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Fourier integral operators of infinite order
on $\mathcal{D}_{L^2}^{\{\sigma\}} \left( \mathcal{D}_{L^2}^{\{\sigma\}'} \right)$ with an application to a
certain Cauchy problem

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Fourier Integral Operators of Infinite Order on $\mathcal{D}_L^{(\sigma)}(\mathcal{D}_L^{(\sigma)'})$
with an Application to a Certain Cauchy Problem.

ROSSELLA AGLIARDI (*)

Introduction.

The aim of this paper is to develop a calculus of Fourier integral operators of infinite order in the spaces $\mathcal{D}_L^{(\sigma)}(\mathcal{D}_L^{(\sigma)'})$ and to apply it to prove some sufficient conditions for a certain Cauchy problem to be well-posed in the above-mentioned spaces.

The calculus we develop here is analogous to the one in [4] in Gevrey classes and in their dual spaces of ultradistributions. As for the spaces $\mathcal{D}_L^{(\sigma)}(\mathcal{D}_L^{(\sigma)'})$ we consider here, we recall that they have been employed many a time in dealing with the Cauchy problem and the propagation of Gevrey singularities. For instance in [15] it is shown that some pseudo-differential and Fourier integral operators of finite order continuously map $\mathcal{D}_L^{(\sigma)}(\mathcal{D}_L^{(\sigma)'})$ to themselves and the same thing is true for the fundamental solution of a hyperbolic equation with constant multiplicities constructed in [14]. Specifically in [14] the hyperbolic equation is reduced to an equivalent system. Therefore at first a fundamental solution is determined for an operator of the form

\[ P = \partial_t - i\lambda(t, x, D_x) + a(t, x, D_x) \]

where the symbol of $\lambda$ is real, $\lambda$ and $a$ are continuous in $t$ with values in some spaces of symbols of Gevrey type $\sigma$ and of order $1$ and $p$ respectively, for some $p \in [0, 1]$. A fundamental solution is found which maps $\mathcal{D}_L^{(\sigma)}(\mathcal{D}_L^{(\sigma)'})$ to itself whenever $\sigma < 1/p$. The well-posedness of the Cauchy problem for an operator of the form (I) is well-known when $\sigma < 1/p$ (see also [11]). A necessary condition for the well-

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This work is dedicated to the memory of Prof. L. Cattabriga.
posedness in the case $a > 1/p$ is proved in [10]. There it is assumed that whenever
\[ a(t, x, \xi) = \bar{a}(t, x, \xi) + \bar{a}(t, x, \xi) \]
with $\bar{a}$ homogeneous in $\xi$ of degree $p$ and order $\bar{a} < p$, the following condition is required for the well-posedness of the Cauchy problem (with initial datum at $t = 0$) when $a > 1/p$:
\[ \Re \bar{a}(0, x, \xi) > 0 \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n. \]

In what follows we shall prove some sufficient conditions for the Cauchy problem to (I) to be well-posed when $a > 1/p$ (see [2] for the case where $\lambda$ and $a$ do not depend on $x$). Moreover we shall confine our discussion to analytical symbols; we refer to [1] for the Gevrey case in the simplified case where $\lambda = 0$. Specifically we shall prove the following

**Theorem.** Let $P$ be an operator of the form (I) where $\lambda$ and $a$ are pseudo-differential operators whose symbols satisfy the following properties:

1) $\lambda(t, x, \xi)$ is real valued and belongs to $C([0, T]; \mathcal{S}^{1,1,1}(\mathbb{R}^{2n}))$ for $t \in [0, T]$ and $\varphi$ solves the eiconal equation:

\[ \begin{cases} 
\nabla_x \varphi(t', s; x, \xi) = \lambda(t, x, \nabla_x \varphi(t', s; x, \xi)), \\
\varphi(s, x; x, \xi) = x \cdot \xi
\end{cases} \quad \text{for } t \in [s, T'], \text{ for a suitable } T' \in [s, T], \]

then we claim that
\[ \lim_{t \to +\infty} q^{-1/\alpha} \int_s^t \Re a(t', x, \nabla_x \varphi(t', s; x, q\eta)) \, dt' > 0 \]

$\forall t \in [s, T']$, uniformly with respect to $x, \eta \in \mathbb{R}^n \times S_{n-1}$, where we assume $\sigma(2p - 1) < 1, \sigma > 1$. Then the Cauchy problem for $P$ with datum at $t = s$ is well-posed in $\mathcal{D}^{(2)}_k(\mathcal{D}^{(2')}_{\mathcal{L}})$.

