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On the Euler Equations with a Singular External Velocity Field.

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ABSTRACT - We study the Euler equations for an incompressible fluid in presence of a singular external velocity field produced by fixed point vortices. We prove the existence and the uniqueness of the solution.

1. Results.

In this paper we study the existence and the uniqueness of the time evolution of a two dimensional incompressible ideal fluid in presence of a singular external velocity field. The motion is governed by the Euler equations:

$$\partial_t \omega(\boldsymbol{x}, t) + ([\boldsymbol{u}(\boldsymbol{x}, t) + E(\boldsymbol{x}, t)] \cdot \nabla(\omega(\boldsymbol{x}, t) = 0 \qquad \boldsymbol{x} = (x_1, x_2) \in R^2 , \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0 \qquad \text{(continuity equation)} , \tag{1.2}$$

$$\omega(\boldsymbol{x}, t) = \partial_1 u_2 - \partial_2 u_1 , \tag{1.3}$$

$$\omega(\boldsymbol{x}, 0) = \omega_0 , \tag{1.4}$$

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where

> $\boldsymbol{u} = (u_1, u_2)$ is the velocity field, ω the vorticity, $\boldsymbol{E}(\boldsymbol{x}, t)$ an external velocity field that is assumed to satisfy the divergence-free condition

$$\nabla \cdot \boldsymbol{E} = 0 . \tag{1.5}$$

If the velocity field decays at infinity, we can reconstruct it by means of ω:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \int \boldsymbol{K}(x - y)\,\omega(\boldsymbol{y}, t)\, d\boldsymbol{y} \tag{1.6}$$

where

$$\boldsymbol{K}(\boldsymbol{x}) = \nabla^{\perp} g(\boldsymbol{x}) , \tag{1.7}$$

$$\nabla^{\perp} = (\partial_2, -\partial_1) , \tag{1.8}$$

$$g(\boldsymbol{x}) = -(1/2\pi) \ln |\boldsymbol{x}| . \tag{1.9}$$

The problem can be studied by the well known iterative method [1] and an existence and uniqueness theorem can be easily established when the external field is smooth and the initial datum is smooth and has a compact support. Equation (1.1) has no meaning for non smooth ω_0 and $\boldsymbol{E}$; however a weak version of the Euler equations can be introduced for the time evolution of a class of non smooth initial data (see for instance [6]). We do not enter here in this (standard) point. We note only that a similar generalization does not work when the external field $\boldsymbol{E}$ is the sum of a smooth part $\boldsymbol{E}_1$ and a singular part $\boldsymbol{E}_2$ produced by N vortices of intensity a_i fixed in the points $\boldsymbol{z}_i$, $i = 1, \dots, N$. We want to study this problem in the present paper. The physical meaning of this model will be discussed in the following section.

We consider eqs. (1.1)-(1.5) in the case in which

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}_1(\boldsymbol{x}, t) + \boldsymbol{E}_2(\boldsymbol{x}) \tag{1.10}$$

where

$$\boldsymbol{E}_1(\boldsymbol{x}, t) \in C_2 \cap L_\infty(R^2 \times [0, T]) \tag{1.11}$$

and

$$(1.12)\qquad \boldsymbol{E}_2(\boldsymbol{x}) = -\sum_{i=1}^{N} a_i(1/2\pi)\,\nabla^{\perp}\ln|\boldsymbol{x}-\boldsymbol{z}_i| = \\ = \sum_{i=1}^{N} a_i\left(-\frac{x_2 - z_{2,i}}{2\pi|\boldsymbol{x}-\boldsymbol{z}_i|^2}, \frac{x_1 - z_{1,i}}{2\pi|\boldsymbol{x}-\boldsymbol{z}_i|^2}\right)$$

where $\boldsymbol{z}_i = (z_{1,i}, z_{2,i})$.

We suppose the initial datum $\omega_0(\boldsymbol{x})$ regular and with a compact support which does not overlap the singularities of $\boldsymbol{E}_2$; that is

$$(1.13)\qquad \omega_0(\boldsymbol{x}) \in C^2(R^2)\,,$$

$$(1.14)\qquad \operatorname{supp}\omega_0(\boldsymbol{x}) \cap \{\boldsymbol{z}_i\} = \emptyset\,; \qquad i = 1, \dots, N\,.$$

The main result of this paper is given by the following theorem.

