

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

DĂNUȚ MARCU

**Note on the gammoids arising from undirected graphs**

*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 83 (1990), p. 1-6

[http://www.numdam.org/item?id=RSMUP\\_1990\\_\\_83\\_\\_1\\_0](http://www.numdam.org/item?id=RSMUP_1990__83__1_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1990, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## Note on the Gammoids Arising from Undirected Graphs.

DĂNUȚ MARCU (\*)

### 1. Introduction.

In this note, we consider strict gammoids, which arise from undirected graphs. We exhibit a minimal example of a strict gammoid, which cannot arise in this way and we interpret the Ingleton and Piff's characterization (see [1]) of the strict gammoids for the undirected case. In a directed graph  $D = (V, F)$ , we say that  $X \subseteq V$  is *linked into*  $Y \subseteq V$ , if there exists a set of mutually disjoint paths in  $D$ , whose set of the initial vertices is  $X$  and whose set of the terminal vertices is a subset of  $Y$ . Given  $A, B \subseteq V$ , the collection of all subsets of  $A$ , which can be linked into  $B$ , is a special type of matroid, known as a *gammoid*. In the case when  $A = V$ , the gammoid is said to be *strict*. This concept translates naturally to an undirected graph  $G$ . One can either replace paths by undirected paths, in the definitions, or one can regard  $G$  as a directed graph, in which each of its edges  $\{u, v\}$  is replaced by two directed edges  $uv$  and  $vu$ . This latter comment was made by Woodall in [3] and he called (strict) gammoids arising from undirected graphs, *undirected (strict) gammoids*. In [3], Woodall gave an example of a strict gammoid, which was not an undirected gammoid, and, in this note, we exhibit a minimal such example and, in passing, we interpret the Ingleton and Piff's characterization of the strict gammoids for the undirected case.

(\*) Indirizzo dell'A.: Str. Pasului 3, Sect. 2, 70241 Bucharest, Romania.

## 2. The main results.

**THEOREM 1.** *Any matroid of rank  $\leq 2$  is an undirected strict gammoid.*

**PROOF.** Let  $M$  be the matroid in question, let  $V$  be its underlying set and let  $X$  be those points of  $V$ , which form independent singletons in  $M$ . Then, the relation « $\sim$ », defined on  $X$  by

$$x \sim y \text{ if } x = y \text{ or if } \{x, y\} \text{ is a circuit of } M, \text{ with } x \neq y,$$

is easily seen to be an equivalence relation on  $X$ . Let its distinct equivalence classes be  $[x_1], \dots, [x_n]$  and let  $G$  be the undirected graph with the vertex set  $V$  and the edge set given by

$$E = \{\{x_i, x_j\} : 1 \leq i < j \leq n\} \cup \left( \bigcup_{1 \leq i \leq n} \{\{x_i, x\} : x \in [x_i]\} \right).$$

Then, it is straightforward to check that  $M$  consists precisely of those subsets of  $V$  linked into  $B$  in  $G$ , where  $B$  is any subset of  $\{x_1, \dots, x_n\}$  of cardinality equal to the rank of  $M$ . Hence,  $M$  is an undirected strict gammoid.  $\square$

A *transversal* of a family of sets  $\mathcal{A} = (A_1, \dots, A_n)$  is a set of  $n$  elements  $\{x_1, \dots, x_n\}$ , with  $x_i \in A_i$ , for each  $i$ . A *partial transversal* of  $\mathcal{A}$  is a transversal of some subfamily of  $\mathcal{A}$ . It is well known that the set of the partial transversal of  $\mathcal{A}$  form a matroid, and one arising in this way is called a *transversal matroid*. In this case,  $\mathcal{A} = (A_1, \dots, A_n)$  is a *presentation* of the matroid and it is well known that a transversal matroid of rank  $n$  has a presentation of a family consisting of precisely  $n$  sets. Of the many presentations of a transversal matroid  $M$  one is called a *minimal presentation*, if it uses the smallest number of sets possible and if none of the sets used can be replaced by a proper subset to give another presentation of  $M$ . Now, a family  $\mathcal{A} = (A_1, \dots, A_n)$  will be called *symmetric*, if there exist distinct  $x_1, \dots, x_n$ , with  $x_i \in A_i$ , for  $1 \leq i \leq n$  and  $x_i \in A_j$  implies that  $x_j \in A_i$ , for  $1 \leq i, j \leq n$ . A transversal matroid will be called *symmetric*, if it possesses such a presentation. So, for example, a transversal matroid of rank  $\leq 2$  is symmetric, a minimal presentation providing

the required symmetric presentation. For, if  $(A_1, A_2)$  is a minimal presentation of a matroid, then it is easy to check that neither  $A_i$  is a subset of the other. Hence, there exist  $x_1 \in A_1 \setminus A_2$  and  $x_2 \in A_2 \setminus A_1$ , from which the symmetry is clear.