This paper is organized as follows. In § 0 we give some preliminary definitions. § 1 is devoted to the development of a calculus for Fourier integral operators of infinite order in $\mathcal{D}^{(2)}_k(\mathcal{D}^{(2')}_{\mathcal{L}})$ which allows us to prove the above mentioned result concerning the well-posedness of the Cauchy problem in these spaces. Indeed in § 2, by applying some
results of § 1, we construct a parametrix for the Cauchy problem by solving the transport equations, as in [3] and [4]. Under our assumptions it turns out to be a Fourier integral operator of infinite order of the kind examined previously. Then a fundamental solution is determined. Finally we give some results concerning the propagation of Gevrey singularities. I wish to thank Prof. D. Mari and Prof. L. Zanghirati for some suggestions.

0. Main notation and definitions.

For $\xi \in \mathbb{R}^n$ we set $\langle \xi \rangle = \sqrt{1 + \sum_{j=1}^{n} \xi_j^2}$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we write $D^\alpha_x = D^\alpha_{x_1} \cdots D^\alpha_{x_n}$, where $D_{x_j} = -i \partial_{x_j}$.

By $\langle D_{x_j} \rangle^N$ we mean a pseudo-differential operator of order $N$ whose symbol is $\langle \xi \rangle^N$.

We recall here the notation concerning symbols of infinite order of Gevrey type that can be found in [3].

We shall say that $p(x, \xi) \in S^0_{\sigma, \mu}(\mathbb{R}^n \times \mathbb{R}^n_{B_x, B_1})$, where $A, B, B_0 > 0$, $\sigma > 1$, $\mu > 1$, if $\forall \varepsilon > 0$ there exists $C_\varepsilon > 0$ such that:

$$
\sup_{x \in \mathbb{R}^n} \| \partial^{\alpha}_{x} D^\beta_x p(x, \xi) \| < C_\varepsilon A^{|\alpha + \beta|} \xi^{|\alpha|} \langle \xi \rangle^{-1 - |\alpha|} \exp \left[ \varepsilon \langle \xi \rangle^{|1/\sigma|} \right]
$$

$\forall \alpha, \beta \in \mathbb{N}^n$, $\forall \xi \in \mathbb{R}^n$, $|\xi| > B_0 + B |x|^\sigma$. We shall write $S^0_{\sigma, \mu}(\mathbb{R}^n \times \mathbb{R}^n_{B_x, B_1})$ for $S^0_{\sigma, \mu}(\mathbb{R}^n \times \mathbb{R}^n_{B_x, B_1})$. We shall denote by $S^0_{\sigma, \mu}$ the space $A, B, B_0 \rightarrow +\infty$

As for formal series of symbols we shall say that $\sum_{j \geq 0} p_j(x, \xi)$ is in $FS^0_{\sigma, \mu}(\mathbb{R}^n \times \mathbb{R}^n_{B_x, B_1})$ if $p_j(x, \xi) \in S^0_{\sigma, \mu}(\mathbb{R}^n \times \mathbb{R}^n_{B_x, B_1})$ and $\forall \varepsilon > 0$ $\exists C_\varepsilon > 0$ such that:

$$
\sup_{x \in \mathbb{R}^n} \| \partial^{\alpha}_{x} D^\beta_x p_j(x, \xi) \| < C_\varepsilon A^{|\alpha + \beta| + \alpha! \langle \beta ! \rangle} \langle \xi \rangle^{-1 - |\alpha| - |\beta|} \exp \left[ \varepsilon \langle \xi \rangle^{|1/\sigma|} \right]
$$

$\forall \xi \in \mathbb{R}^n$, $|\xi| > B_0 + B (|x| + j)^\sigma$.

We shall give the following definition of equivalence of formal series of symbols. We shall write that $\sum_{j \geq 0} p_j(x, \xi) \sim 0$, if $\forall \varepsilon > 0$ $\exists C_\varepsilon > 0$ and $A, B, B_0 \geq 0$ such that:

$$
\sup_{x \in \mathbb{R}^n} \| \partial^{\alpha}_{x} D^\beta_x \sum_{j \leq s} p_j(x, \xi) \| < C_\varepsilon A^{|\alpha + \beta| + \alpha! \langle \beta ! \rangle} \langle \xi \rangle^{-1 - |\alpha| - |\beta|} \exp \left[ \varepsilon \langle \xi \rangle^{|1/\sigma|} \right]
$$

$\xi \in \mathbb{R}^n$, $|\xi| > B_0 + B (|x| + s)^\sigma$. 

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In [15] it is shown how to construct a «true» symbol from a series of formal symbols denoted by \( \sum_{i \geq 0} p_i \), i.e. \( p(x, \xi) \in S^0_{b, \sigma, \mu} \) is found in order to have \( p \sim \sum_{i \geq 0} p_i \).