THEOREM. *For any $T > 0$ and $0 \leqslant t \leqslant T$, we consider the Euler equations* (1.1)-(1.5) *with the external field $\boldsymbol{E}(\boldsymbol{x})$ given by* (1.10)-(1.12) *and with initial vorticity ω_0 satisfying conditions* (1.13), (1.14). *Then they have a solution unique in the class of function $C^2(R^2)$ with a compact support.*

PROOF. The difficulty of the proof lies in the singularity of the velocity field $\boldsymbol{E}_2$. Otherwise, substituting $\boldsymbol{E}$ by a mollified version $\boldsymbol{E}_\varepsilon$:

$$(1.15)\qquad \boldsymbol{E}_\varepsilon(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x}) \text{ if } \min_i |\boldsymbol{x}-\boldsymbol{z}_i| \geqslant \varepsilon \text{ and } \boldsymbol{E}_\varepsilon \in C^2 \cap L_\infty(R^2, [0, T])$$

the Theorem can be proved by the standard iterative method used in ref. (1). We do not repeat them here but we keep in mind the result under the assumption (1.15).

Of course, as long as supp $\omega(\boldsymbol{x}, t)$ remains far away from the point vortices the external field felt by the fluid is smooth as in the previous case (1.15). But, a priori, the fluid can arrive in $\boldsymbol{z}_i$, where the field becomes singular and the methods of ref. (1) fail to be valid. In this note we will prove that this cannot happen.

Actually the external field $\boldsymbol{E}$ and the velocity field produced by the fluid can push the fluid near the vortices $\boldsymbol{z}_i$. In this region the fluid particles keep turning around $\boldsymbol{z}_i$ very fastly and the particle paths become quasi-circular while the time-average of the velocity field becomes very small due to the divergence-free of the velocity field and the Gauss-Green Lemma.

We give a simple rigorous proof of this fact. (A proof that perhaps does not outline the real average property.) For the sake of simplicity we suppose the presence of only a single vortex of intensity one in the origin

$$z_i = \mathbf{0}\,. \tag{1.16}$$

As usual, we introduce the characteristics of eqs. (1.1). We denote by $x(t, x_0)$ the trajectory of the particle of fluid initially in x_0, which satisfy the equations

$$\mathring{x}(t, x_0) = u(x, t) + E(x, t)\,, \tag{1.17}$$

$$u(x, t) = \int K(x - y)\,\omega(x, t)\,dy\,, \tag{1.18}$$

$$x(0, x_0) = x_0\,. \tag{1.19}$$

The solution is obtained by the invariance of the vorticity along the particle paths:

$$\omega(x, t) = \omega\big(x(-t, y), 0\big)\,. \tag{1.20}$$

For the moment we suppose that a unique solution exists in $[0, T]$ and we follow the time evolution of a particle initially in the support of the vorticity:

$$x(0, x_0) = x_0 \in \operatorname{supp} \omega_0\,. \tag{1.21}$$

We prove that this particle path cannot arrive in the origin, that is

$$d = \inf_{\substack{x_0 \in \operatorname{supp} \omega_0 \\ 0 \leqslant t \leqslant T}} |x(t, x_0)| > 0 \tag{1.22}$$

for any $T > 0$.

To prove that d is strictly positive, we introduce a suitable Liapunov function $H(x, t)$:

$$H(x, t) = -(1/2\pi) \ln |x| + \Psi(x, t) \tag{1.23}$$

where $\Psi = \Psi_1 + \Psi_2$ is the stream function defined as

$$\nabla^\perp \Psi_1(\boldsymbol{x}, t) = \boldsymbol{E}_1(\boldsymbol{x}, t)\,, \tag{1.24}$$

$$\Psi_2(\boldsymbol{x}, t) = -(1/2\pi)\int \ln|\boldsymbol{x}-\boldsymbol{y}|\omega(\boldsymbol{y}, t)\,d\boldsymbol{y}\,. \tag{1.25}$$

