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Periodic Solutions for a Class of Autonomous Hamiltonian Systems.

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1. - Introduction.

In this paper we shall be concerned with the existence of T -periodic solutions of Hamiltonian systems $\dot{p} = -H'_q(p, q)$, $\dot{q} = H'_p(p, q)$ when H is of the form

$$(1) \quad H(p, q) = U(p) + V(q)$$

so that the above equations of motion became

$$(2) \quad \dot{p} = -V'(q), \quad \dot{q} = U'(p).$$

Hamiltonians of the form (1) occupy a central position in the general theory of Hamiltonian systems. Moreover, in applications to concrete problems, p and q play substantially distinct roles. In fact, in many classical problems, the term $U(p)$ has the form $(\frac{1}{2})|p|^2$ or, more in general, is a positive definite quadratic form. Hence $U(p)$ is strictly convex. On the contrary, a wide freedom in the choice of the potential $V(q)$ is required. For Hamiltonians of the special form $|p|^2/2 + V(q)$, Hamilton's equation reduces to Newton's equation $\ddot{q} + V'(q) = 0$. Here, the higher order term is a linear operator. The natural nonlinear generalization of the above class (which shall be our

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main model) consists in Hamiltonians of the form $(1/\alpha)|p|^\alpha + V(q)$.

Throughout this paper, c_i ($i \in \mathbb{N}$) denote positive constants. We shall prove the following result.

THEOREM A. *Let $U, V \in C^1(\mathbb{R}^n, \mathbb{R})$, U strictly convex, V everywhere nonnegative. Assume that there are positive constants $\alpha \in]1, +\infty[$, $\mu > \alpha/(\alpha - 1)$, and r such that the following conditions hold.*

- (H₁) $c_1|p|^\alpha \leq U(p) \leq c_2|p|^\alpha$, for all $p \in \mathbb{R}^n$,
 (H₂) $\alpha U(p) \leq U'(p) \cdot p$, for all $p \in \mathbb{R}^n$,
 (H₃) $0 < \mu V(q) \leq V'(q) \cdot q - c_3$, for all $|q| \geq r$.

Then, for each $T > 0$, the problem (2) has infinitely many T -periodic non trivial solutions.

By setting $\alpha = 2$ and by considering the particular case $U(p) = (\frac{1}{2})|p|^2$, we reobtain a result of Benci (theorem 3.7 [B]), which in turn generalizes a result of Rabinowitz (theorem 2.61 [R1]). For $\alpha \neq 2$ theorem A is substantially different from all the results available to us. Note that (in theorem A): (i) the potential V is superquadratic at infinity when $1 < \alpha < 2$; (ii) the potential V could be subquadratic, quadratic or superquadratic at infinity, when $\alpha > 2$; (iii) no growth assumptions are made for small $|q|$; (iv) V is not necessarily convex. Remarks (i), (ii) and (iii) show that our assumptions are quite different from those made by Rabinowitz in his well known theorems on Hamiltonian systems (see [R3], [R4] for references).

Each one of the remarks (i)-(iv) show also that our assumptions are entirely different from those of Clarke's theorems 1.1 and 1.2 in reference [C2]. Note, in particular, that Clarke requires that $\mu < \alpha/(\alpha - 1)$, instead of $\mu > \alpha/(\alpha - 1)$. Our assumptions are also entirely different from those of Brezis and Coron theorem 2 [BC]. Hamiltonians of the particular form (1) satisfy the condition (6) of reference [BC] if $\alpha > 2$ and $\mu > 2$ (note that in theorem A, if $\alpha > 2$, μ can be smaller than 2); and under these assumptions theorem A gives T -periodic solutions for small T and theorem 2 in [BC] gives T -periodic solutions for large T . Note finally that, in references [BC] and [C2], the Hamiltonians are assumed to be convex but minimality of the period is proved.

We limit ourselves to give only the strictly necessary references.

For a complete bibliography and usefull comments we refer the reader to [R3].