We recall also that whenever \( p \sim 0 \) in \( F S^0_{b, \sigma, \mu} \) then \( \exists A, B_0, C > 0 \) and \( h > 0 \) such that:

\[
\sup_{x \in \mathbb{R}^n} |D_x^\beta p(x, \xi)| \leq CA|\beta|^\sigma \exp \left[-h|\xi|^{1/\sigma}\right] \quad \forall \beta \in \mathbb{N}^n, \quad \forall |\xi| > B_0.
\]

Now we recall here a few definitions concerning the spaces which are of interest in this work. We refer to [8] and [14] for a more detailed outline. For notational convenience we shall often omit to point out the domain when \( \mathbb{R}^n \) is meant.

Let \( \mathcal{D}^{[\ell]}_{L^2, \epsilon} = \{ f \in L^2; \exp[\epsilon \langle \xi \rangle^{1/\sigma}] \cdot (\mathcal{F}f)(\xi) \in L^2 \} \) where \( \epsilon > 0, \sigma > 1 \). (By \( \mathcal{F}f \) we denote the \( L^2 \)-Fourier transform of \( f \)).

Let \( \mathcal{D}^{[\ell]' }_{L^2, \epsilon} \) be the dual space of the Hilbert space \( \mathcal{D}^{[\ell]}_{L^2, \epsilon} \) and denote by \( \mathcal{F} \) the transpose of the operator \( \mathcal{F} : \{ v \in L^2; \exp[\epsilon \langle \xi \rangle^{1/\sigma}] v(\xi) \in L^2 \} \rightarrow \mathcal{D}^{[\ell]}_{L^2, \epsilon} \).

By using the notation \( \mathcal{D}^{[\ell]}_{L^2, \epsilon} \) for any real number \( \epsilon \) and by denoting both \( \mathcal{F}f \) and \( \mathcal{F}f \) by \( \mathcal{F} \), we get

\[
\mathcal{D}^{[\ell]}_{L^2, -\epsilon} = \mathcal{D}^{[\ell]}_{L^2, \epsilon}
\]

and we denote the norm in these spaces by

\[
\|f\|_{\mathcal{D}^{[\ell]}_{L^2, \epsilon}} = \|\exp[\epsilon \langle \xi \rangle^{1/\sigma}] \mathcal{F}(\xi)\|_{L^2(\mathbb{R}^n)}.
\]

Afterwards we define:

\[
\mathcal{D}^{[\ell]}_{L^2} = \lim_{\epsilon \rightarrow 0^+} \mathcal{D}^{[\ell]}_{L^2, \epsilon}, \quad \mathcal{D}^{[\ell]}_{L^2} = \lim_{\epsilon \rightarrow +\infty} \mathcal{D}^{[\ell]}_{L^2, \epsilon},
\]

\[
\mathcal{D}^{[\ell]'}_{L^2} = \lim_{\epsilon \rightarrow 0^+} \mathcal{D}^{[\ell]'}_{L^2, \epsilon}, \quad \mathcal{D}^{[\ell]'}_{L^2} = \lim_{\epsilon \rightarrow +\infty} \mathcal{D}^{[\ell]'}_{L^2, \epsilon}.
\]

The following spaces will also be used:

\[
S_{\alpha, \epsilon} = \{ f \in S; \exp[\epsilon \langle \xi \rangle^{1/\sigma}] \mathcal{F}(\xi) \in S \}, \quad \epsilon > 0, \sigma > 1.
\]
which are Fréchet spaces with the semi-norms:

\[ |f|_{s_\alpha} = \sup_{\xi \in \mathbb{R}^n} \sup_{|\alpha| + k - 1} |\xi|^k \partial_\xi^\alpha (\exp[\xi|\xi|^{1/\sigma}] f(\xi)) \], \quad l = 0, 1, \ldots .

Finally we shall denote by \( \mathcal{S}_{\sigma} \) the space of all \( f \in C^\infty(\mathbb{R}^n) \) satisfying:

\[ \sup_{\xi \in \mathbb{R}^n} |\partial_\xi^\alpha f(x)| < C_0 C_1 \alpha! \sigma \quad \text{for some} \quad C_0, C_1 > 0, \quad \sigma > 1 . \]

1. Fourier integral operators of infinite order on \( \mathcal{D}_L^{(q)} \) (\( \mathcal{D}_L^{(q')} \)).

In the following pages we shall study operators with amplitude in \( S^{\sigma}_{\sigma_0, \mu} \) and phase \( \varphi \in S^{1, \sigma_0, \mu}(\mathbb{R}^n \times \mathbb{R}^n, 0; A(\varphi)) \) satisfying:

(i) \[ \sum_{|\alpha + \beta| \leq 2} \sup_{\xi, \tilde{\xi} \in \mathbb{R}^n} \frac{|\tilde{\xi}|}{|\xi|} \frac{|\partial_\xi^\alpha \partial_{\tilde{\xi}}^\beta D^2_{\mathcal{F}}(\varphi(x, \xi) - \varphi(x, \tilde{\xi}))|}{|\xi|} < \tau \quad \text{for some} \quad \tau \in [0, 1[ \]

(see [9], Def. 1.2 in Cap. 10).

