Hölder regularity in non autonomous degenerate abstract parabolic equations

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Hölder Regularity in Non Autonomous Degenerate Abstract Parabolic Equations.

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ABSTRACT - We prove optimal Hölder regularity results for a class of non-autonomous degenerate parabolic equations, in general Banach space.

0. Introduction.

We consider a parabolic evolution equation in general Banach space $X$:

\[
\begin{cases}
    u'(t) = \varphi(t) A(t) u(t) + f(t), & 0 < t < T, \\
    u(0) = u_0.
\end{cases}
\]

Here «parabolic» means that for every $t \in [0, T]$, the operator $A(t): D(A(t)) \subset X \to X$ generates an analytic semigroup in $X$. The domains $D(A(t))$ may possibly be not constant and not dense in $X$. We assume that the family $\{A(t): 0 \leq t \leq T\}$ satisfies some conditions guaranteeing that there exists an evolution operator for problem

\[
\begin{cases}
    v'(t) = A(t) v(t), & 0 < t \leq T, \\
    v(0) = v_0.
\end{cases}
\]

The function $\varphi: [0, T] \to \mathbb{R}$ is continuous and nonnegative, and it is allowed to vanish at $t = 0$ and at $t = T$. Therefore (0.1) is an abstract degenerate parabolic initial value problem. We assume that

\[\varphi(t) = O(t^\beta) \quad \text{as } t \to 0,\]

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and

\[ \varphi(t) = O(T - t)^{\beta_1} \quad \text{as} \quad t \to T, \quad \text{with} \quad \beta, \beta_1 \geq 0. \]

Then we can state precise regularity results (mainly, Hölder regularity results) for the solution of (0.1). The solution is a strong, classical, or strict one (see Def. 2.1, 3.1, 3.6) according to the regularity of the data. We get a representation formula for the solution by setting

\[ \tau = \dot{u}(t) = \int_0^t \varphi(\sigma) \, d\sigma \quad \text{and} \quad u(t) = w(\dot{u}(t)) \quad \text{in} \quad (0.1), \]

so that (0.1) becomes

\[
\begin{cases}
    w'(\tau) = A(\dot{\phi}^{-1}(\tau))w(\tau) + f(\dot{\phi}^{-1}(\tau))/\varphi(\dot{\phi}^{-1}(\tau)), \quad 0 < \tau < \dot{\phi}(T), \\
    w(0) = u_0.
\end{cases}
\]

Now, \( w \) is given by the variation of constants formula

\[ w(\tau) = G(\tau, 0)u_0 + \int_0^{\tau} G(\tau, \sigma)f(\dot{\phi}^{-1}(\sigma))/\varphi(\dot{\phi}^{-1}(\sigma)) \, d\sigma, \]

so that

\[ (0.3) \quad u(t) = G(\dot{\phi}(t), 0)u_0 + \int_0^t G(\dot{\phi}(t), \dot{\phi}(s))f(s) \, ds, \quad 0 \leq t \leq T. \]

Here \( G(\sigma, \tau) \) is the evolution operator associated to the family \( \{A(\dot{\phi}^{-1}(\tau), \tau) \in [0, \dot{\phi}(T)]\} \). Our results are shown by a careful study of formula (0.3).

The literature on the subject is not very rich. In a previous paper ([5]) we studied the case where \( A(t) = A \) is independent of time, and \( \varphi(t) > 0 \) for \( t > 0 \). Weak solutions to (0.1) are considered in [9] and [12], in the case where \( X \) is a Hilbert space, and \( D(A(t)) = D \) is constant and dense in \( X \). A certain class of degenerate equations could be studied also by means of the «sum of operators» method of [6]. However, such a method seems to be more fruitful in singular equations than in degenerate ones (see also [7], [8]). The paper is structured as follows. Section 1 is devoted to notation and preliminary estimates on the evolution operator relevant to problem (0.2). Section 2 deals with classical solvability of (0.1) and regularity properties of the classical solutions, whereas strict and strong solutions are studied in Section 3. Finally, in Section 4 we apply some of the
abstract results to a class of initial boundary value problems for second order degenerate parabolic equations.

1. Notation and preliminaries.

Let $T > 0$ and let $X$ be any Banach space with norm $\| \cdot \|$. We shall use in the sequel the following functional spaces $B([0, T]; X)$ (the space of all bounded functions $f: [0, T] \rightarrow X$ endowed with the sup norm $\| \cdot \|_{\infty}$), $\text{Lip}([0, T]); X$ (the space of all Lipschitz continuous $f: [0, T] \rightarrow X$), $C([0, T]; X), C^\omega([0, T]; X), C([0, T]; X)$ with the usual meanings and norms. We consider a continuous function $\varphi: [0, T] \rightarrow \mathbb{R}$ such that $\varphi(t) > 0$ for $t \in [0, T]$. We shall see that the behaviour of the solution of (0.1) depends heavily on the behaviour of $\varphi$ as $t \rightarrow 0$ and as $t \rightarrow T$.

Therefore we assume:

\[
\begin{aligned}
\varphi \in C([0, T]), \\
k t^\beta \leq \varphi(t) \leq K t^\beta, & \quad 0 \leq t \leq T/2 \\
k (T - t)^\beta_1 \leq \varphi(t) \leq K (T - t)^\beta_1, & \quad T/2 < t \leq T,
\end{aligned}
\]

with $\beta \geq 0$, $\beta_1 \geq 0$ and $0 < k \leq K$.

It is convenient to introduce the notation

\[
\begin{aligned}
\varphi(t, s) &= \int_s^t \varphi(r) \, dr, \\
\varphi(t) &= \varphi(t, 0), & \quad 0 \leq s \leq t \leq T.
\end{aligned}
\]

Due to assumption (1.1), we get:

\[
\begin{aligned}
\frac{k}{\beta + 1} t^{\beta + 1} \leq \varphi(t) & \leq \frac{K}{\beta + 1} t^{\beta + 1}, \\
\frac{k}{\beta + 1} t^\beta (t - s) \leq \varphi(t, s) \leq K t^\beta (t - s), & \quad 0 \leq s \leq t \leq T/2, \\
\frac{k}{\beta_1 + 1} (T - s)^{\beta_1} (t - s) \leq \varphi(t, s) \leq K (T - s)^{\beta_1} (t - s), & \quad T/2 < s \leq t \leq T.
\end{aligned}
\]

Let

\[
\begin{aligned}
g(t, s) &= \begin{cases} 
\frac{t^\beta}{(T - s)^{\beta_1}}, & \text{if } T/2 < s \leq t \leq T, \\
t^\beta, & \text{if } 0 \leq s \leq t \leq T/2.
\end{cases}
\end{aligned}
\]
Then from (1.1) and (1.4) it follows:

\[
\begin{cases}
k g(t, t) \leq \varphi(t) \leq K g(t, t) & \text{for } 0 \leq t \leq T, \\
g(\tau, s) \leq g(t, s), \\
g(t, \tau) \leq g(t, s),
\end{cases}
\]

both for \(0 \leq s \leq \tau \leq t \leq T/2\) and for \(T/2 < s \leq \tau \leq t \leq T\);

and from (1.2), (1.3) and (1.4) it follows:

\[
\begin{cases}
k_1 t^{\beta + 1} \leq \varphi(t) \leq K_1 t^{\beta + 1} & \text{for } 0 \leq t \leq T/2, \\
k_1 g(t, s)(t - s) \leq \psi(t, s) \leq K g(t, s)(t - s) \leq K_1 g(t, s)(t - s)
\end{cases}
\]

both for \(0 \leq s \leq t \leq T/2\) and for \(T/2 < s \leq t \leq T\),

\[
\psi(t, \tau) \leq \psi(t, s)
\]

both for \(0 \leq s \leq \tau \leq t \leq T/2\) and for \(T/2 < s \leq \tau \leq t \leq T\),

with \(k_1 = \min \{k/(\beta + 1), k/(\beta_1 + 1)\}\), \(K_1 = K/(\beta + 1)\).