It is easy to see that $|\boldsymbol{H}(\boldsymbol{x}, t)|$ becomes infinite if and only if $\boldsymbol{x} = 0$. In fact the velocity field produced by the fluid itself is bounded:

$$\begin{aligned}|\boldsymbol{u}_{\text{fluid}}| &= \Big|\int_{\operatorname{supp}\omega(\boldsymbol{x},t)} (-1/2\pi)\nabla^\perp \ln|\boldsymbol{x}-\boldsymbol{y}|\omega(\boldsymbol{y}, t)\,d\boldsymbol{y}\Big| \leqslant \\ &\leqslant \sup_{\boldsymbol{y}\in R^2}|\omega(\boldsymbol{y}, t)|\Big|\int_{\operatorname{supp}\omega(\boldsymbol{x},t)}(1/2\pi)\nabla^\perp\ln|\boldsymbol{x}-\boldsymbol{y}|\,d\boldsymbol{y}\Big| \leqslant \\ &\leqslant \sup_{\boldsymbol{x}\in R^2}|\omega_0(\boldsymbol{y})|(1/2\pi)\int_0^{\varrho_0}(1/p)\int_0^{2\pi}\varrho\,d\varrho\,d\theta < \infty\end{aligned} \tag{1.26}$$

where

$$\pi\varrho_0^2 = \text{meas supp }\omega(\boldsymbol{y}, t) = \text{meas supp }\omega_0(\boldsymbol{x})$$

(meas denotes the Lebesgue measure). Moreover in the (1.26) we have used the equality (consequence of (1.20))

$$\sup_{\boldsymbol{y}\in R^2}|\omega(\boldsymbol{y}, t)| = \sup_{\boldsymbol{x}\in R^2}|\omega_0(\boldsymbol{y})|\,. \tag{1.27}$$

Moreover for large $|\boldsymbol{x}|$, $\boldsymbol{E}(\boldsymbol{x}, t)$ is bounded, so that the fluid remains in bounded regions during finite time, and in this region

$$\big|\ln|\boldsymbol{x}|\big| \to \infty \quad \text{only if } |\boldsymbol{x}| \to 0\,. \tag{1.28}$$

Furthermore $\Psi_1(\boldsymbol{x})$ is bounded in a bounded region, because of $\nabla^\perp\Psi_1(\boldsymbol{x})$ is bounded.

Finally we note that

$$\begin{aligned}|\Psi_2(\boldsymbol{x}, t)| &= \Big|(-1/2\pi)\int \ln|\boldsymbol{x}-\boldsymbol{y}|\omega(\boldsymbol{y}, t)\,d\boldsymbol{y}\Big| \leqslant \\ &\leqslant \sup_{\boldsymbol{x}\in R^2}|\omega_0(\boldsymbol{x})|\Big|\int_0^{\varrho_0} d\varrho\,\varrho\ln\varrho < \infty\,.\end{aligned} \tag{1.29}$$

Therefore condition (1.14) implies that $H(\boldsymbol{x}, 0)$ is bounded. We want to prove that $H(\boldsymbol{x}, t)$ remains bounded for every time which implies (1.22).

For any t, $0 \leqslant t \leqslant T$, we have

$$|H(\boldsymbol{x}, t)| \leqslant |H(\boldsymbol{x}, 0)| + T \sup_{0 \leqslant t \leqslant T} |dH(\boldsymbol{x}, t)/dt| \tag{1.30}$$

along the curve $\boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{x}_0)$.

Hence it is enough to prove that

$$|dH(\boldsymbol{x}, t)/dt| < \text{const} < \infty . \tag{1.31}$$

We have

$$\begin{aligned} dH(\boldsymbol{x}, t)/dt &= \\ &= \nabla[(-1/2\pi) \ln |\boldsymbol{x}| + \Psi_1(\boldsymbol{x}) + \Psi_2(\boldsymbol{x}, t)] \cdot \mathring{\boldsymbol{x}}(t, \boldsymbol{x}_0) + \\ &+ \partial_t \Psi(\boldsymbol{x}, t) = \partial_t \Psi(\boldsymbol{x}, t) \end{aligned} \tag{1.32}$$

by the equations of motion (1.17)-(1.19) and the identity

$$\nabla f(\boldsymbol{x}) \cdot \nabla^{\perp} f(\boldsymbol{x}) = 0 . \tag{1.33}$$

Equation (1.32) means that the « potential energy » H is a constant of motion a part the explicit time dependence of the velocity field.