In this paragraph the action of the above-mentioned operators on the spaces \( S_{\sigma_0} \) and \( \mathcal{D}_L^{(q)} \) is investigated. Notice that the operator defined by:

(1.1) \[ P_\varphi u(x) = \int \exp[i\varphi(x, \xi)] p(x, \xi) \tilde{u}(\xi) \varphi \xi \quad \forall u \in S_{\sigma_0} \text{ or } \forall u \in \mathcal{D}_L^{(q)} \]

is well-defined. Moreover we have the following

**Theorem 1.1.** If \( p(x, \xi), \varphi(x, \xi) \) are as above, then \( \forall \epsilon > 0 \) there exists \( \delta_\epsilon > 0 \) such that \( \forall \delta \in [0, \delta_\epsilon] \) the operator \( P_\varphi \) defined by (1.1) is a continuous map from \( S_{\sigma_0} \) to \( S_{\sigma_0, \delta} \) and from \( \mathcal{D}_L^{(q)} \) to \( \mathcal{D}_L^{(q, \delta)} \).

Therefore \( P_\varphi \colon \mathcal{D}_L^{(q)} \to \mathcal{D}_L^{(q, \delta)} \) is a continuous map.

Moreover \( P_\varphi \) extends to a continuous operator from \( \mathcal{D}_L^{(q')} \) to \( \mathcal{D}_L^{(q', \delta')} \).

Now let us consider \( \sigma \)-regularizing operators.

If \( r(x, \xi) \) has the property:

(i) \[ \sup_{\xi \in \mathbb{R}^n} |D^2_\xi r(x, \xi)| < C A \beta \sigma \exp[-h|\xi|^{1/\sigma}] \]

with \( C, A, h > 0, \forall \beta \in \mathbb{N}^n, \forall \xi \in \mathbb{R}^n, |\xi| \) sufficiently large, then the operator defined by

\[ R_\varphi u(x) = \int \exp[i\varphi(x, \xi)] r(x, \xi) \tilde{u}(\xi) \varphi \xi \quad \forall u \in \mathcal{D}_L^{(q)} \]

extends to an operator defined on \( \mathcal{D}_L^{(q')} \).
Furthermore it follows that
\[
\sup_{x \in \mathbb{R}^n} |D_x^\beta R_\varphi u(x)| \leq C |\beta|!^\sigma \quad \text{for every } u \in \mathcal{D}^{\sigma'}_R,
\]
that is, $R_\varphi$ maps $\mathcal{D}^{\sigma'}_R$ to $\mathcal{E}^\sigma_0$.

More precisely we can prove:

**Theorem 1.2.** If $r(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ has the property

\[
(\text{ii}) \sup_{x \in \mathbb{R}^n} |\partial_\xi^\beta D_\xi^\beta r(x, \xi)| \leq C |\beta|!^\sigma \exp \left(-h |\xi|^{1/\sigma}\right)
\]

for $|\xi| \geq A_1 + B|x|^\alpha$ and with $h > 0$, then the operator $R_\varphi$ defined on $\mathcal{D}^{\sigma'}_R$ by

\[
R_\varphi u(x) = \int \exp [i\varphi(x, \xi)] r(x, \xi) \hat{u}(\xi) d\xi
\]
is a continuous map from $\mathcal{D}^{\sigma'}_R$ to $\mathcal{D}^{\sigma}_L$.

**Theorem 1.3** (composition of a pseudo-differential and a Fourier integral operator). Let $p_1(x, \xi) \in S^0_{\sigma, 1} (\mathbb{R}^n \times \mathbb{R}^n_{\sigma, 0})$, $p_2(x, \xi) \in S^0_{\sigma, \mu} (\mathbb{R}^n \times \mathbb{R}^n_{\sigma, 0})$, and the phase $\varphi(x, \xi) \in S^1_{\sigma, \mu} (\mathbb{R}^n \times \mathbb{R}^n_{\sigma, 0})$ with property (i).

Let

\[
P_{1}(u) = \int \exp [i\varphi(x, \xi)] p_1(x, \xi) \hat{u}(\xi) d\xi;
\]

\[
P_{2}(u) = \int \exp [i\varphi(x, \xi)] p_2(x, \xi) \hat{u}(\xi) d\xi
\]
for every $u \in \mathcal{D}^{\sigma}_L$.