Throughout the paper we shall assume that for each \(t \in [0, T]\), \(A(t): D(A(t)) \subset X \rightarrow X\) generates an analytic semigroup \(e^{sA(t)}\), \(s \geq 0\) in \(X\). The domains \(D(A(t))\) may change with \(t\), however the resolvent sets \(\rho(A(t))\) are assumed to contain a common sector

\[
S_{\delta_0} = \{z \in \mathbb{C}: |\arg z| \leq \delta_0\} \cup \{0\} \quad \forall t \in [0, T],
\]

with \(\delta_0 \in (\pi/2, \pi]\).

Moreover we shall assume, as in [2],[3]:

There exist \(C > 0\), \(h \in \mathbb{N}\), \(\alpha_1, \ldots, \alpha_h, \delta_1, \ldots, \delta_h\), with \(0 \leq \delta_i < \alpha_i \leq 2\), such that

\[
\|A(t) R(\lambda, A(t))(A(t)^{-1} - A(s)^{-1})\|_{L(X)} \leq C \sum_{i,j=1}^{h} (t-s)^{\alpha_i} |\lambda|^{\delta_i - 1}
\]

\(\forall \lambda \in S_{\delta_0} \setminus \{0\}, \forall 0 \leq s \leq t \leq T\).

\(D(A(t))\) is endowed with the graph norm and its closure in \(X\) is denoted by \(\overline{D(A(t))}\). We refer to [13] and [3] for all properties of \(e^{sA(t)}\) and of the interpolation spaces \(D_{A(t)}(\varphi, \infty), D_{A(t)}(\varphi + 1, \infty), D_{A(t)}(\varphi)\) and \(D_{A(t)}(\varphi + 1)\), \(0 < \varphi < 1\). We set \(D_{A(t)}(0, \infty) = X\) and \(D_{A(t)}(1, \infty) = D(A(t)) \forall t \in [0, T]\). Set

\[
B(\tau) = A(\varphi^{-1}(\tau)), \quad 0 \leq \tau \leq \varphi(T).
\]
Then the family \( \{B(\tau)\} \) satisfies assumption (1.7). More precisely, we have (due to (1.1) and (1.3)):

\[
\begin{align*}
\|B(\tau)R(\lambda, B(\tau))[B(\tau)^{-1} - B(\sigma)^{-1}]\|_{L(X)} & \leq \\
& \leq \left(\frac{C(\sigma - \tau)^{\alpha_i/(\beta + 1)}}{\lambda}\right)^\delta \lambda^{-1/2} & \text{if } 0 \leq \sigma \leq \tau \leq \phi(T/2), \\
& \leq \left(\frac{C(\sigma - \tau)^{\alpha_i/(\beta + 1)}}{\lambda}\right)^\delta \lambda^{-1/2} & \text{if } \phi(T/2) < \sigma \leq \tau \leq \phi(T).
\end{align*}
\]

We assume

\[
(1.10) \quad \delta = \min \left\{ \alpha_i/(\beta + 1) - \delta_i : 1 \leq i \leq h \right\} \in [0, 1], \\
\quad \quad \text{with } \beta = \max \{\beta, \beta_1 \}.
\]

By [2],[3] there exists an evolution operator \( G(\tau, \sigma) \) relevant to the family \( \{B(\tau) = A(\phi^{-1}(\tau)) : 0 \leq \tau \leq \phi(T)\} \). It can be represented as

\[
G(\tau, \sigma) = e^{(\tau - \sigma)B(\sigma)} + \int_{\sigma}^{\tau} Z(r, \sigma) dr, \quad 0 \leq \sigma \leq \tau \leq \phi(T)
\]

(see [2]). Here \( Z(r, \sigma) \in L(X) \) for \( 0 \leq \tau \leq r \leq \phi(T) \), and for \( 0 \leq s < \tau < t \leq \phi(T) \) we have:

\[
(1.12) \quad \|Z(t, s)\|_{L(D_{\beta_0}(\sigma, \infty), X)} \leq c_1(\delta)(t - s)^{\delta + \delta - 1}, \quad \delta \in [0, 1],
\]

\[
(1.13) \quad \|Z(t, s) - Z(\tau, \sigma)\|_{L(D_{\beta_0}(\delta, \infty), X)} \leq c_2(\delta, \eta)(t - \tau)^{\eta}(t - s)^{\delta + \delta - 1} - \eta
\]

with \( \delta \in [0, 1], \quad \eta \in \max \{\delta - 1, 0\}, \quad \delta' \);

\[
(1.14) \quad \|Z(t, s) - Z(\tau, \sigma)\|_{L(D_{\beta}(\sigma), X)} \leq c_3(\eta)(t - \tau)^{\eta}, \quad \eta \in ]0, \delta[;
\]

where \( \delta \) is defined in (1.10) (see [2], Lemma 2.2).

Moreover we have (recall that \( A(\cdot) = B(\phi(\cdot)) \)):

\[
(1.15) \begin{cases}
\quad \text{if } \alpha, \delta \in [0, 1], \quad 0 \leq t \leq T, \quad 0 < \xi \leq \phi(T), \\
\quad \|e^{A(t)}\|_{L(D_{\beta_0}(\delta, \infty), X)} \leq M_0(\delta); \\
\quad \|A(t)^m e^{A(t)}\|_{L(D_{\beta_0}(\delta, \infty), D_{\alpha_0}(\alpha, \infty))} \leq M_1(\alpha, \delta, m) \xi^{-(m + \alpha - \delta)} \\
\quad \text{with } m \geq 1;
\end{cases}
\]
Let $0 \leq s < t \leq \phi(T)$, and let $a$ be defined in (1.10). We have:

\begin{equation}
\|B(t)^m e^{B(t)} - B(s)^m e^{B(s)}\|_{L(X)} \leq M_2 (m) \sum_{i=1}^{k} \phi(t, s)^{\frac{\alpha}{\beta + 1}} \xi^{-m + \xi_i}.
\end{equation}

(see [3, Lemmas 1.8 and 1.10] and our assumptions (1.9), (1.10)).

Let $0 < s < r < t < \phi(T)$, and let $\delta$ be defined in (1.10). We have:

\begin{equation}
\|G(t, s)\|_{L(D_{B0}(\delta, \alpha), D_{B0}(\alpha, \infty))} \leq c_4 (\alpha, \delta)(t - s)^{\delta - \alpha};
\end{equation}

\begin{equation}
\|B(t) G(t, s)\|_{L(D_{B0}(\delta, \alpha), D_{B0}(\alpha, \infty))} \leq c_5 (\alpha, \delta)(t - s)^{\delta - \alpha - 1};
\end{equation}

\begin{equation}
\|G(t, s) - G(\tau, s)\|_{L(D_{B0}(\delta, \alpha), x)} \leq c_6 (\delta)(t - \tau)^{\delta}, \quad \delta \in [0, 1];
\end{equation}

(see [10, Lemmas 4.1 and 4.2]).

\begin{equation}
\|B(t) G(t, s)\|_{L(D_{B0}(\delta + 1, \alpha), D_{B0}(\alpha, \infty))} \leq c_7 ;
\end{equation}

(see [3, Thm. 6.1]).

The following lemma will be useful in the sequel.

**Lemma 1.1.** Let $y \in X$. Then $\int_{0}^{t} G(t, \sigma) y d\sigma \in D(B(t)) \forall t > 0$, and

\begin{equation}
\|B(t) \int_{0}^{\tau} G(t, \sigma) y d\sigma\| \leq c\|y\| \forall t \in [0, \phi(T)];
\end{equation}

\begin{equation}
\|B(t) \int_{0}^{\tau} G(t, \sigma) y d\sigma - B(\tau) \int_{0}^{\tau} G(\tau, \sigma) y d\sigma\| \leq c(\alpha) \frac{(t - \tau)^{\alpha}}{(\tau - \alpha)^{\alpha}},
\end{equation}

for each $\alpha < \tau < t \leq \phi(T)$, $0 < \alpha < 1$.