We evaluate $\partial_t \Psi$:

$$\partial_t \Psi = \partial_t \Psi_1 + \partial_t \Psi_2 . \tag{1.34}$$

The regularity properties of $\boldsymbol{E}_1$ imply that $\partial_t \Psi_1$ is bounded. Moreover

$$|\partial_t \Psi_2(\boldsymbol{y}, t)| = \left| \int (-1/2\pi) \ln |\boldsymbol{x} - \boldsymbol{y}| \, \partial_t \omega(\boldsymbol{y}, t) \, d\boldsymbol{y} \right| = \tag{1.35}$$

$$= \left| \int (1/2\pi) \ln |\boldsymbol{x} - \boldsymbol{y}| [\boldsymbol{u}(\boldsymbol{y}, t) + \boldsymbol{E}(\boldsymbol{y}, t)] \cdot \nabla \omega(\boldsymbol{y}, t) \, d\boldsymbol{y} \right| . \tag{1.36}$$

We suppose, for the moment, that condition (1.22) holds so that the integrand is smooth and we can integrate by parts. The boundary term vanishes and by (1.2) and (1.5) we obtain:

$$= \left| (1/2\pi) \nabla_y \ln |\boldsymbol{x} - \boldsymbol{y}| \cdot [\boldsymbol{u}(\boldsymbol{y}, t) + \boldsymbol{E}_1(\boldsymbol{y}, t) + \boldsymbol{E}_2(\boldsymbol{y})] \omega(\boldsymbol{y}, t) \, d\boldsymbol{y} \right| . \tag{1.37}$$

We study these terms separately:

$$\left|\int (1/2\pi)\,\boldsymbol{\nabla}_y \ln|\boldsymbol{x}-\boldsymbol{y}|\cdot[\boldsymbol{u}(\boldsymbol{y},t)+\boldsymbol{E}_1(\boldsymbol{y},t)]\omega(\boldsymbol{y},t)\,d\boldsymbol{y}\right| \leqslant \tag{1.38}$$

$$\leqslant (1/2\pi)\sup_{\boldsymbol{y}\in R^2}|\omega(\boldsymbol{y},t)|\sup_{\boldsymbol{y}\in R^2}[\boldsymbol{u}(\boldsymbol{y},t)+\boldsymbol{E}_1(\boldsymbol{y},t)]\int_{\operatorname{supp}\omega(\boldsymbol{y},t)} d\boldsymbol{y}\,|\boldsymbol{\nabla}_y \ln|\boldsymbol{x}-\boldsymbol{y}|| \leqslant$$

$$\leqslant \text{const}\int_0^{\varrho_0} d\varrho < \infty\,.$$

We study the last term.

$$\int (1/2\pi)\big|[\boldsymbol{\nabla}_y \ln|\boldsymbol{x}-\boldsymbol{y}|\cdot\boldsymbol{E}_2(\boldsymbol{y})]\omega(\boldsymbol{y},t)\big|\,d\boldsymbol{y} = \tag{1.39}$$

$$= (2\pi)^{-2}\int\big|[\boldsymbol{\nabla}_y \ln|\boldsymbol{x}-\boldsymbol{y}|\cdot\boldsymbol{\nabla}_{\boldsymbol{y}}^{\perp}\ln|\boldsymbol{y}|]\omega(\boldsymbol{y},t)\big|\,d\boldsymbol{y} \leqslant$$

$$\leqslant (2\pi)^{-2}\sup_{\boldsymbol{y}\in R^2}|\omega(\boldsymbol{y},t)|\int_{\operatorname{supp}\omega(\boldsymbol{y},t)}\big|\boldsymbol{\nabla}_y \ln|\boldsymbol{x}-\boldsymbol{y}|\cdot\boldsymbol{\nabla}_{\boldsymbol{y}}^{\perp}\ln|\boldsymbol{y}|\big|\,d\boldsymbol{y} \leqslant$$

$$\leqslant \text{const}\int_{\operatorname{supp}\omega(\boldsymbol{y},t)}|\boldsymbol{x}-\boldsymbol{y}|^{-1}|\sin\alpha|\,|\boldsymbol{y}|^{-1}\,d\boldsymbol{y}$$

where α denotes the angle between $\boldsymbol{y}$ and $\boldsymbol{y}-\boldsymbol{x}$.

We consider two cases: $|\boldsymbol{x}|\geqslant 2\varrho_0$ or $|\boldsymbol{x}|<2\varrho_0$.