Then there exists an operator $Q_\varphi$ defined on $\mathcal{D}^{\sigma}_L$ by

\[
Q_\varphi u(x) = \int \exp [ib(x, \xi)] q(x, \xi) \hat{u}(\xi) d\xi
\]
and such that $q(x, \xi) \sim \sum_{i>0} g_i(x, \xi)$, where

\[
g_i(x, \xi) = \sum_{|\gamma| = i} 1/\gamma ! D_\xi^\gamma \left((\partial_\xi^\gamma p_1(x, \xi) \tilde{\varphi} \tilde{\varphi} (x, y, \xi)) p_2(y, \xi)\right)_{y = x}
\]
with

\[
\tilde{\varphi} \varphi (x, y, \xi) = \int_0^1 \nabla \varphi (y + t(x - y), \xi) d\theta,
\]
and there exists an operator $R_\varphi$ continuously mapping $D^{(\sigma)}_{L^2}$ to $D^{(\sigma)}_{L^2}$ such that:

$$P_1(x, D_x)(P_2 \varphi(x, D_x)u(x)) = Q_\varphi(x, D_x)u(x) + R_\varphi(x, D_x)u(x) \quad \forall u \in D^{(\sigma)}_{L^2}.$$  

Wave front sets in Gevrey classes. When $u \in D^{(\sigma)}_{L^2}'$, we give the following definition of wave front set.

**Definition** (see [15]). Let $u \in D^{(\sigma)}_{L^2}$ with $\sigma > 1$; let $\sigma_1 \geq \sigma$. We say that a point $(x_0, \xi_0) \in \mathcal{T}^\ast(R^n)$ does not belong to $WF_{(\sigma)}(u)$ when there exists a symbol $\alpha(x, \xi)$ in $S^{0,-\sigma}$ with $\alpha(x_0, \theta \xi_0) \neq 0 (\theta > 1)$ such that $\alpha(x, D_x)u(x) \in \mathcal{E}_b^{(\sigma)}(R^n)$.

**Remark.** In [14] Taniguchi has shown that this definition is equivalent to the one given by Hörmander (1971) in the case where $u \in \mathcal{E}'$.

For Fourier integral operators of the kind analyzed above we have the following result:

**Theorem 1.4.** If $p$ and $\varphi$ satisfy the properties listed previously and moreover $\varphi(x, \xi)$ is homogeneous in $\xi$ of degree 1 for $|\xi|$ sufficiently large, then for every $\sigma_1 \geq \sigma$ we have:

$$WF_{(\sigma_1)}(P \varphi u) \subset \{(x, \theta \nabla_x \varphi(x, \xi)); \quad \theta > 1, (\nabla_x \varphi(x, \xi), \xi) \in WF_{(\sigma)}(u), |\xi| \text{ sufficiently large}\}.$$  

2. **An application of Fourier integral operators of infinite order to the investigation of sufficient conditions for a certain Cauchy problem to be well-posed in $D^{(\sigma)}_{L^2}$.**

In this section we consider operators of the form:

$$P = \partial_t - i \lambda(t, x, D_x) + a(t, x, D_x)$$

where $\lambda$ and $a$ are pseudo-differential operators whose symbols are in $C([0, T]; C^{-\infty}(R^n))$ and satisfy the following properties:

$\lambda(t, x, \xi)$ is real valued and belongs to $\mathcal{B}([0, T]; \tilde{S}^{1,1}(R^n \times R^n))$;

$a(t, x, \xi)$ is in $\mathcal{B}([0, T]; \tilde{S}^{0,1}(R^n \times R^n))$ with $p \in [0, 1]$ and verifies:

$$(2.1) \lim_{\varepsilon \to +\infty} \text{Re} a(t, x, \varepsilon \eta) > 0 \text{ for every } t \in [0, T] \text{ and uniformly with respect to } (x, \eta) \in R^n \times S_{n-1}, \text{ where we assume } \sigma(2p - 1) < 1, \sigma > 1.$$
REMARK 2.1. Actually in dealing with the Cauchy problem with initial datum at \( t = s \), \( s \in [0, T] \), it is sufficient to claim:

\[
\lim_{q \to +\infty} \int_{t}^{\infty} q^{-1/s} \text{Re} \, a(t', x, \nabla_x \varphi(t', s; x, q_\eta)) \, dt' > 0, \quad \forall t \in [s, T'],
\]
uniformly with respect to \((x, \eta) \in \mathbb{R}^n \times S_{n-1}\), where \( \varphi \) and \( T' \) are determined in the following Prop. 2.1.