**Proof.** It is an easy consequence of Propositions 2.1(iv) and 2.6(iii)(d) of [3] (with $\mu = 0$), together with the representation formula (1.22) of [3].

Let us state other estimates which will be used throughout the paper.
LEMMA 1.2. Let $\delta$ be defined in (1.10). Then, both for $0 \leq s < \tau < t \leq T$ and for $T/2 < s < \tau < t \leq T$ we have:

$$(1.23) \quad \|B(\phi(t)) G(\phi(t), \phi(s))\|_{L(D_{A_0}(\delta, \infty), D_{A_0}(\delta, \infty))} \leq$$

$$\leq C_0(\delta, \delta) g(t, s)^{\delta - \gamma - 1} (t - s)^{\delta - \gamma - 1}, \quad \delta \in [0, 1], \quad \alpha \in [0, \delta];$$

$$(1.24) \quad \|G(\phi(t), \phi(s)) - G(\phi(\tau), \phi(s))\|_{L(\infty)} \leq C_1(\nu) \frac{(t - \tau)^{\nu}}{(\tau - s)^{\nu}}, \quad \nu \in [0, 1];$$

$$(1.25) \quad \|B(\phi(t)) G(\phi(t), \phi(s)) - B(\phi(t')) G(\phi(t'), \phi(s))\|_{L(D_{A_0}(\delta, \infty), X)} \leq$$

$$\leq \frac{1}{g(\tau - s)^{1 - \delta}} \left\{ C_2(\delta) \left[ \frac{1}{(\tau - s)^{1 - \delta}} - \frac{1}{(t - s)^{1 - \delta}} \right] + \right.$$  

$$+ C_3(\delta, \delta, \gamma) g(t, \tau)^{\gamma} g(\tau, s)^{\delta - \gamma} \frac{(t - \tau)^{\gamma}}{(\tau - s)^{1 + \gamma - \delta - \gamma}} \right\}, \quad \delta \in [0, 1], \quad \eta \in [\max\{\delta + \delta - 1, 0\}, \delta];$$

$$(1.26) \quad \|B(\phi(t)) G(\phi(t), \phi(s)) - B(\phi(t')) G(\phi(t'), \phi(s))\|_{L(D_{A_0}(\delta), X)} \leq$$

$$\leq C_4(\nu) \frac{(t - \tau)^{\nu}}{(\tau - s)^{\nu}} + C_5(\delta, \gamma) g(t, \tau)^{\gamma} (t - \tau)^{\gamma}, \quad \nu \in [0, 1], \quad \eta \in [0, \delta];$$

$$(1.27) \quad \|B(\phi(t)) G(\phi(t), \phi(s)) - B(\phi(t')) G(\phi(t'), \phi(s))\|_{L(D_{A_0}(\delta + 1, \infty), X)} \leq$$

$$\leq C_6(\delta) g(t - s)^{\delta} (t - \tau)^{\delta} + C_7(\gamma) g(t, \tau)^{\gamma} (t - \tau)^{\gamma}, \quad \delta \in [0, 1], \quad \eta \in [0, \delta].$$

PROOF. (1.23) is a simple consequence of (1.18), (1.2), (1.4) and (1.6). To show (1.24), ..., (1.27), we consider either $0 \leq s < \tau < t \leq T/2$ or $T/2 < s < \tau < t \leq T$. (1.24), for $\nu = 0$, holds with $C_1 = 2 c_4$ thanks to (1.17). For $\nu > 0$, thanks to (1.23), (1.4) and (1.5), we have

$$\|G(\phi(t), \phi(s)) - G(\phi(\tau), \phi(s))\|_{L(\infty)} =$$

$$= \int_{\tau}^{t} \phi(r) B(\phi(r)) G(\phi(r), \phi(s)) \, dr \|_{L(\infty)} \leq KC_0 \int_{\tau}^{t} \frac{g(r, \tau) \, dr}{g(r, s)(r - s)} \leq$$

$$\leq KC_0 \frac{1}{(\tau - s)^{1 - \nu}} \int_{\tau}^{t} \frac{dr}{(r - s)^{\nu}} \leq \frac{KC_0 (t - \tau)^{\nu}}{\nu (\tau - s)^{\nu}}, \quad 0 < \nu \leq 1.$$
Concerning (1.25) and (1.26), thanks to [2, (2.10)] for \( \delta \in [0,1] \) we have:

\[
\|B(\phi(t)) G(\phi(t), \phi(s)) - B(\phi(\tau)) G(\phi(\tau), \phi(s))\|_{L(D_{A_0}(\delta, \infty), X)} \leq
\]

\[
\leq \|B(\phi(s)) [e^{\int_{\tau}^{t} B(\phi(s))} - e^{\int_{\tau}^{s} B(\phi(s))}]\|_{L(D_{A_0}(\delta, \infty), X)} + \|Z(\phi(t), \phi(s)) - Z(\phi(\tau), \phi(s))\|_{L(D_{A_0}(\delta, \infty), X)}.
\]

By (1.15), (1.4), (1.5) and (1.6) we have:

\[
\|B(\phi(s)) [e^{\int_{\tau}^{t} B(\phi(s))} - e^{\int_{\tau}^{s} B(\phi(s))}]\|_{L(D_{A_0}(\delta, \infty), X)} =
\]

\[
= \left\| \int_{\tau}^{t} \phi(r) B(\phi(s))^2 e^{\int_{\tau}^{r} B(\phi(s))} dr \right\|_{L(D_{A_0}(\delta, \infty), X)} \leq
\]

\[
\leq M_1(\delta) \int_{\tau}^{t} \frac{\phi(r) dr}{\psi(r, s)^2} \leq \frac{\gamma M_1(\delta)}{K_1^2 - \delta} g(\tau, s)^{1 - \delta} \int_{\tau}^{t} \frac{dr}{(r - s)^2}. \]

Now if \( \delta \in [0,1] \)

\[
\int_{\tau}^{t} \frac{dr}{(r - s)^2} = \frac{1}{1 - \delta} \left[ \frac{1}{(r - s)^{1 - \delta}} - \frac{1}{(t - s)^{1 - \delta}} \right]
\]

and from (1.13), (1.4), (1.6) it follows:

\[
\|Z(\phi(t), \phi(s)) - Z(\phi(\tau), \phi(s))\|_{L(D_{A_0}(\delta, \infty), X)} \leq
\]

\[
\leq c_2(\delta, \eta) \psi(t, \tau)^{\gamma} \psi(\tau, s)^{\delta + \delta - \gamma - 1}, \quad \gamma \in [\max\{\delta + \delta - 1, 0\}, \delta[\]

\[
\leq c_2(\delta, \eta) K^{\delta + \delta - 1} g(t, \tau)^{\gamma} g(\tau, s)^{\delta + \delta - \gamma - 1} (t - \tau)^{\gamma} (\tau - s)^{\delta + \delta - \gamma - 1}. \]