2. - Proofs.

Without loss of generality we will assume that $V(0) = 0$. Let T be a fixed positive number and denote by $\| \cdot \|$ and $\| \cdot \|'$ the norms in $L^\beta(0, T; \mathbb{R}^n)$ and in $L^\alpha(0, T; \mathbb{R}^n)$, respectively. We set $\beta = \alpha/(\alpha - 1)$. Moreover,

$$E = \left\{ u \in L^\beta(0, T; \mathbb{R}^n) : \int u = 0 \right\},$$

where $\int u$ stands for $\int_0^T u(t) dt$. This abbreviated notation will be systematically used in the sequel. We set

$$B_\varrho = \{ u \in E : \| u \| \leq \varrho \}, \quad \partial B_\varrho = \{ u \in E : \| u \| = \varrho \}.$$

Define

$$(3) \quad Pu(t) = \int_0^t u(\tau) d\tau, \quad \forall t \in [0, T].$$

Clearly, $Pu(0) = Pu(T) = 0$, for every $u \in E$. The map P defines an isomorphism between E and the Sobolev space $W_0^{1,\beta}(0, T; \mathbb{R}^n)$.

The Legendre transform in \mathbb{R}^n of $U(p)$ is defined by

$$G(u) = \text{Sup} \{ u \cdot p - U(p) : p \in \mathbb{R}^n \}.$$

We recall that $G'(u) = p$ if and only if $U'(p) = u$, and that

$$(4) \quad \begin{cases} c_4 |u|^\beta \leq G(u) \leq c_5 |u|^\beta, \\ G'(u) \cdot u \leq \beta G(u), \\ |G'(u)| \leq c_6 |u|^{\beta-1}, \end{cases}$$

for all $u \in \mathbb{R}^n$. On the other hand, it readily follows, from (H^3) , that

$$(5) \quad \begin{cases} V'(q) \cdot q \geq \mu V(q) - c_7, \\ V(q) \geq c_8 |q|^\mu - c_9, \end{cases}$$

for all $q \in \mathbb{R}^n$.

One has the following result.

THEOREM 1. *Let (u, y) be a critical point of the functional*

$$(6) \quad f(u, y) = \int [G(u) - V(Pu + y)],$$

which is defined on the Banach space $E \oplus \mathbb{R}^n$. Then, the pair $(p, q) = (G'(u), Pu + y)$ is a T -periodic solution of problem (2).

This result is proved by applying the «dual action principle» (see Clarke [C1] and Clarke and Ekeland [CE]) only just to those variables with respect to which the hamiltonian is convex. Before proving the lemma, let us introduce some notations. The symbol \langle, \rangle denotes the duality pairing between the dual of a Banach space and the Banach space itself. The scalar product in \mathbb{R}^n is denoted either by $x \cdot y$ or by $\langle x, y \rangle$. Furthermore, f' denotes the (Fréchet) derivative of f , and f'_u, f'_v denote the partial derivatives with respect to u and y , respectively.

PROOF OF THEOREM 1. By taking into account that Pv is a periodic function, one easily proves that

$$(7) \quad \begin{aligned} \langle f'_u(u, y), v \rangle &= \\ &= \int G'(u) \cdot v - V'(Pu + y) \cdot Pv = \int [G'(u) + PV'(Pu + y)] \cdot v \end{aligned}$$

for every $u, v \in E, y \in \mathbb{R}^n$. Moreover,

$$(8) \quad \langle f'_v(u, y), x \rangle = - \left(\int V'(Pu + y) \right) \cdot x, \quad \forall x \in \mathbb{R}^n.$$

In particular,

$$f'(u, y) = \left(G'(u) + PV'(Pu + y), - \int V'(Pu + y) \right) \in L^x \oplus \mathbb{R}^n,$$

and

$$(9) \quad \langle f'(u, y), (v, x) \rangle = \int G'(u) \cdot v - \int V'(Pu + y) \cdot (Pv + x).$$

Note that $f \in C^1(E \oplus \mathbb{R}^n, \mathbb{R}^n)$.