REMARK 2.2. When \( \sigma < 1/p \) the assumption (2.1) is trivially true and it is well-known that the Cauchy problem for \( P \) is well-posed in \( D_{\sigma}^{(T')} \). We henceforth confine our discussion to the case where \( \sigma > 1/p \). We recall now the following

PROPOSITION 2.1 (see [4], [14]). If \( \lambda \) is as above, then there exists \( T' > 0 \) and there exists a solution \( \varphi(t, s) \) of the eiconal equation

\[
\begin{cases}
\partial_t \varphi(t, s; x, \xi) = \lambda(t, x, \nabla_x \varphi(t, s; x, \xi)) \\
\varphi(s, s; x, \xi) = x \cdot \xi
\end{cases}
\]

where \( \varphi(x, \xi) - x \cdot \xi \in C^1([0, T')^2; C^\infty(R^{2n})) \cap B^1([0, T')^2; \tilde{S}^{1,1}(R^{2n})) \cap \mathcal{F}(c_0 |t - s|) \) for some \( c_0 > 0 \). (For the definition of \( \mathcal{F}(\tau) \) we refer to [9].)

REMARK 2.3. In the proof of Prop. 2.1 it is shown that if \( T' \) is sufficiently small, a solution \((p, q)\) of

\[
\begin{cases}
\dot{q} = -\nabla_\xi \lambda(t, q, p), \\
\dot{p} = \nabla_x \lambda(t, q, p)
\end{cases}
\]

and

\[(q, p)(t = s) = (y, \eta)\]

can be found with the following properties:

\[
q(t, s; y, \eta) - y \in B^1([0, T']^2; \tilde{S}^{0,1,1}) \cap C^1([0, T']^2; C^\infty(R^{2n}))
\]

\[
p(t, s; y, \eta) \in B^1([0, T']^2; \tilde{S}^{1,1,1}) \cap C^1([0, T']^2; C^\infty(R^{2n}))
\]

\[
|q(t, s; y, \eta) - y| < C_1 |t - s|;
\]

\[
|p(t, s; y, \eta) - \eta| < C_1 |t - s|
\]

for some \( C_1 > 0 \).

By choosing a suitable \( T' \) it can also be proved that there exists the inverse function \( y = Y(t, s; x, \eta) \) of \( x = q(t, s; y, \eta) \) and

\[
|Y(t, s; x, \eta) - x| < C_2 |t - s|
\]
for some $C \geq 0$,

$$Y(t, s; x, \eta) = x \in \mathcal{B}([0, T']); \bar{S}^{0, 1, 0} \cap C([0, T'); \mathcal{C}^{\infty}(R^{2n})).$$

**Theorem 2.1.** If $P$ is as above, then for every $s \in [0, T]$ there exists $T' \in [s, T]$ and there exists $e(t, s; x, \xi)$ continuously differentiable up to the first order with respect to $s, t$ with values in $\mathcal{C}^{\infty}(R^n \times R^n)$ and belonging to $\bar{S}^{\infty, \sigma, \mu}(R^n \times R^n)$ uniformly with respect to $t$ and $s$, and with some $\mu \geq 1$, satisfying the following properties. If $\varphi$ is the solution of (2.1), then the Fourier integral operator defined by:

$$(E_{\varphi}u)(x) = \int \exp [i\varphi(t, s; x, \xi)] e(t, s; x, \xi) \bar{u}(\xi) d\xi \quad \forall u \in \mathcal{D}^{[q]}_x$$

satisfies

$$\begin{cases} PE_{\varphi}(t, s) = R_{\varphi}(t, s) & \forall t \in [s, T] \\ E_{\varphi}(s, s) = I \quad \text{(identity operator)} \end{cases}$$

where $R_{\varphi}(t, s)$ is a continuous map from $\mathcal{D}^{[q]}_{t, s}$ to $\mathcal{D}^{[q]}_{t, s}$, for every $t$ and $s$.

**Proof.** In view of Th. 1.3 it is sufficient to determine $e(t, s; x, \xi) \sim \sum_{h \geq 0} e_h(t, s; x, \xi)$ such that $e(s, s; x, \xi) = 1$ and that the following term vanishes:

$$\partial_t e(t, s; x, \xi) - \frac{1}{2} \sum_{j, k = 1, \ldots, n} \partial^2_{s, \xi_j} \lambda(t, x, \nabla \varphi(t, s; x, \xi)) \cdot$$

$$\cdot \partial^2_{s, \xi_j} \varphi(t, s; x, \xi) e(t, s; x, \xi) - i \sum_{j = 1, \ldots, n} (\partial_\xi \lambda)(t, x, \nabla \varphi(t, s; x, \xi)) \cdot$$

$$\cdot D_{\xi_j} e(t, s; x, \xi) - i \sum_{|\gamma| \geq 2} 1/\gamma! D^\gamma_{\xi}(\partial^\gamma \lambda)(t, x, \nabla \varphi(t, x, z, \xi)) \cdot e(t, s; x, \xi) \bigg|_{z = x} +$$

$$+ \sum_{|\gamma| \geq 0} 1/\gamma! D^\gamma_{\xi}(\partial^\gamma \varphi(t, x, \xi)) \cdot e(t, s; x, \xi) \bigg|_{z = x}$$

where

$$\tilde{\nabla} \varphi(t, x, z, \xi) = \frac{1}{0} \int \nabla \varphi(t, z + \theta(x - z), \xi) d\theta.$$ 