Hence (1.25) holds. If \( \delta = 1 \) and \( \nu \in [0,1] \) we get:

\[
\int_{\tau}^{t} \frac{dr}{r - s} \leq \frac{(t - \tau)^{\nu}}{\nu(\tau - s)^{\nu}}
\]

and from (1.14), (1.4), (1.6) it follows, for \( \eta \in [0,\delta[\):

\[
\|Z(\phi(t), \phi(s)) - Z(\phi(\tau), \phi(s))\|_{L(D_{A_0}(\delta), X)} \leq
\]

\[
\leq c_3(\eta) \psi(t, \tau)^{\gamma} < c_3(\eta) K^{\gamma} g(t, \tau)^{\gamma} (t - \tau)^{\gamma};
\]

hence (1.26) holds. Finally from [2, (2.10)], (1.15), (1.14), (1.4), (1.5)
and (1.6) it follows:
\[ \|B(\phi(t)) G(\phi(t), \phi(s)) - B(\phi(\tau)) G(\phi(\tau), \phi(s))\|_{L^1((\delta + 1, \infty), X)} \leq \]
\[ \leq \|e^{\varphi(t, s)} B(\phi(s)) - e^{\varphi(\tau, s)} B(\phi(s))\|_{L^1((\delta + 1, \infty), X)} \cdot \]
\[ \sup_{0 < s < T} \|B(\phi(s))\|_{L^1((\delta + 1, \infty), X)} + \]
\[ + \|Z(\phi(t), \phi(s)) - Z(\phi(\tau), \phi(s))\|_{L^1((\delta, \infty), X)} \leq \]
\[ \leq \text{const} \left( \int_{\tau}^{t} \varphi(r) B(\phi(s)) e^{\varphi(r, s)} B(\phi(s)) dr \right)_{L^1((\delta, \infty), X)} + c_3(\gamma) \psi(t, \tau)^\gamma \leq \]
\[ \leq \text{const} M_1(\delta) \int_{\tau}^{t} \varphi(r) dr_{\psi(r, s)^{1-\delta}} + Kc_3(\gamma) g(t, \tau)^\gamma (t - \tau)^\gamma \leq \]
\[ \leq \text{const} \frac{KM_1(\delta)}{k_1^{1-\delta}} g(t, s)^{\delta} \int_{\tau}^{t} \frac{dr}{(r - s)^{1-\delta}} + Kc_3(\gamma) g(t, \tau)^\gamma (t - \tau)^\gamma \]
and (1.27) holds.

We are now in position to state the main properties of the function
\[ w(t) = G(\phi(t), 0)x, \quad 0 \leq t \leq T, \quad x \in X. \]
To simplify some statements, we introduce the following notation:
\[ B([a, b]; D_{A(\delta)}(\delta, \infty)) = \]
\[ = \left\{ u; [a, b] \to X: u(t) \in D_{A(\delta)}(\delta, \infty) \quad \forall t \in [a, b]; \quad \sup_{a \leq t \leq b} \|u(t)\|_{L^1((\delta, \infty), X)} < +\infty \right\}, \]
for \( a < b, \quad 0 < \delta < 2, \quad \delta \neq 1. \)

**Proposition 1.3.** Let \( \delta \) be defined in (1.10). For each \( x \in X \), we have:
\[ w \in B([0, T]; X); \]
\[ w \in B([\varepsilon, T]; D_{A(\delta + 1, \infty)}), \quad \forall \varepsilon \in ]0, T[; \]
\[ Aw \in C^\eta([\varepsilon, T]; X), \quad \forall \varepsilon \in ]0, T[; \quad \eta < \delta; \]
PROOF. (1.30), (1.34) and (1.35) follow from [2, Thm. 2.3 (i) and (v) and Thm. 4.1 (ii)]. By (1.23) and (1.4) for \(0 \leq t \leq T/2\) we have:

\[
\begin{align*}
\text{In particular, setting} & \quad \omega(t) = \phi(t)A(t)w(t) = \phi(t)B(\phi(t))G(\phi(t), 0)x, \quad \forall t \in [0, T]; \\
\text{we have:} & \quad x \in D(A(0)) \iff w \in C([0, T]; X) \quad \text{and} \quad w(0) = x; \\
\end{align*}
\]

Therefore by (1.38) and (1.20) we have, for \(T/2 \leq t \leq T\):

\[
\begin{align*}
\|B(\phi(t))G(\phi(t), 0)x\|_{D_{A(t)}(\omega, \infty)} &= \\
&= \|B(\phi(t))G(\phi(t), \phi(T/2))\|_{D_{A(t)}(\omega, \infty)} \leq C_\gamma \|\bar{x}\|_{D_{A(t)}(\omega, \infty)} \\
\end{align*}
\]

and (1.31) holds. By (1.25) \(Aw \in C^\gamma([\epsilon, T/2]; X);\) if \(T/2 < \tau < t \leq T\), by
(1.38), (1.27) and (1.4) we have:
$$\|A(t) w(t) - A(\tau) w(\tau)\| =$$
$$= \|B(\phi(t)) G(\phi(t), \phi(T/2)) \bar{x} - B(\phi(\tau)) G(\phi(\tau), \phi(T/2)) \bar{x}\| \leq$$
$$\leq \{ C_6 (T/2)^{\delta_1} (t - \tau)^\gamma + C_7 (T - \tau)^{\delta_2} (t - \tau)^\gamma \} \|\bar{x}\|_{D(A(T/2))} (|\delta + 1|, \infty), \quad \gamma < \delta.$$

Hence (1.32) holds. If $B_1 = 0$, (1.33) follows from [2, Thm. 2.3 (v)], (1.34) and (1.1). If $B_1 > 0$, $w \in C^1([0, T]; X)$ and by (1.34), (1.38), (1.23) and (1.1) we get:
$$\lim_{t \to T^-} \|\phi(t) B(\phi(t)) G(\phi(t), 0) x\| =$$
$$= \lim_{t \to T^-} \|\phi(t) B(\phi(t)) G(\phi(t), \phi(T/2)) \bar{x}\| = C_0 \|\bar{x}\|_{D(A(T/2))} \lim_{t \to T^-} \phi(t) = 0.$$

Therefore $w \in C^1([0, T]; X)$. Let us show (1.36). The second equivalence follows from [3, thm. 6.1] and (1.33). Concerning the first equivalence, if $\beta > 0$, thanks to (1.23) we have:
$$x \in D(A(0)) \Rightarrow \lim_{t \to 0} \|\phi(t)^{1 - 1/(\beta + 1)} B(\phi(t)) G(\phi(t), 0) x\| = 0$$
and thanks to [2, thm. 4.1 (iv)] we have:
$$x \in D(A(0)(1/(\beta + 1)))) \Rightarrow \exists \lim_{t \to 0} \|\phi(t)^{1 - 1/(\beta + 1)} B(\phi(t)) G(\phi(t), 0) x\| = y,$$
and for each $t \in ]0, T]$
$$\|\phi(t)^{1 - 1/(\beta + 1)} B(\phi(t)) G(\phi(t), 0) x\| \leq \text{const} \|x\|_{D(A(0)(1/(\beta + 1)), \infty)}.$$

Since $D(A(0))$ is dense in $D(A(0)(1/(\beta + 1)))$, $y = 0$.
Hence fixed $\varepsilon > 0, \exists \delta \leq T/2: \forall t < \delta$
$$\|B(\phi(t)) G(\phi(t), 0) x\| \leq \varepsilon / \phi(t)^{\delta/(\beta + 1)};$$
so that
$$\|w(t)\| = \|\phi(t) B(\phi(t)) G(\phi(t), 0) x\| \leq K t^{\delta} \varepsilon / \phi(t)^{\delta/(\beta + 1)} \leq K \varepsilon / k_1^{\delta/(\beta + 1)}$$
and $\lim_{t \to 0} w(t) = 0$. By (1.33) $w \in C^1([0, T]; X)$. Conversely
$$w' \in C([0, T/2]; X) \Rightarrow \exists \lim_{t \to 0} w'(t) = \lim_{t \to 0} \phi(t) B(\phi(t)) G(\phi(t), 0) x = y.$$
Let us show that $y = 0$. Fix $A \in \mathcal{A}$; then we have:

$$\begin{align*}
[\lambda - B(0)]^{-1} y = \lim_{t \to 0} [\lambda - B(\phi(t))]^{-1} \phi(t) B(\phi(t)) G(\phi(t), 0) x = \\
= \lim_{t \to 0} \lambda [\lambda - B(\phi(t))]^{-1} \phi(t) G(\phi(t), 0) x - \lim_{t \to 0} \phi(t) G(\phi(t), 0) x = 0.
\end{align*}$$

Therefore $y = 0$. Fixed $\varepsilon > 0$, for $t$ close to 0, we have: $\|\phi(t) B(\phi(t)) G(\phi(t), 0) x\| < \varepsilon$, and consequently

$$\frac{\|G(\phi(t), 0) x - G(\phi(\tau), 0) x\|}{[\phi(t) - \phi(\tau)]^{1/(\beta + 1)}} \leq \frac{\int_t^\tau \varepsilon dr}{k_1^{1/(\beta + 1)} t^{1/(\beta + 1)} (t - \tau)^{1/(\beta + 1)}} \leq \left[ \int_t^\tau \varepsilon dr \right] [k_1^{1/(\beta + 1)} (t - \tau)] = \varepsilon / k_1^{1/(\beta + 1)}.$$

Hence, thanks to [2, Thm. 4.2 (iv)], $x \in D_{A(0)} (1/(\beta + 1))$.