**THEOREM 2.** *The duals of the undirected strict gammoids are precisely the symmetric transversal matroids.*

**PROOF.** In [1], Ingleton and Piff have shown that the duals of the transversal matroids are precisely the strict gammoids. More particularly, it follows, from a version of their result in [2, pag. 217], that if  $M$  (on the set  $V$ ) has the presentation  $\mathcal{A} = (A_1, \dots, A_n)$  and a transversal  $\{x_1, \dots, x_n\}$ , with  $x_i \in A_i$  for each  $i$ , and if  $D = (V, F)$  is the directed graph given by

$$F = \bigcup_{1 \leq i \leq n} \{\{x_i, x\} : x \in A_i \setminus \{x_i\}\},$$

then  $X \subseteq V$  is linked into  $B = V \setminus \{x_1, \dots, x_n\}$  if and only if  $V \setminus X$  contains a transversal of  $\mathcal{A}$ . It is therefore easy to check that, in the special case when  $\mathcal{A}$  is symmetric (and the  $x_i$ 's are chosen accordingly), the same result holds for the corresponding undirected graph. Hence, the dual of a symmetric transversal matroid is an undirected strict gammoid. Conversely, if the dual of  $M$  is an undirected strict gammoid and consists of the sets linked into  $B$  in the undirected graph  $G = (V, E)$ , then, from the same result referred to above, it can be deduced that  $V \setminus B$  has exactly  $n$  distinct elements  $x_1, \dots, x_n$  and that  $M$  is the transversal matroid with the presentation  $\mathcal{A} = (A_1, \dots, A_n)$ , where

$$A_i = \{x_i\} \cup \{x : \{x_i, x\} \in E\}, \quad 1 \leq i \leq n.$$

It is clear that  $\mathcal{A}$  is symmetric, and the result follows.  $\square$

We have remarked above that the transversal matroids of rank  $\leq 2$  are symmetric, and we now see that sufficiently small transversal matroids of rank 3 are also symmetric.

**THEOREM 3.** *A transversal matroid of rank 3, on a set of 6 or fewer points, is symmetric.*

PROOF. Let  $M$  be the matroid in question and let  $(A_1, A_2, A_3)$  be a minimal presentation of  $M$ . Then, in particular, if

$$|A_1 \cup A_2 \cup A_3| = m(\leq 6),$$

it follows that

$$(1) \quad |A_i| \leq m - 2 \leq 4, \quad \text{for each } i$$

and

$$(2) \quad A_i \not\subseteq A_j, \quad \text{if } i \neq j.$$

Now, in cases, we exhibit a symmetric presentation of  $M$ .

*Case 1.*  $A_1 \not\subseteq A_2 \cup A_3$ ,  $A_2 \not\subseteq A_1 \cup A_3$  and  $A_3 \not\subseteq A_1 \cup A_2$ .

In this case, of course, there exist

$$x_1 \in A_1 \setminus (A_2 \cup A_3), \quad x_2 \in A_2 \setminus (A_1 \cup A_3) \quad \text{and} \quad x_3 \in A_3 \setminus (A_1 \cup A_2)$$

and it is clear that  $(A_1, A_2, A_3)$  is symmetric.

*Case 2.*  $A_1 \subseteq A_2 \cup A_3$ ,  $A_2 \not\subseteq A_1 \cup A_3$  and  $A_3 \not\subseteq A_1 \cup A_2$ .

In this case, there exist

$$x_2 \in A_1 \setminus A_3 = (A_1 \cap A_2) \setminus A_3 \quad \text{and} \quad x_3 \in A_1 \setminus A_2 = (A_1 \cap A_3) \setminus A_2.$$

If there exists  $x_1 \in A_1 \cap A_2 \cap A_3$ , then the symmetry of  $(A_1, A_2, A_3)$  is clear. So, we may assume that  $A_1 \cap A_2 \cap A_3 = \emptyset$ , such that  $|A_1| = |A_1 \cap A_2| + |A_1 \cap A_3|$ . If  $|A_1 \cap A_2| \geq 2$ , then there exist distinct  $x'_1, x'_2 \in A_1 \cap A_2 = (A_1 \cap A_2) \setminus A_3$  and  $x'_3 \in A_3 \setminus (A_1 \cup A_2)$  and, again, the symmetry of  $(A_1, A_2, A_3)$  is clear. So, finally, we may suppose that  $|A_1 \cap A_2| \leq 1$  and, similarly, that  $|A_1 \cap A_3| \leq 1$ . Then, using (2), it is easy to see that there exist four elements  $x''_1, x''_2, x''_3, x''_4$ , such that  $A_1 = \{x''_1, x''_2\}$ ,  $\{x''_1, x''_3\} \subseteq A_2 \setminus A_3$  and  $\{x''_2, x''_4\} \subseteq A_3 \setminus A_2$ . If we now replace the element  $x''_1$  of  $A_2$  by  $x''_2$ , we get a symmetric presentation of  $M$ , with representatives  $x''_1, x''_3$  and  $x''_4$ .

*Case 3.*  $A_1 \subseteq A_2 \cup A_3$ ,  $A_2 \subseteq A_1 \cup A_3$  and  $A_3 \not\subseteq A_1 \cup A_2$ .