In the first case the integral is trivially bounded:

$$(1.39) \leqslant \text{const}\int_{\operatorname{supp}\omega(\boldsymbol{y},t)}|\boldsymbol{x}-\boldsymbol{y}|^{-1}|\boldsymbol{y}|^{-1}\,d\boldsymbol{y} \leqslant \text{const}\int_0^{\varrho_0}(\varrho\varrho_0)^{-1}\varrho\,d\varrho < \infty\,. \tag{1.40}$$

In the second case the double integral plays a fundamental rôle:

$$(1.40) \leqslant \text{const}\int_0^{2\varrho_0} dr\int_0^{2\pi} d\theta\,\frac{|\boldsymbol{x}|\,|\sin\theta|}{\big|r^2+|\boldsymbol{x}|^2-2r|\boldsymbol{x}|\cos\theta\big|} \tag{1.41}$$

where we have used polar coordinates centered in the origin such that $\boldsymbol{y}=\boldsymbol{x}$ corresponds to $\theta=0$ and we have also used the identity

$$|\boldsymbol{x}|\,|\sin\theta| = |\boldsymbol{y}-\boldsymbol{x}|\,|\sin\alpha|\,. \tag{1.42}$$

We perform the integral. We put $z = 2r|\boldsymbol{x}|\cos\theta$ and we have

$$(1.43)\qquad (1.41) \leqslant \text{const}\int_0^{2\varrho_0}(r)^{-1}\,dr\int_{-2r|\boldsymbol{x}|}^{2r|\boldsymbol{x}|}dz\,\frac{1}{r^2+|\boldsymbol{x}|^2-z} =$$

$$= \text{const}\int_0^{2\varrho_0}(r)^{-1}\ln\left\{\frac{r+|\boldsymbol{x}|}{r-|\boldsymbol{x}|}\right\}^2 dr = \text{const}\int_0^{2\varrho_0|\boldsymbol{x}|^{-1}}(z')^{-1}\ln\left\{\frac{z'+1}{z'-1}\right\}^2 dz' <$$

$$< \text{const}\int_0^{\infty}(z')^{-1}\ln\left\{\frac{z'+1}{z'-1}\right\}^2 dz' = C < \infty$$

where C is a constant independent of $|\boldsymbol{x}|$.

The last integral is bounded because of the singularity in $z'=1$ is integrable and the integrand vanishes at infinity as z'^{-2}.

This achieves the proof of (1.22).

The further steps for a rigorous proof are straightforward. We substitute $\boldsymbol{E}$ by $\boldsymbol{E}_\varepsilon$ so that the solution do exist unique. The solution we look for in this paper corresponds to $\varepsilon \to 0$. In fact when $\varepsilon \leqslant d$ all solutions corresponding to different $\boldsymbol{E}_\varepsilon$ satisfy the same property (1.22) and then they coincide.

2. Comments.

a) The regularity properties on ω_0 are not essential and the technique of this paper can be applied to a weak form of the Euler equations with initial data in $L_1 \cap L_\infty$ as well. (For a weak form of the Euler equations without an external field with such initial data see for instance [6].)

b) In the present paper we have assumed that the fluid moves in $D = R^2$, but the result holds unchanged in bounded domain: of course we must suppose that the vortices $\boldsymbol{z}_i$ do not belong to the boundary ∂D.

c) The hypothesis (1.14) is essential in our proof at most for the uniqueness. The existence of the solution can be proved in general by compactness arguments starting from the solutions relative to the field $\boldsymbol{E}_\varepsilon$ and using the Ascoli-Arzelà Lemma as $\varepsilon \to 0$.

c') The present technique still work if we replace condition (1.14) by the assumption of the existence of a neighborhood of $\boldsymbol{z}_i$ in which $\omega_0(\boldsymbol{x})$ is equal to $\omega_0(\boldsymbol{z}_i)$.

d) The present problem is a mathematical schematisation of the interaction between point vortices and a fluid with smooth vorticity (the so called vorttex-wave problem). Here we have supposed the vortices fixed. Another scheme (perhaps more realistic) supposes that a vortex moves in the velocity field produced by the smooth vorticity and by other vortices (but not by itself). In this case we can prove a result analogous to Theorem I assuming as Liapunov function the distance between the particle path and the vortices and using the quasi-Lipschitz property of the velocity field [7]. Generalization like *c*) can also be done [7].

Finally we note that rigorous connections between point vortices and Euler equations have been widely studied [2, 3, 4, 5, 6, 8].

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