Concerning (1.37), if $x \in D_{A(0)} (e^{1/(\beta + 1)}, \infty)$, then by (1.23), (1.1) and (1.4), for $p > 0$ and $0 < t \leq T/2$ we have:

$$\|\phi(t) B(\phi(t)) G(\phi(t), 0) x\|_{D_{A(0)}(\beta, \infty)} \leq KC_0(\beta)\|x\|_{D_{A(0)}(\beta + 1/(\beta + 1), \infty)}$$

so that $w'$ is bounded in $D_{A(0)}(\beta, \infty)$, thanks to (1.31). If $\beta = 0$, then $w'$ is bounded in $D_{A(0)}(\beta, \infty)$, thanks to [3, Thm. 6.1] and (1.31).

In the case where $\varphi$ is Hölder continuous, we can show other regularity properties of $w'$.

**Proposition 1.4.** Let $\varphi \in \mathcal{C}^\gamma([0, T])$, $0 < \gamma < 1$, $\gamma \leq \beta$ if $\beta > 0$, $\gamma \leq \beta_1$ if $\beta_1 > 0$ (see (1.1)), and let $\delta$ be defined in (1.10). Then

(1.39) $x \in X \Rightarrow w' \in \mathcal{C}^\gamma([\varepsilon, T]; X), \quad 0 < \varepsilon < T, \; \eta \leq \gamma, \; \eta < \delta$;

(1.40) $\begin{cases} 
\text{if } \beta > 0, \; (1 + \beta - \gamma)/(\beta + 1) < \delta \leq 1 \\
\quad x \in D_{A(0)}(\delta, \infty) \Rightarrow w' \in \mathcal{C}^\gamma([0, T]; X) \\
\quad \text{with } \eta \leq \gamma - (\beta + 1)(1 - \delta), \; \eta < \delta;
\end{cases}$

(1.41) $\begin{cases} 
\text{if } \beta = 0, \; 0 < \delta < 1 \\
\quad x \in D_{A(0)}(\delta + 1, \infty) \Rightarrow w' \in \mathcal{C}^\gamma([0, T]; X) \\
\quad \text{with } \eta \leq \min \{\gamma, \delta\}, \; \eta < \delta.
\end{cases}$
PROOF. From (1.25) we obtain that \( B(\phi(t))w(t) = B(\phi(t))G(\phi(t), 0)x \) is \( \eta \)-Hölder continuos in \([s, T]\) for \( 0 < \varepsilon < T, \eta < \delta \); then (1.36) is a consequence of the Hölder continuity of \( \phi \). Let us show that (1.40) holds. In the case \( \delta < 1 \), thanks to (1.23), (1.25), (1.1) and (1.4) for \( 0 \leq \tau < t \leq T/2 \) we have:

\[
\|w'(t) - w'(&\tau)\| \leq |\varphi(t) - \varphi(&\tau)| \|B(\phi(t))G(\phi(t), 0)x\| + \\
+ \varphi(&\tau)\|B(\phi(t))G(\phi(&\tau), 0)x - B(\phi(&\tau))G(\phi(&\tau), 0)x\| \leq \\
\leq \left\{ C_0(\varphi)_C \frac{(t - &\tau)^\gamma}{t^\beta(1 + 1 - \delta)} + KC_2(\varphi) \left[ \frac{1}{t^{1-\delta(\beta + 1)}} - \frac{1}{t^{1-\delta(\beta + 1)}} \right] + \\
+ C_3(\varphi, \varphi, \eta) \frac{\tau^\delta t^\beta(1 - &\tau)^\gamma}{t^\beta(1 + 1)(\gamma - \delta - &\delta) + 1} \right\} \|x\|_{D_{AB}(&\phi, \infty)} \leq \\
\leq \left\{ C_0(\varphi)_C \frac{(t - &\tau)^\gamma}{t^\beta(1 + 1 - \delta)} + KC_2(\varphi)[t^{\delta(\beta + 1) - 1} - t^{\delta(\beta + 1) - 1}] + \\
+ KC_3(\varphi, \varphi, \eta)(t - &\tau)^\gamma t^\beta(1 + 1)(\lambda - \delta - &\delta) - 1 \right\} \|x\|_{D_{AB}(&\phi, \infty)}. 
\]

In the case \( \delta = 1 \), thanks to (1.23), (1.26), (1.1) and (1.4), for \( 0 \leq \tau < t \leq T/2 \) we have:

\[
\|w'(t) - w'(&\tau)\| \leq \\
\leq \left\{ C_0(\varphi)_C \frac{(t - &\tau)^\gamma}{t^\beta(1 + 1 - \delta)} + KC_4(\varphi) \tau^\delta - \gamma (t - &\tau)^\gamma + KC_5(\varphi, \varphi, \eta) t^\beta(1 - &\tau)^\gamma \right\} \|x\|_{D_{IA(0)}} 
\]

and \( w' \in C^\gamma([0, T/2]; X) \), since \( \gamma \leq \gamma \). For \( T/2 < \tau < t \leq T \), from (1.38), (1.23), (1.27), (1.1) and (1.4) we get:

\[
\|w'(t) - w'(&\tau)\| \leq |\varphi(t) - \varphi(&\tau)| \|B(\phi(t))G(\phi(t), \phi(T/2))x\| + \\
+ \varphi(&\tau)\|B(\phi(t))G(\phi(&\tau), \phi(T/2))x - B(\phi(&\tau))G(\phi(&\tau), \phi(T/2))x\| \leq \\
\leq C_0(\varphi)_C(t - &\tau)^\gamma \|x\|_{D_{IA(T/2)}} + K(T - &\tau)^\delta. \\
\cdot \left\{ C_5(\varphi)(T/2)^{\rho_1}(t - &\tau)^\gamma + C_1(\varphi)(T - &\tau)^{\rho_1}(t - &\tau)^\gamma \right\} \|x\|_{D_{IA(T/2)}(\delta + 1, \infty)} 
\]
and (1.40) holds. In the same way, (1.41) follows from (1.23), (1.27),
(1.1), and (1.4).

In the sequel we shall need also the following lemmas.

**Lemma 1.5.** Let $\mu \geq 0$, $0 < \alpha < 1$, $\nu > 0$ and let $T/2 \leq a \leq b \leq t < T$. Then it holds:

\[ (1.42) \quad \int_a^b (T - s)^{-\mu} (t - s)^{\alpha - 1} \, ds \leq \frac{(b - a)^{\mu - \alpha}}{\alpha - \mu}, \quad \text{if } \mu < \alpha; \]

\[ (1.43) \quad \int_a^b (T - s)^{-\mu} (t - s)^{\alpha - 1} \, ds \leq \]

\[ \leq (T - t)^{\alpha - \mu} \int_0^\infty (1 + y)^{-\mu} y^{\alpha - 1} \, dy, \quad \text{if } \mu > \alpha; \]

\[ (1.44) \quad \int_a^b (T - s)^{-\alpha} (t - s)^{\alpha - 1} \, ds \leq \]

\[ \leq (T/2)^\nu (T - t)^{-\nu} \int_0^\infty (1 + y)^{-(\alpha + \nu)} y^{\alpha - 1} \, dy. \]

**Proof.** Let \( \int_a^b (T - s)^{-\mu} (t - s)^{\alpha - 1} \, da =: I \). In the case $\mu < \alpha$ we get:

\[ I \leq \int_a^b (t - s)^{\alpha - \mu - 1} \, ds = \frac{(b - a)^{\mu - \alpha}}{\alpha - \mu}, \]

and (1.42) holds.