Therefore each $e_h$ can be determined inductively as a solution of the following Cauchy problem:

$$(T_0) \begin{cases} \partial_t e_0(t, s) = \sum_{j = 1}^n (\partial_\xi \lambda)(t, x, \nabla \varphi(t, s; x, \xi)) \partial_{s, \xi_j} e_0(t, s) + \\
\quad + g(t, s; x, \xi) e_0(t, s) = 0, \\
e_0(s, s) = 1, \end{cases}$$
where
\[ g(t, s; x, \xi) = - \sum_{i, k = 1, \ldots, n} \frac{1}{2} \partial_{\xi_i}^2 \psi(t, x, \nabla_\omega \varphi(t, s; x, \xi)) \partial_{\omega_{x_i}} \varphi(t, s; x, \xi) + \]
\[ + a(t, x, \nabla_\omega \varphi(t, s; x, \xi)) \]
and in the case \( h = 1, 2, \ldots \)

Putting \( \psi(t, \omega; y, \xi) = \psi(t, \omega; y, \xi), \xi, \omega = 0, 1, \ldots \) and solving the corresponding transport equations, we can prove inductively the following estimate (by applying for instance Lemma 4.2, p. 56 in [5]):

\[ (T_h) \]
\[ \left| \partial_t e_h(t, s) - \sum_{i=1}^{n} (\partial_{\xi_i} \psi)(t, x, \nabla_\omega \varphi(t, s; x, \xi)) \partial_{\omega_i} e_h(t, s) + \right. \]
\[ + g(t, s; x, \xi) e_h(t, s) + \]
\[ + \sum_{r=0}^{h-1} \left( - \frac{i}{r!} D^r_\xi \left\{ (\partial^r \psi)(t, x, \nabla_\omega \varphi(t, s; x, \xi)) \right\}_{s=x} + \right. \]
\[ + \sum_{|\gamma|=r} 1/\gamma! D^r_\xi \left\{ (\partial^r \psi)(t, x, \nabla_\omega \varphi(t, s; x, \xi)) \right\}_{s=x} \right) = 0, \]
\[ e_h(s, s) = 0. \]

Putting \( \psi_t(t, s; y; \xi) = e_t(t, s; q(t, s; y, \xi), \xi), \ h = 0, 1, \ldots \) and solving the corresponding transport equations, we can prove inductively the following estimate (by applying for instance Lemma 4.2, p. 56 in [5]):

\[ (I_h) \]
\[ \left| \partial^\sigma_\xi \partial^\beta_\omega \psi(t, s; y, \xi) \right| < C_\epsilon \exp \left[ \epsilon \langle \xi \rangle^{1/2}(t - s) \right] A^*|\alpha + \beta| + 2\alpha. \]
\[ \cdot (|\alpha + \beta| + 2h)!/h! \langle \xi \rangle^{-|\alpha| - h} \langle \xi \rangle^{(v-1/2)(|\alpha + \beta| + 2h)} \sum_{l=\max(|\alpha + \beta|, 1)}^{\alpha + \beta + 2h} \left\{ C_0(t - s) \langle \xi \rangle^{1/2} \right\}^l \]

which is true for every \( \epsilon' > 0 \), with suitable positive constants \( A^*, B_0^*, B_1^*, C_0 \) and for \( |\xi| > B_0^* + B_1^*(|\alpha| + h) \).

From \( (I_h) \) it follows that \( \forall \epsilon > 0, \forall \mu' > 0 \) we have:

\[ \left| \partial^\sigma_\xi \partial^\beta_\omega \psi(t, s; y, \xi) \right| < C_\epsilon' \exp \left[ \epsilon \langle \xi \rangle^{1/2} \right] A^*|\alpha + \beta| + h \langle \xi \rangle^{-|\alpha| - h}(\alpha + \beta)!^{\sigma + \mu'} \langle \xi \rangle^{2\sigma - 1 + \mu'} \]

Since \( 2\sigma - 1 < \sigma \) (by assumption (2.i)), we take \( \mu' = \sigma + 1 - 2\sigma p \) and conclude with \( \sum_{h=0}^{h} \psi(t, s) \in FS^{\infty, \sigma, \sigma + 1 - 2\sigma p} \) with respect to \( t \) and \( s \).

