbol{G}}'_s$ be the half-range Fourier sine transform of $\boldsymbol{G}'$, namely

$$\hat{\boldsymbol{G}}'_s(\omega) = \int_0^\infty \boldsymbol{G}'(s) \sin(\omega s)\, ds \,.$$

Upon choosing $\boldsymbol{E}_1 = \boldsymbol{E}_2 = \boldsymbol{E}$, it follows immediately from (5.6) that

$$\boldsymbol{E} \cdot \hat{\boldsymbol{G}}'_s(\omega)\boldsymbol{E} \leqslant 0 \,, \qquad \omega \in R^{++} \,. \tag{5.7}$$

REMARK. By virtue of the Fourier integral theorem we have

$$G(s)-G(0)=-\frac{2}{\pi}\int_0^\infty \frac{\cos(\omega s)-1}{\omega}\,\hat{G}'_s(\omega)\,d\omega\,.$$

Then in view of (5.7) we obtain

$$E\cdot[G(0)-G(s)]E\geqslant 0\,,$$

for every $s\in R^+$ and $E\in \mathfrak{J}_{\text{sym}}$. Hence it follows the negative semidefiniteness of $G'(0)$ and the positive semidefiniteness of $G(0)-G(\infty)$; the last result traces back to Coleman [11].

The main result to emerge from this paper is that the condition (5.4), which is a restriction on the relaxation function G, is also sufficient for the validity of the second law; thus (5.6), along with the symmetry of $G(0)$, embodies all the restrictions placed by the second law. The sufficiency property is proved as follows.

THEOREM 3. If G satisfies the condition (5.4) for every $\omega\in R^{++}$ then for every $\varepsilon>0$ there exists $\eta_\varepsilon>0$ such that the inequality (4.3) holds for every η_ε-approximate cycle.

PROOF. As a first step we prove that the cycles satisfy (4.4). If (σ, P) is a cycle then σ represents a periodic history of period d_P. Now, each periodic history $E^t_{\tau_n}\in \mathfrak{T}$, with period $\tau_n=n\alpha$, can be expressed through its Fourier series as

$$E^t_{\tau_n}(s)=\sum_{k=0}^{\infty}\left(A_k\cos[k\omega_n(t-s)]+B_k\sin[k\omega_n(t-s)]\right)$$

where $\omega_n=2\pi/\tau_n=2/n\alpha$. Consider the state $\sigma^n=(E_{\tau_n}(t), E_{\tau_n})$ and the process P^n of duration τ_n defined as

$$P^n(t)=\dot{E}_{\tau_n}(t)\,,\qquad t\in[0,\tau_n)\,.$$

Then the pair (σ^n, P^n) is a cycle. In such a case the work L turns out to be given by

$$L = \int_0^{\tau_n} \dot{\boldsymbol{E}}_{\tau_n}(t)\cdot \boldsymbol{G}(0)\,\boldsymbol{E}_{\tau_n}(t)\,dt + \\ + \int_0^{\tau_n}\int_0^{\infty} \sum_{k=1}^{\infty}\sum_{h=0}^{\infty} k\omega_n[-\boldsymbol{A}_k \sin(k\omega_n t) + \boldsymbol{B}_k \cos(k\omega_n t)]\cdot \\ \cdot \boldsymbol{G}'(s)\{\boldsymbol{A}_h \cos[h\omega_n(t-s)] + \boldsymbol{B}_h \sin[h\omega_n(t-s)]\}\,ds\,dt\,.$$

The first integral vanishes because $\boldsymbol{E}_{\tau_n}(\tau_n) = \boldsymbol{E}_{\tau_n}(0)$ and $\boldsymbol{G}(0)$ is symmetric. The integration term by term of the double series shows that the only non-vanishing terms are those with $h = k$. Then we are left with

$$L = \sum_{k=1}^{\infty} k\omega_n \int_0^{\infty}\left[\int_0^{\tau_n} \sin^2(k\omega_n t)\,dt\right]\cdot \\ \cdot[-\boldsymbol{A}_k\cdot \boldsymbol{G}'(s)\,\boldsymbol{A}_k \sin(k\omega_n s) - \boldsymbol{A}_k\cdot\boldsymbol{G}'(s)\,\boldsymbol{B}_k\cos(k\omega_n s)]\,ds + \\ + \sum_{k=1}^{\infty} k\omega_n \int_0^{\infty}\left[\int_0^{\tau_n}\cos^2(k\omega_n t)\,dt\right]\cdot \\ \cdot[\boldsymbol{B}_k\cdot\boldsymbol{G}'(s)\,\boldsymbol{A}_k\cos(k\omega_n s) - \boldsymbol{B}_k\cdot\boldsymbol{G}'(s)\,\boldsymbol{B}_k\sin(k\omega_n s)]\,ds$$

whence

$$L = -\pi\sum_{k=1}^{\infty} k\int_0^{\infty}[\boldsymbol{A}_k\cdot\boldsymbol{G}'(s)\,\boldsymbol{A}_k + \boldsymbol{B}_k\cdot\boldsymbol{G}'(s)\,\boldsymbol{B}_k]\sin(k\omega_n s)\,ds - \\ -\pi\sum_{k=1}^{\infty} k\int_0^{\infty}\boldsymbol{A}_k\cdot[\boldsymbol{G}'(s) - \boldsymbol{G}'^{T}(s)]\,\boldsymbol{B}_k\cos(k\omega_n s)\,ds\,.$$

The hypothesis that (5.6) holds yields $L \geqslant 0$. Then, owing to (5.5), it is proved that (4.4) follows from (5.6) in the case of periodic histories.