In the case $\mu > \alpha$, (1.43) follows setting $(t - s) = (T - t) y$ in $I$.

In the case $\mu = \alpha$, for each $\nu > 0$, from (1.43) it follows:

\[ I \leq (T/2)^\nu (T - t)^{-\nu} \int_0^\infty (1 + y)^{-(\alpha + \nu)} y^{\alpha - 1} \, dy, \]

and (1.44) holds.
Lemma 1.6. Let \( \varphi \in C([0, T]) \) and let \( \delta \) be defined in (1.10). Then it holds:

\[
B(\hat{\varphi}(t)) \int_a^t \varphi(s) G(\hat{\varphi}(t), \hat{\varphi}(s)) \, ds - \left[ e^{\psi(t, a)B(\hat{\varphi}(t))} y - y \right] =
\]

\[
= O((t - a)^\delta);
\]

if \( \varphi \in C^\gamma([0, T]), \ 0 < \gamma < 1, \ \gamma \leq \beta \) if \( \beta > 0, \ \gamma \leq \beta_1 \) if \( \beta_1 > 0 \),

(see (1.1)), then

\[
B(\hat{\varphi}(t)) \int_a^b \varphi(s) G(\hat{\varphi}(t), \hat{\varphi}(s)) \, ds \left\|_{L(\mathbb{X})} \right. \leq
\]

\[
\begin{cases}
C_7(\gamma, \delta) \left[ \frac{(b - a)^\gamma}{t^\beta} + t^\delta (b - a)^\delta + 1 \right], \\
C_8(\gamma, \delta, T), \ \text{for} \ T/2 < a \leq b \leq t \leq T, \ \beta_1 = 0;
\end{cases}
\]

\[
\begin{cases}
C_9(\beta_1, \gamma, \delta, \nu, T) \left[ \frac{1}{(T - t)^\beta_1 - \gamma + \nu} + (T - t)^\delta_1 (b - a)^\delta + 1 \right], \\
\end{cases}
\]

for \( T/2 < a < b < t < T, \ \beta_1 > 0; \)

with \( \nu = 0 \) if \( \beta_1 > \gamma, \ \nu > 0 \) if \( \beta_1 = \gamma; \)

both for \( 0 \leq a < b \leq t \leq T/2, \) and for \( T/2 < a \leq b \leq t < T. \)

Proof. From [2, (2.10)] it follows, for \( 0 \leq s < t \leq T: \)

\[
B(\hat{\varphi}(t)) G(\hat{\varphi}(t), \hat{\varphi}(s)) = \left[ B(\hat{\varphi}(s)) e^{\psi(t, s)B(\hat{\varphi}(s))} - B(\hat{\varphi}(t)) e^{\psi(t, s)B(\hat{\varphi}(t))} \right] +
\]

\[
+B(\hat{\varphi}(t)) + B(\hat{\varphi}(s)) e^{\psi(t, s)B(\hat{\varphi}(s))} =: I_1 + I_2 + I_3.
\]

By (1.16), (1.12) and (1.10) we have:

\[
\left\| I_1 \right\|_{L(\mathbb{X})} + \left\| I_2 \right\|_{L(\mathbb{X})} \leq M_2 \sum_{i=1}^h \left[ \varphi(t, s) \chi_i / (\delta_i + 1) / \varphi(t, s)^{1 + \delta_i} \right] +
\]

\[
+ c_1 \varphi(t - s)^{\delta_i - 1} \leq (hM_2 + c_1) \varphi(t, s)^{\delta_i - 1}.
\]
Hence by (1.48), (1.5) and (1.6) we get both for $0 \leq a \leq t \leq T/2$ and for $T/2 < a \leq t \leq T$:

\[
(1.49) \quad \int_a^t \varphi(s) \left| I_1 + I_2 \right|_{L^\infty} ds \leq \left( hM_2 + c_1 \frac{K}{k_1^{1-\varepsilon}} \right) \int_a^t \frac{g(s, s) ds}{g(t, s)^{1-\varepsilon} (t-s)^{1-\varepsilon}} = O((t-a)^\varepsilon).
\]

Moreover from [13, (1.4)], for $y \in X$ it follows:

\[
(1.50) \quad \int_a^t \varphi(s) I_3 y ds = e^{\varphi(t, s)B(\varphi(t))} y - y.
\]

Therefore by (1.49) and (1.50), (1.45) holds. Let us show (1.46).

From [2, (2.10)] for $0 \leq s \leq t < T$ it follows:

\[
(1.51) \quad B(\varphi(t)) G(\varphi(t), \varphi(s)) = \left[ B(\varphi(s)) e^{\varphi(t, s)B(\varphi(t))} - B(\varphi(t)) e^{\varphi(t, s)B(\varphi(t))} \right] + Z(\varphi(t), \varphi(s)) + \frac{1}{\varphi(t)} \left[ \varphi(s) B(\varphi(t)) e^{\varphi(t, s)B(\varphi(t))} + (\varphi(t) - \varphi(s)) B(\varphi(t)) e^{\varphi(t, s)B(\varphi(t))} \right] =: I_1 + I_2 + \frac{1}{\varphi(t)} \left[ \varphi(t) I_3 + I_4 \right].
\]

Let $0 \leq a \leq b \leq t < T$. From (1.48) it follows:

\[
(1.52) \quad \left\| \int_a^b (I_1 + I_2) ds \right\|_{L^\infty} \leq (hM_2 + c_1) \int_a^b \varphi(t, s)^{\varepsilon-1} ds;
\]

from (1.15) it follows:

\[
(1.53) \quad \left\| \int_a^b \varphi(s) I_3 ds \right\|_{L^\infty} = \left\| e^{\varphi(t, a)B(\varphi(t))} - e^{\varphi(t, b)B(\varphi(t))} \right\|_{L^\infty} \leq 2M_0,
\]

and

\[
(1.54) \quad \left\| \int_a^b I_4 ds \right\|_{L^\infty} \leq M_1 \left[ \varphi \right] \int_a^b (t-s)^{\gamma} \varphi(t, s)^{-1} ds.
\]
Therefore by (1.51), ..., (1.54) we get:

$$\begin{align*}
(1.55) \quad \left\| B(\phi(t)) \int_{a}^{b} G(\phi(t), \phi(s)) \, ds \right\|_{L(X)} & \leq \\
& \leq (hM_2 + c_1) \int_{a}^{b} \varphi(t, s)^{\gamma - 1} \, ds + \frac{2M_0}{\varphi(t)} + M_1[\varphi]_{C^{0}} \frac{1}{\varphi(t)} \int_{a}^{b} (t - s)^{\gamma} \varphi(t, s)^{1 - \gamma} \, ds.
\end{align*}$$

From (1.6) and (1.4) it follows:

$$\begin{align*}
(1.56) \quad \int_{a}^{b} \varphi(t, s)^{\gamma - 1} \, ds & \leq \left[ \frac{1}{(k_1^{1 - \gamma} \varphi(t, t)^{1 - \gamma})} \right] \int_{a}^{b} (t - s)^{\gamma - 1} \, ds = \\
& = \begin{cases} 
(b - a)^{\gamma} / [\delta k_1^{1 - \gamma} t^{\gamma (1 - \gamma)}] & \text{if } 0 \leq a < b \leq t \leq T/2, \\
(b - a)^{\gamma} / [\delta k_1^{1 - \gamma} (T - t)^{\gamma (1 - \gamma)}] & \text{if } T/2 < a \leq b \leq t < T;
\end{cases}
\end{align*}$$

By (1.6) we have, both for $0 \leq a < b \leq t \leq T/2$ and for $T/2 < a \leq b \leq t < T$:

$$\begin{align*}
(1.57) \quad \int_{a}^{b} (t - s)^{\gamma} \varphi(t, s)^{-1} \, ds & \leq k_1^{-1} \int_{a}^{b} g(t, s)^{-1} (t - s)^{\gamma - 1} \, ds =: J.
\end{align*}$$