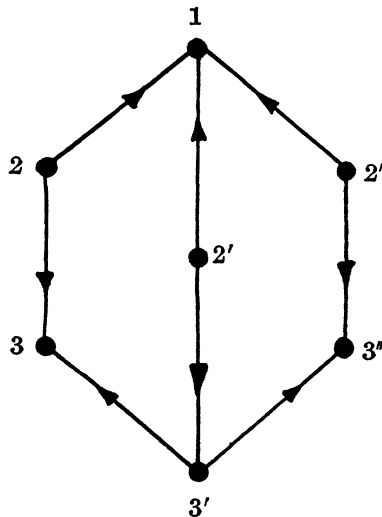
In this case, there exist distinct  $x_1, x_2 \in (A_1 \cup A_2 \cup A_3) \setminus A_3 = A_1 \setminus A_3 = (A_1 \cap A_2) \setminus A_3$  and  $x_3 \in A_3 \setminus (A_1 \cup A_2)$  and, again, the symmetry of  $(A_1, A_2, A_3)$  is clear.

*Case 4.*  $A_1 \subseteq A_2 \cup A_3$ ,  $A_2 \subseteq A_1 \cup A_3$  and  $A_3 \subseteq A_1 \cup A_2$ .

It is not difficult to see that, in this case, any subset of  $A_1 \cup A_2 \cup A_3$ , which has cardinality at most three and is dependent, must be contained in two of the sets  $A_1, A_2$  and  $A_3$  and be disjoint from the third. But then, (1) and (2) lead to a contradiction, in this particular case. Hence, every subset of  $A_1 \cup A_2 \cup A_3$ , of cardinality at most three, is in  $M$  and so  $M$  has the symmetric presentation  $(A_1 \cup A_2 \cup A_3, A_1 \cup A_2 \cup A_3, A_1 \cup A_2 \cup A_3)$ .  $\square$

It is immediate, from the above results, that a strict gammoid, which is not an undirected gammoid, must be of rank at least 3 and on a set of at least 7 elements; below, we present such a gammoid of rank precisely 3 and on a set of precisely 7 elements.

**EXAMPLE.** A minimal strict gammoid, which is not an undirected gammoid. Let  $M$  be the strict gammoid of sets linked into 1, 3, 3', in the directed graph illustrated in the figure:



Then, the circuits of  $M$ , of cardinality 3, are precisely  $\{1, 2, 3\}$ ,  $\{1, 2', 3'\}$ ,  $\{1, 2'', 3''\}$  and  $\{3, 3', 3''\}$ ; all other sets of cardinality  $\leq 3$  are independent. This example (and the verification below that  $M$  is not an undirected gammoid) is not dissimilar to Woodall's, in [3].

Assume that  $M$  is an undirected gammoid, consisting of the subsets of  $\{1, 2, 3, 1', 2', 3', 1'', 2'', 3''\} (\subseteq V)$  linked into a set  $B$  of cardinality 3, in the undirected graph  $G = (V, E)$ . Then, since  $\{3, 3', 3''\}$  is a circuit of  $M$ , it follows, from the Menger's Theorem, that there exist  $x, y \in V$ , such that every path from  $\{3, 3', 3''\}$  to  $B$ , in  $G$ , uses at least one of  $x$  and  $y$ . This means that, in addition, every path from  $\{3, 3', 3''\}$  to  $\{1, 2, 1', 2', 1'', 2''\}$  uses at least one of  $x$  and  $y$ , since, for example, the existence of a path from 3 to 1 avoiding  $x$  and  $y$ , together with the independence of  $1, 3', 3''$ , would imply the existence of a path from 3 to  $B$  avoiding  $x$  and  $y$ . Now, let us call a path from  $v$  to  $\{x, y\}$ , which meets  $\{x, y\}$  only at its terminal vertex, a  $v$ - $x$  path or a  $v$ - $y$  path, depending upon which member of  $\{x, y\}$  it uses. Then, since  $\{3, 1, 2'\} \in M$  but  $\{3', 1, 2'\} \notin M$ , it follows that either there exists a 3- $x$  path but no 3'- $x$  path or that there exists a 3- $y$  path but no 3'- $y$  path. Let us assume the former. A similar argument, applied to  $\{3', 1, 2\} \in M$  and  $\{3, 1, 2\} \notin M$ , shows that there exists a 3'- $y$  path but no 3- $y$  path. Similar arguments, with respect to the pairs 3, 3'' and 3', 3'', show that there exists no 3''- $x$  path and no 3''- $y$  path and, hence, no path from 3'' to  $B$ .

This contradiction shows that  $M$  is not an undirected gammoid.

#### REFERENCES

- [1] A. W. INGLETON - M. J. PIFF, *Gammoids and transversal matroids*, J. Comb. Theory Ser. B, **15** (1973), pp. 51-68.
- [2] D. J. A. WELSH, *Matroid Theory*, Academic Press, London - New York, 1976.
- [3] D. R. WOODALL, *The induction of matroids by graphs*, J. London Math. Soc., **10** (1975), pp. 27-35.

Manoscritto pervenuto in redazione il 13 marzo 1988.