As a second step, completing the proof of the theorem, we show that (5.6) ensures the validity of (4.3). Let $\sigma \in \Sigma$ be a state and $\boldsymbol{E}_t$ the corresponding history. For every $\eta > 0$ consider a process P_η

such that the final state $\sigma^f = \hat{\varrho}(\sigma, P_\eta)$ belongs to $\mathcal{O}_\eta(\sigma)$ namely

$$\|\sigma^f - \sigma\| < \eta .$$

In conjunction with the state σ^f and the associated history $\boldsymbol{E}_f^t$ define the periodic history $\tilde{\boldsymbol{E}}_f^t$, with period $\tau = d_{P_\eta}$, as in (5.8) and denote the corresponding state by $\tilde{\sigma}^f$. Then

$$\|\tilde{\sigma}^f - \sigma^f\| < \beta\eta \tag{5.11}$$

with

$$\beta = \sqrt{\gamma} \sum_{m=1}^{\infty} \frac{1}{(1 + m\tau)^{1+\varkappa/2}} .$$

To prove (5.11) let $n \geqslant 1$ be any integer and let σ^n be the state corresponding to the history $\boldsymbol{E}_n^t$ defined as

$$\boldsymbol{E}_n(t-s) = \begin{cases} \tilde{\boldsymbol{E}}_f(t-s) , & s \in [0, n\tau] , \\ \boldsymbol{E}_f(t-s+n\tau) , & s \in [n\tau, \infty) . \end{cases}$$

Of course $\sigma^1 = \sigma^f$. Observe now that

$$\|\tilde{\sigma}^f - \sigma^f\| = \lim_{n\to\infty} \|\sigma^n - \sigma^f\| \leqslant \lim_{n\to\infty} \sum_{m=1}^{n-1} \|\sigma^{m+1} - \sigma^m\| .$$

Because $\boldsymbol{E}_n(t) = \boldsymbol{E}_{n+1}(t)$ for any value of n we have

$$\|\sigma^{n+1} - \sigma^n\|^2 = \Big|\int_0^\infty \boldsymbol{E}_{n+1}(t-s) - \boldsymbol{E}_n(t-s)|^2 h(s)\, ds =$$

$$= \int_0^\infty |\boldsymbol{E}_f(t-s) - \boldsymbol{E}(t-s)|^2 h(s + n\tau)\, ds .$$

By virtue of (2.2) it follows that

$$\int_0^\infty |\boldsymbol{E}_f(t-s) - \boldsymbol{E}(t-s)|^2 h(s + n\tau)\, ds <$$

$$< \frac{k}{(1 + n\tau)^{2+\varkappa}} \int_0^\infty |\boldsymbol{E}_f(t-s) - \boldsymbol{E}(t-s)|^2 h(s)\, ds .$$

Hence we obtain

$$\|\tilde{\sigma}^f - \sigma^f\| = \lim_{n\to\infty} \|\sigma^n - \sigma^f\| \leqslant$$
$$\leqslant \lim_{n\to\infty} \sum_{m=1}^{n-1} \|\sigma^{m+1} - \sigma^m\| < \lim_{n\to\infty} \sum_{m=1}^{n-1} \frac{\sqrt{\gamma}}{(1+m\tau)^{1+\varkappa/2}}\,\eta$$

whence the inequality (5.11). As a consequence

$$\|\sigma - \tilde{\sigma}^f\| \leqslant \|\sigma - \sigma^f\| + \|\sigma^f - \tilde{\sigma}^f\| < (1+\beta)\eta\,.$$

Owing to the continuity of $L(\cdot, P)$, for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$|L(\sigma, P_\eta) - L(\tilde{\sigma}^f, P_\eta)| < \varepsilon \tag{5.12}$$

whenever $\hat{\varrho}(\sigma, P_\eta) \in \mathcal{O}_\eta(\sigma)$. On the other hand, because $(\tilde{\sigma}^f, P_\eta)$ is a cycle, it follows that

$$L(\tilde{\sigma}^f, P_\eta) \geqslant 0\,.$$

Hence, in view of (5.12), we obtain (4.2) and (4.3). □

Based on Theorem 3 we observe that the second law for linear viscoelastic materials may be stated in terms of cycles only. In such a case the second law is expressed by the inequality (4.4).

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