Moreover by (1.4) we have:

$$\begin{align*}
(1.58) \quad J = k_1^{-1} t^{-\beta} \int_{a}^{b} (t - s)^{\gamma - 1} \, ds = \frac{(b - a)^{\gamma}}{\gamma k_1 t^{\beta}} & \quad \text{if } 0 \leq a < b \leq t \leq T/2;
(1.59) \quad J = k_1^{-1} \int_{a}^{b} (T - s)^{\beta} (t - s)^{\gamma - 1} \, ds & \quad \text{if } T/2 < a \leq b \leq t < T.
\end{align*}$$

Hence if $0 \leq a < b \leq t < T/2$, by (1.55), ..., (1.58) we have:

$$\begin{align*}
\left\| B(\phi(t)) \int_{a}^{b} G(\phi(t), \phi(s)) \, ds \right\|_{L(X)} & \leq \\
& \leq \frac{K(hM_2 + c_1)(b - a)^{\gamma} t^{\beta \gamma}}{\delta k_1^{1 - \gamma} \varphi(t)} + M_1[\varphi]_{C^{0}} \frac{(b - a)^{\gamma}}{\gamma k_1 \varphi(t) t^{\beta}} + \frac{2M_0}{\varphi(t)},
\end{align*}$$

and the first inequality of (1.46) follows with

$$C_\gamma(\gamma, \varphi) = \max \left\{ \frac{K(hM_2 + c_1)}{[\delta k_1^{1 - \gamma}]}, \frac{M_1[\varphi]_{C^{0}}}{[\gamma k_1]}, \frac{2M_0}{\varphi(t)} \right\}.$$
If $T/2 < a \leq b \leq t \leq T$ and $\beta_1 = 0$, by (1.55), (1.56), (1.57), (1.59) and (1.42) we have:

$$
\left\| B(\phi(t)) \int_a^b G(\phi(t), \phi(s)) \, ds \right\|_{L(X)} \leq [C_7(\gamma, \delta)/\varphi(t)][(b - a)^{\gamma} + (b - a)^{\delta} + 1] \leq [C_7(\gamma, \delta)/k][T^{\gamma} + T^{\delta} + 1],
$$

and the second inequality of (1.46) follows.

Finally, if $T/2 < a \leq b < t \leq T$ and $\beta_1 > 0$, by (1.55), (1.56), (1.57), (1.59), (1.43) and (1.44) we have:

$$
\left\| B(\phi(t)) \int_a^b G(\phi(t), \phi(s)) \, ds \right\|_{L(X)} \leq \frac{K(hM_2 + c_1)(b - a)^{\delta}(T - t)^{2\beta_1}}{\delta k_1^{1 - \delta} \varphi(t)} +
$$

$$
+ \frac{M_1[\varphi]c^{(T/2)^{\gamma}}}{k_1} \int_0^{+\infty} (1 + y)^{-\beta_1 - \nu} y^{\gamma - 1} \, dy \cdot \frac{(T - t)^{\gamma - \beta_1 - \nu}}{\varphi(t)} + \frac{2M_0}{\varphi(t)},
$$

with $\nu = 0$ in the case $\beta_1 > \gamma$ and $\nu > 0$ in the case $\beta_1 = \gamma$.

Hence the third inequality of (1.46) follows with

$$
C_9(\beta_1, \gamma, \delta, \nu, T) =
$$

$$
= \max \left\{ \frac{K(hM_2 + c_1)}{\delta k_1^{1 - \delta}}, \frac{M_1[\varphi]c^{(T/2)^{\gamma}}}{k_1} \int_0^{+\infty} (1 + y)^{-\beta_1 - \nu} y^{\gamma - 1} \, dy, 2M_0 \right\}.
$$

The last inequality is a trivial consequence of the previous inequalities and of (1.4). We introduced it to simplify some statements in the sequel. ■

2. The classical solution.

DEFINITION 2.1. Let $f \in C([0, T]; X)$. A function $u \in C([0, T]; X)$ is said to be a classical solution of (0.1) in the interval $[0, T]$ if $u \in C^1([0, T]; X)$, $t \rightarrow A(t) u(t)$ belongs to $C([0, T]; X)$ and (0.1) holds.

Arguing exactly as in [3, Prop. 3.7(ii)] we get that $x \in D(A(0))$ is a necessary condition in order that problem (0.1) has a classical solution $u$ such that $\|A(t) u(t)\| \leq \text{const} \, t^{-\mu}$, $\mu \in [0, 1 + \delta]$.
In Section 1 we showed (see (1.33), (1.34) and (1.35)) that if \( x \in D(A(0)) \) then the function \( w \) defined in (1.28) is a classical solution of (0.1) for \( f \equiv 0 \). In the general case, arguing exactly as in [2, Thm. 5.2], we get that if problem (0.1) has a classical solution \( u \), then \( u \) is given by the representation formula (0.3). Consequently, the classical solution of (0.1) is unique. We shall show that if \( f \) is either Hölder continuous with values in \( X \), or bounded with values in some interpolation space, then the function \( u \) given by (0.3) is in fact a classical solution of (0.1). We begin by studying the function

\[
\begin{cases}
  v(t) = \int_a^t G(\dot{\phi}(t), \dot{\phi}(s)) f(s) \, ds, \\
  \text{with either } a = 0 \text{ and } 0 \leq t \leq T/2, \text{ or } a = T/2 \text{ and } T/2 < t \leq T.
\end{cases}
\]

We recall that, by (0.3) we have:

\[
u(t) = G(\dot{\phi}(t), 0) x + \int_0^t G(\dot{\phi}(t), \dot{\phi}(s)) f(s) \, ds \quad \text{if } 0 \leq t \leq T/2,
\]

and

\[
u(t) = G(\dot{\phi}(t), \dot{\phi}(T/2)) \bar{x} + \int_{T/2}^t G(\dot{\phi}(t), \dot{\phi}(s)) f(s) \, ds \quad \text{if } T/2 < t \leq T
\]

with \( \bar{x} = u(T/2) \).

**Proposition 2.2.** For every continuous \( f: [0, T] \rightarrow X \), \( v \) enjoys the following properties: for \( 0 < \alpha < 1 \) we have:

\[
v \in C^\alpha([0, T]; X);
\]

\[
v \in B([\varepsilon, T - \varepsilon]; D_{A(\alpha)}(\alpha, \infty)) \quad \varepsilon \in ]0, T/2[;
\]

\[
\begin{cases}
  \text{if } \beta > 0, \quad v \in B([0, T/2]; D_{A(\alpha)}(1/(\beta + 1)), \infty), \\
  \text{if } \beta = 0, \quad v \in B([0, T/2]; D_{A(\alpha)}(\alpha, \infty)), \\
  \text{if } 0 < \alpha < 1/(\beta + 1), \quad v \in B([T/2, T]; D_{A(\alpha)}(\alpha, \infty)).
\end{cases}
\]
PROOF. (2.2) is a simple consequence of estimates (1.17) and (1.24). By (1.17), (1.6) and (1.5) we get, for $0 < t < T$:

$$\| v(t) \|_{D_{A(0)}(\alpha, \infty)} = \left\| \int_{a}^{t} G(\phi(t), \phi(s)) f(s) \, ds \right\|_{D_{A(0)}(\alpha, \infty)} \leq \cdots$$

so that (2.3) holds. Concerning (2.4), by (1.17), (1.6) and (1.4) we have

$$\| v(t) \|_{D_{A(0)}(\alpha, \infty)} \leq \cdots$$

for $0 \leq t \leq T/2$:

$$\| v(t) \|_{D_{A(0)}(\alpha, \infty)} \leq \cdots$$

for $T/2 < t < T$, if $\alpha < 1/ (\beta_1 + 1)$:

$$\| v(t) \|_{D_{A(0)}(\alpha, \infty)} \leq \cdots$$

thanks to (1.42). Hence $v$ is bounded with values in $D_{A(0)}(\alpha, \infty)$ as stated in (2.4).

