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On Chains of Purely Transcendental Field Extensions.

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ABSTRACT - Conditions are given under which the union of an ascending chain of purely transcendental extensions of a ground field k is again a purely transcendental extension of k .

In an attempt to solve a problem posed by Anderson and Ohm on semi-valuations of group rings [1], we came across the following problem. Suppose that $K_0 \subset K_1 \subset \dots \subset K_n \subset \dots$ is a countable ascending chain of fields K_n such that (i) each K_n is a purely transcendental extension of K_0 , and (ii) K_n is algebraically closed in K_{n+1} , for each $n \geq 0$. Is the union $K = \bigcup K_n$ likewise a purely transcendental extension of K_0 ? Though the answer to this question was not needed in the solution of the Anderson-Ohm problem (cf. Bastos-Viswanathan [2]), we have found the question interesting enough to look into it.

In this note we give a complete answer to this question under the hypothesis that K is the field of quotients of a group algebra (of a torsion-free abelian group). We will prove in Theorem 1 that in this case the answer is in the affirmative. However, the situation changes if uncountable chains are admitted as is shown by Example 2.

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Our results rely heavily on well-known theorems on torsion-free abelian groups. The connection with abelian group theory can be utilized to establish certain properties of what we call almost purely transcendental extensions; see Theorem 3.

1. - The main result.

For a field k and a torsion-free abelian group A , the group algebra $k[A]$ is a domain. Its field of quotients is denoted by $k(A)$. If A is a free abelian group then $k(A)$ is a purely transcendental extension of k and a basis of A is a transcendence base for $k(A)|k$.

Our main goal is to prove the following theorem.

THEOREM 1. - *Let k be any field, A a torsion-free abelian group, and B a subgroup of A . Suppose that*

$$(1) \quad k(B) = K_0 \subset K_1 \subset \dots \subset K_n \subset \dots \quad (n < \omega)$$

is an ascending chain of fields such that

- (i) K_n is a purely transcendental extension of K_0 ($n < \omega$);
- (ii) K_n is algebraically closed in K_{n+1} ($n < \omega$);
- (iii) $K = \bigcup_{n < \omega} K_n = k(A)$.

Then A/B is a free abelian group and $k(A)|k(B)$ is a purely transcendental extension.

PROOF. The chain (1) gives rise to an ascending chain of abelian groups in the following way. Consider the multiplicative groups K_n^* of the fields $K_n \text{ mod } K_0^*$. From (1) we obtain an ascending chain

$$(2) \quad (1) = K_0^*/K_0^* \subset K_1^*/K_0^* \subset \dots \subset K_n^*/K_0^* \subset \dots \quad (n < \omega).$$

Conditions (i)-(iii) translate into (i*)-(iii*) as follows.

- (i*) All K_n^*/K_0^* are free abelian groups.

In fact, it is well known that a polynomial ring over any field with an arbitrary (even infinite) number of indeterminates is a UFD.

Therefore, the multiplicative group of its field of quotients mod that of the ground field is freely generated by a set of irreducible polynomials over K_0 .

(ii*) K_n^*/K_0^* is a pure subgroup of K_{n+1}^*/K_0^* ($n < \omega$).

This is obvious, since because of (ii) the m -th roots of elements of K_n which are contained in K_{n+1}^* are already in K_n^* . (Thus (ii) can be weakened by assuming only that K_n is closed under taking roots of elements in K_{n+1} .)

(iii*) We have $\bigcup_{n < \omega} (K_n^*/K_0^*) = K^*/K_0^*$.

A well-known theorem by Hill [5] states that the union of a countable ascending chain of free abelian groups where each member is a pure subgroup in its successor is itself free. Hence K^*/K_0^* is a free abelian group. Now observe that A/B is a subgroup of K^*/K_0^* ; therefore it is a free abelian group.

If A/B is free, then B is a direct factor of A , $A = B \times F$ for a free subgroup F of A . It follows that $k(A) = [k(B)](F)$. Since the adjunction of a free abelian group is nothing else than a purely transcendental extension, the proof is completed. \square

One wonders if the last theorem can be improved. As we have noticed in the introduction, we do not know if the condition $K = k(A)$ can be dropped. (Of course, the freeness of K^*/K_0^* follows, but this alone does not imply that $K|K_0$ is a purely transcendental extension.) But we do know that the countability of the chain (1) is relevant. In fact, the following example shows that Theorem 1 is in general not valid for uncountable chains of fields.

EXAMPLE 2. Let

$$A_0 = 0 \subset A_1 \subset \dots \subset A_\alpha \subset \dots \quad (\alpha < \omega_1)$$

be a continuous well-ordered ascending chain of countable free abelian groups A_α (continuity means $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$ for limit ordinals α and ω_1 is the first uncountable ordinal) where each A_α is pure in $A_{\alpha+1}$. Suppose that the set

$$E = \{\alpha < \omega_1 \mid A_{\alpha+1}/A_\alpha \text{ is not free}\}$$

is stationary in ω_1 . (It is easy to construct such a chain, e.g. by taking a projective resolution of $\mathbf{Q}: 0 \rightarrow F_0 \rightarrow F_1 \rightarrow \mathbf{Q} \rightarrow 0$ and embedding A_α in $A_{\alpha+1}$ in the very same way F_0 is embedded in F_1 , for each $\alpha < \omega_1$.) It is well-known (see Eklof [3]) that under these circumstances, $A = \bigcup_{\alpha < \omega_1} A_\alpha$ is not a free abelian group.

Now, for any field k , define $K_\alpha = k(A_\alpha)$. Then the chain

$$K_0 \subset K_1 \subset \dots \subset K_\alpha \subset \dots \subset K = \bigcup_{\alpha < \omega_1} K_\alpha \quad (\alpha < \omega_1)$$

satisfies conditions (i), (ii), (iii) above. Since A is not free, $K|K_0$ cannot be a purely transcendental extension. \square

2. – Almost purely transcendental extensions.

A closer look at the preceding example reveals that, though the union K is not a purely transcendental extension of K_0 , it is close to being one: every subfield of K that contains K_0 and has countable transcendence degree over K_0 can be embedded in a countably generated purely transcendental extension of K_0 in K .

Let us call an extension K of a field K_0 *almost purely transcendental* if the transcendence degree of K over K_0 is an infinite cardinal λ and every subextension L of K_0 in K with transcendence degree $< \lambda$ can be embedded in a purely transcendental extension of K_0 in K .

We are now going to verify (for the needed set-theoretical concepts, see [7]):

THEOREM 3. (a) *Let λ be an uncountable regular cardinal which is not weakly compact. Then every field K_0 has an almost purely transcendental extension K of transcendence degree λ which is not purely transcendental.*

(b) *If λ is a singular cardinal, then every almost purely transcendental extension of transcendence degree λ is necessarily purely transcendental.*

PROOF. (a) Gregory [4] has proved that for every regular, not weakly compact cardinal λ there is a non-free abelian group A of cardinality λ all of whose subgroups of cardinalities $< \lambda$ are free.

Choose $K = K_0(A)$ with such an A . Then K is not a purely transcendental extension of K_0 , since K^*/K_0^* contains the non-free subgroup A . Now if L is a subextension in K , of transcendence degree $\kappa < \lambda$ over K_0 , then an easy cardinality argument shows that $L \subset K_0(B)$ for a subgroup B of A , of cardinality κ . As B must be free, $K_0(B)$ is a purely transcendental extension of K_0 that contains L .

(b) This part was proved by Hodges [6], p. 218. It is a consequence of his version of Shelah's compactness theorem for singular cardinals. We refer to [6] for details. \square

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