In view of Th. 1.1.21 in [4] it is also \( \sum_{h=0}^{h} \psi(t, s) \in FS^{\infty, \sigma, \sigma + 1 - 2\sigma p} \) uniformly with respect to \( t, s \). Thus the theorem is proved.

**Remark 2.4.** The operator \( R_\psi \) in Th. 2.1 can be regarded as a pseudo-differential operator, say \( \tilde{R} \), whose amplitude is \( \tilde{R}(x, \xi) = \exp \left[ ip(x, \xi) - ip \cdot \xi \right] r(x, \xi) \), if \( r(x, \xi) \) is the amplitude of \( R \). Moreover \( \tilde{R}(x, \xi) \) has the same properties as \( r(x, \xi) \).

Therefore, as we proved in [1], we have the following
LEMMA 2.1. If \( \bar{R} \) is as in Remark 2.4 then there exists a solution of
\[
\bar{R}(t, s) = - F(t, s) - \int_s^t \bar{R}(t, \tau) F(\tau, s) \, d\tau
\]
and \( F \) continuously maps \( C([s, T']); \, \mathbb{D}^{(a''')}_{L^q} \) to \( C([s, T') \); \( \mathbb{D}^{(a')}}_{L^q} \).

THEOREM 2.2. If \( E_\varphi(t, s) \) and \( R_\varphi(t, s) \) are as in Th. 2.1 and \( F(t, s) \) is as in Lemma 2.1, then
\[
E_\varphi(t, s) = E_\varphi(t, s) + \int_s^t E_\varphi(t, \tau) F(\tau, s) \, d\tau
\]
is a fundamental solution for the Cauchy problem for \( P \).

If \( g \in \mathbb{D}^{(a')}_{L^q}, f \in C([0, T'); \mathbb{D}^{(a''')}_{L^q}) \) (respectively \( g \in \mathbb{D}^{(a''')}_{L^q}, f \in C([0, T]; \mathbb{D}^{(a''')}_{L^q}) \), then for every \( s \in [0, T[ \) there exists \( T' \in ]s, T] \) such that \( \forall t \in [s, T'] \)
\[
u(t, x) = E_\varphi(t, s)g + \int_s^t E_\varphi(t, \tau) f(\tau, \cdot) \, d\tau
\]
is the solution of the Cauchy problem
\[
\begin{align*}
Pu(t, \cdot) &= f(t, \cdot) \quad \text{in } [s, T'] \times \mathbb{R}^n \\
u(s, \cdot) &= g
\end{align*}
\]
and \( u \) belongs to \( C^1 \) as a function of \( t \) with values in the space \( \mathbb{D}^{(a')}_{L^q} \) (respectively \( \mathbb{D}^{(a''')}_{L^q} \)).

Moreover, whenever \( \lambda(t, x, \xi) \) is homogeneous in \( \xi \) for \( |\xi| \) large, then for every \( f \) in \( C([0, T]; \mathbb{D}^{(a''')}_{L^q}) \) and for every \( g \) in \( \mathbb{D}^{(a''')}_{L^q} \), the solution \( u(t, x) \) of the problem (C) (with \( s \in [0, T[, \, T' \) sufficiently small and \( t \in [s, T'] \) satisfies
\[
WF_{(\varnothing)}(u(t, \cdot)) \subset \{(q(t, s; y, \eta), \theta p(t, s; y, \eta)); (y, \eta) \in WF_{(\varnothing)}(g); \theta \geq 1 \text{ and } |\eta| \text{ sufficiently large}\}, \quad \forall t \in [s, T'],
\]
where \( q, p \) are solutions to
\[
\begin{align*}
\dot{q} &= - \nabla_\xi \lambda(t, q, p) \quad \dot{p} = \nabla_\eta \lambda(t, q, p) \quad (q, p)(t = s) = (y, \eta).
\end{align*}
\]

PROOF. In view of Th. 2.1 and Lemma 2.1 we have \( P\bar{E}_\varphi(t, s) = 0 \) and \( u(t, x) \) defined in (2.2) satisfies all the claims above. Uniqueness
of the solution follows by a standard argument where the transpose of $P$ is considered (see [4]).

Finally (2.2)' follows from Th. 1.4 and Th. 2.2.

REFERENCES

1] R. Agliardi, Pseudo-differential operators of infinite order on $\mathcal{D}^{\infty}_P(\mathcal{D}^{\infty}_P)$, and applications to the Cauchy problem for some elementary operators, to appear on Ann. di Mat. Pura e Appl.


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