If (u, y) is a critical point, it follows from (8) that

$$(10) \quad \int V'(Pu + y) = 0.$$

Moreover, (7) shows that $\int [G'(u) + PV'(Pu + y)] \cdot v = 0, \forall v \in E$, or equivalently that there exists $z \in \mathbb{R}^n$ such that

$$(11) \quad G'(u) + PV'(Pu + y) = z, \quad \forall t \in [0, T].$$

Define

$$(12) \quad \begin{cases} p = G'(u) = z - PV'(Pu + y), \\ q = Pu + y. \end{cases}$$

Due to (10), p and q are T -periodic.

Moreover, $\dot{p} = -V'(Pu + y) = -V'(q)$, and $\dot{q} = u = U'(p)$. //

Now, with the aid of Theorem 1, we will prove that the functional f has non trivial critical points. Hence Theorem A holds. Before proving Theorem A, let us make the following remarks:

REMARK 1. The above results also apply if

$$H(p, q) = U(p_1, \dots, p_k, q_{k+1}, \dots, q_n) + V(q_1, \dots, q_k, p_{k+1}, \dots, p_n),$$

where U and V are as in theorem 2, and $0 \leq k \leq n$. This is easily shown by doing the change of variables $q_j \rightarrow -p_j, p_j \rightarrow q_j, j = k + 1, \dots, n$.

REMARK 2. It is worth noting that the functional $f(u, y)$ is invariant under the S^1 -action of $\mathcal{A} = \{A_s : s \in \mathbb{R}\}$ which is defined on $E \oplus \mathbb{R}^n$ by

$$(13) \quad A_s(u, y) = \left(u(t + s), y + \int_0^s u(\tau) d\tau \right).$$

One easily verifies that $A_{s+T}(u, y) = A_s(u, y)$ and that $A_r A_s(u, y) = A_{r+s}(u, y)$ (we assume that the elements $u \in E$ are extended as T -periodic functions over the entire real line). Moreover, straight-

forward calculations show that

$$(14) \quad f(A_s(u, y)) = f(u, y), \quad \forall (u, y) \in E \oplus \mathbb{R}^n, \quad \forall s \in \mathbb{R}.$$

The fixed points under the action of A are precisely the elements $(0, y)$, for $y \in \mathbb{R}^n$.

Due to the above S^1 -invariance, it seems possible to apply Fadell, Husseini, Rabinowitz Theorem 3.14 [FHR] to show that f has an unbounded sequence of critical values. However the corresponding sequence of T -periodic solutions could coincide with some in the (T/m) -periodic solutions furnished by theorem $A(m \in \mathbb{N})$.

In the sequel we will prove theorem A by applying Rabinowitz's Theorem 5.3 [R4] to the functional f . Alternately, we could apply the theorem 1.1 in reference [R2]. In order to apply Rabinowitz's theorem it is sufficient to prove that f satisfies the following hypothesis.

$$(15) \quad f|_{\mathbb{R}^n} \leq 0,$$

$$(16) \quad \text{There are positive constants } \varrho, \theta \text{ such that } f(u, 0) \geq \theta \text{ if } \|u\| = \varrho.$$

$$(17) \quad \text{For each finite dimensional subspace } \tilde{E} \text{ of } E \oplus \mathbb{R}^n \text{ there exists a constant } R = R(\tilde{E}) \text{ such that } f(u, y) \leq 0 \text{ wherever } \|u\| + |y| \geq R, (u, y) \in \tilde{E} \text{ (}^1\text{)}.$$

$$(18) \quad \text{The functional } f \text{ verifies the Palais-Smale condition.}$$

Condition (15) is trivially verified. Conditions (16), (17), and (18) will be proved in the sequel.

LEMMA 1. *Under the hypothesis of theorem A the condition (16) is fulfilled.*

PROOF. We shall denote by $\|\cdot\|_\infty$ the usual norm on the space $L^\infty(0, T; \mathbb{R}^n)$. To show that

$$\int [G(u) - V(Pu)] \geq \theta \quad \text{for all } u \in \partial B_\varrho$$

(¹) In particular the assumption (I5) of Theorem 5.3 [R4] holds. See also Remark 5.5 (iii) there.

it is sufficient to prove that, for every $u \in \partial B_\varrho$, one has

$$c_4 \int [|u|^\beta - V(Pu)] \geq \theta .$$

Let c_{10} be a positive constant such that $|Pv|_\infty \leq c_{10} \|v\|$ for all $v \in E$. By assuming that $\varrho \leq c_{10}^{-1}$ one gets, for every $t \in [0, T]$,

$$|V(Pu(t))| \leq |Pu(t)| |\omega(P(u(t)))| \leq |Pu(t)|^\beta |\omega(Pu(t))| ,$$

where $\lim_{|q| \rightarrow 0} \omega(q) = \omega(0) = 0$. It readily follows that

$$\left| \int V(Pu) \right| \leq c_{11} \max_{0 \leq t \leq T} |\omega(Pu(t))| \|u\|^\beta .$$

In particular,

$$c_4 \int [|u|^\beta - V(Pu)] \geq \left(c_4 - c_{11} \max_{0 \leq t \leq T} |\omega(Pu(t))| \varrho^\beta \right) .$$

Since $|Pu|_\infty \leq c_{10} \varrho$ we conclude that

$$c_4 - c_{11} \max_{0 \leq t \leq T} |\omega(Pu(t))| > 0$$

if $\varrho = \|u\|$ is small enough. //

LEMMA 2. *Under the assumptions of theorem A, condition (17) is fulfilled.*

PROOF. One easily verifies that

$$(19) \quad [(u, y)] \equiv \|Pu + y\|_\mu$$

is a norm in $E \oplus \mathbb{R}^n$, where $\|\cdot\|_\mu$ stands for the usual norm in the space $L^\mu(0, T; \mathbb{R}^n)$. Let u_1, \dots, u_k be linearly independent vectors in E , and denote by E_k the subspace generated by these vectors. Set $\tilde{E} = E_k \oplus \mathbb{R}^n$. Since \tilde{E} is finite dimensional, there exists a positive constant $K = K(\tilde{E})$ such that

$$(20) \quad K(\|u\| + |y|) \leq \|Pu + y\|_\mu, \quad \forall (u, y) \in \tilde{E} .$$

By using (5)₂, (4)₁ and (20) one proves that

$$\begin{aligned} f(u, c) &\leq c_5 \|u\|^\beta - c_8 \|Pu + y\|^\mu - c_9 T \leq \\ &\leq c_5 (\|u\| + |y|)^\beta - c_8 K^\mu (\|u\| + |y|)^\mu, \end{aligned}$$

for every $(u, y) \in \tilde{E}$. The thesis follows, since $\mu > \beta$. //

Finally we prove the Palais-Smale condition.

LEMMA 3. *Let $(u_m, y_m) \in E \oplus \mathbf{R}^n$ be a sequence such that*

$$f(u_m, y_m) \leq M, \quad \forall m \in \mathbf{N},$$

and $f'(u_m, y_m) \rightarrow 0$ as $m \rightarrow +\infty$. Then (u_m, y_m) is a bounded sequence in $E \oplus \mathbf{R}^n$. Moreover, there exists a convergent subsequence in $E \oplus \mathbf{R}^n$.

PROOF. In the sequel we denote by $E' = \{w \in L^\alpha(0, T; \mathbf{R}^n) : \int w = 0\}$ the dual space of E , and by $\|P\|$ the norm of the linear operator $P: E \rightarrow L^\beta(0, T; \mathbf{R}^n)$. For convenience, we set $\varepsilon_m = f'_u(u_m, y_m)$, $\delta_m = f'_y(u_m, y_m)$. By assumption one has $\|\varepsilon_m\|_{E'} \rightarrow 0$, $|\delta_m| \rightarrow 0$, as $m \rightarrow +\infty$.

By using formulae (9) with $(u, y) = (v, x) = (u_m, y_m)$, and by taking into account (4)₂ and (5)₁, it readily follows

$$\langle \varepsilon_m, u_m \rangle + \langle \delta_m, y_m \rangle \leq \beta \int G(u_m) - \mu \int V(Pu_m + y_m) + c_7 T.$$

The above estimate, the assumption

$$\int G(u_m) - \int V(Pu_m + y_m) \leq M,$$

the boundedness of the sequences $\|\varepsilon_m\|_{E'}$ and $|\delta_m|$, and the condition $\mu > \beta$, imply that

$$(21) \quad \begin{cases} \int V(Pu_m + y_m) \leq c_{12} + c_{13} (\|u_m\| + |y_m|), \\ \int G(u_m) \leq M + c_{12} + c_{13} (\|u_m\| + |y_m|). \end{cases}$$

From (4)₁ and (21)₂ it follows that

$$(22) \quad \|u_m\|^\beta \leq c_{14} + c_{15}(\|u_m\| + |y_m|).$$

On the other hand,

$$\int |y_m|^\beta \leq 2^{\beta-1} \int (1 + |Pu_m + y_m|^\mu) + 2^{\beta-1} \|P\|^\beta \|u_m\|^\beta.$$

This inequality, together with (5)₂, (21)₁ and (22) yields

$$(23) \quad |y_m|^\beta \leq c_{16} + c_{17}(\|u_m\| + |y_m|).$$

The estimates (22), (23) show that $\|u_m\|$ and $|y_m|$ are uniformly bounded.

Now we prove the second part of the lemma. From (7) one gets

$$\langle \varepsilon_m, v \rangle = \int [G'(u_m) + PV'(Pu_m + y_m)] \cdot v,$$

for every $v \in E$. Hence

$$(24) \quad \left| \int [G'(u_m) + PV'(Pu_m + y_m)] \cdot v \right| \leq \|\varepsilon_m\|_{E'} \|v\|.$$

On the other hand, from (8) it follows that $|\int V'(Pu_m + y_m)| = |\delta_m|$, and from (4) it follows

$$\left| \int G'(u_m) \right| \leq c_8 T^{1/\beta} \|u_m\|^{\beta-1}.$$

Consequently, the mean value of $G'(u_m) + V'(Pu_m + y_m)$ is uniformly bounded with respect to m . Hence, along a suitable subsequence, one has

$$(25) \quad \lim_{m \rightarrow +\infty} \frac{1}{T} \int [G'(u_m) + V'(Pu_m + y_m)] = \xi_0 \in \mathbb{R}^n.$$

Equations (24) and (25) imply that

$$(26) \quad \lim_{m \rightarrow +\infty} \|G'(u_m) + PV'(Pu_m + y_m) - \xi_0\|' = 0.$$

Therefore, by setting $z_m = G'(u_m)$, $\xi_0 - PV'(Pu_m + y_m) = z$, one has $z_m \rightarrow z$ in L^α . Moreover, $u_m = U'(z_m)$, a.e. in $]0, T[$. A well known

Krasnoselskii's theorem shows that U' is a continuous map from L^α into L^β (note that assumption (H1) implies that $|U'(p)| \leq c|p|^{\alpha-1}$, $\forall p \in \mathbb{R}^n$; argue as in [E], lemma 1). Hence, $u_m \rightarrow U'(z)$ in L^β . The convergence of y_m along some subsequence is obvious. //

The existence of infinitely many T -periodic solutions follows by a well known argument, since each (T/m) -periodic solution ($m \in \mathbb{N}$) is T -periodic. We don't know if our solution has T as the minimal period.

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