Now we state two existence and uniqueness results for the classical solution of (0.1). The first one details with the case where $f$ has values in same interpolation space.

**Theorem 2.3.** Let $f \in C([0, \tau]; X) \cap B([0, T]; D_{A(0)}(\alpha, \infty))$, $0 < \alpha < 1$ and let $x \in D(A(0))$. Then the function $u$ defined in (0.3) is the unique classical solution of (0.1). Moreover for $\epsilon \in ]0, T/2[$, $\beta \geq 0$ and $\gamma \leq \alpha$, $\gamma < \delta$, we have:

(2.5) $Au \in C^\gamma([\epsilon, T - \epsilon]; X)$ if $\beta_1 > 0$, $Au \in C^\gamma([\epsilon, T]; X)$ if $\beta_1 = 0$;

(2.6) $u', Au \in B([\epsilon, T - \epsilon]; D_{A(0)}(\vartheta, \infty))$, $\vartheta = \min \{\alpha, \delta\}$, $\epsilon \in ]0, T/2[$.

**Proof.** Since $u = w + v$, with $w$ given by (1.28), it is sufficient to show that $v$ is the classical solution of (0.1) with $x = 0$. From (2.2) and
(2.1) it follows that \( v \in C([0, T]; X) \) and \( v(0) = 0 \). From (1.23) and (1.5) we get:

\[
(2.7) \quad \|A(t) v(t)\| = \left\| B(\dot{\phi}(t)) \int_a^t G(\phi(t), \phi(s)) f(s) \, ds \right\| \leq \\
\leq \frac{C_0(\alpha)}{g(t, t)^{1-\alpha}} \int_a^t (t-s)^{\alpha-1} \, ds \sup_{0 \leq s \leq T} \|f(s)\|_{D_{\text{at}}(\alpha, \infty)} \leq \\
\leq \frac{C_0(\alpha)}{ag(t, t)^{1-\alpha}} (t-\alpha)^{\alpha} \sup_{0 \leq s \leq T} \|f(s)\|_{D_{\text{at}}(\alpha, \infty)}
\]

and \( v(t) \in D(A(t)) \) for each \( 0 < t < T \). By (2.1), (1.23) and (1.25), both for \( 0 \leq \alpha < \tau < t \leq T/2 \) and for \( T/2 \leq \alpha < \tau < t < T \) we have:

\[
(2.8) \quad \|A(t) v(t) - A(\tau) v(\tau)\| \leq \left\| \int_a^\tau B(\dot{\phi}(t)) G(\phi(t), \phi(s)) f(s) \, ds \right\| + \\
\left\| \int_a^\tau [B(\dot{\phi}(t)) G(\phi(t), \phi(s)) - B(\dot{\phi}(\tau)) G(\phi(\tau), \phi(s))] f(s) \, ds \right\| \leq \\
\leq \left\{ \frac{C_0(\alpha)}{g(t, t)^{1-\alpha}} \int_a^\tau \frac{ds}{(t-s)^{1-\alpha}} + \frac{C_2(\alpha)}{g(\tau, \tau)^{1-\alpha}} \int_a^\tau \frac{1}{(\tau-s)^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}} \right\} ds + \\
+ \frac{C_3(\alpha, \delta, \eta) g(t, \tau)^{\alpha-\eta}(t-\tau)^{\delta-\eta}}{g(\tau, \tau)^{1-\alpha}} \left( \int_a^\tau \frac{ds}{(\tau-s)^{1+\gamma-\alpha-\delta}} \right) \right\}.
\]

\[
\sup_{0 \leq s \leq T} \|f(s)\|_{D_{\text{at}}(\alpha, \infty)} \leq \left\{ \frac{C_0(\alpha)(t-\tau)^{\alpha}}{ag(t, t)^{1-\alpha}} + \frac{2C_2(\alpha)(t-\tau)^{\alpha}}{ag(\tau, \tau)^{1-\alpha}} + \\
+ \frac{C_3(\alpha, \delta, \eta) g(t, \tau)^{\alpha-\eta}(t-\tau)^{\delta-\eta}}{(\alpha + \delta - \eta) g(\tau, \tau)^{1-\alpha}} (t-\tau)^{\gamma}(t-\tau)^{\delta-\eta} \right\} \sup_{0 \leq s \leq T} \|f(s)\|_{D_{\text{at}}(\alpha, \infty)},
\]

and taking into account (1.4) we get, for \( \varepsilon \in ]0, T/2[\):

\[
(2.9) Av \in C^\gamma([\varepsilon, T-\varepsilon]; X) \quad \text{if} \quad \beta_1 > 0, \quad Av \in C^\gamma([\varepsilon, T]; X) \quad \text{if} \quad \beta_1 = 0.
\]

Hence (2.5) follows from (1.32).
Let us show that $v$ is differentiable for $0 < t < T$. For $0 < \varepsilon < 1$ set:

\begin{equation}
(2.10) \quad v_\varepsilon(t) = \int_0^t G(\phi(t), \phi(s)) f(s) \, ds.
\end{equation}

As $\varepsilon$ goes to 1, $v_\varepsilon$ converges uniformly to $v$ on each compact subset of $]0, T[$; moreover $v_\varepsilon$ is differentiable in $]0, T[$ with

\begin{equation*}
v_\varepsilon'(t) = \varepsilon G(\phi(t), \phi(t\varepsilon)) f(t\varepsilon) + \varphi(t) B(\phi(t)) \int_0^t G(\phi(t), \phi(s)) f(s) \, ds.
\end{equation*}

Therefore, since $f \in C([0, T]; D(A(t)))$, $v_\varepsilon' \to \varphi(\cdot) A(\cdot)v(\cdot) + f(\cdot)$ as $\varepsilon \to 1$.

Hence $v$ is differentiable in $]0, T[$ with $v'(t) = \varphi(t) A(t)v(t) + f(t)$. Since $\varphi \in C([0, T])$, by (2.5) it follows $v' \in C([0, T]; X)$. Summing up, we find that $u = w + v$ is a classical solution of (0.1). Concerning (2.6), by (1.31) $w \in B([\varepsilon, T]; D_A(\sigma + \varepsilon, \infty))$. Moreover setting, for $0 < s < t < T$ and $0 < \sigma < \tau < \phi(T)$, $\phi(s) = \sigma$ and $\phi(t) = \tau$ in (2.1), we have:

\begin{equation}
(2.11) \quad A(t)v(t) = A(\phi^{-1}(\tau)) \int_{\phi(a)}^{\tau} G(\tau, \sigma) f(\phi^{-1}(\sigma)) \frac{d\sigma}{\varphi(\phi^{-1}(\sigma))},
\end{equation}

\begin{equation*}
= B(\tau) \int_{\phi(a)}^{\tau} G(\tau, \sigma) \tilde{f}(\tau) \, d\sigma = B(\tau) \tilde{u}(\tau).
\end{equation*}

By (1.1) and (1.3) for $0 \leq t \leq T/2$ and $0 \leq \sigma \leq \phi(T/2)$ it follows:

\begin{equation*}
\varphi(\phi^{-1}(\sigma)) \geq k(\phi^{-1}(\sigma))^{\beta} \geq (k/k_1^{\beta/(\beta + 1)}) \sigma^{\beta/(\beta + 1)};
\end{equation*}

hence

\begin{equation}
(2.12) \quad \sigma \to \sigma^{\beta/(\beta + 1)} \tilde{f}(\sigma) \in B([0, \phi(T/2)]; X) \cap B([0, \phi(T/2)]; D_{B(\cdot)}(\sigma, \infty)).
\end{equation}

Plugging (2.12) in (2.11) and applying Prop. 3.1 (vi) of [3] with $x = 0$, $\mu = \beta/(\beta + 1) < 1$, we get, if $\varepsilon = \min \{x, \varepsilon\}$:

\begin{equation}
(2.13) \quad \tau \to \tau^{\beta + \beta/(\beta + 1)} B(\tau) \tilde{v}(\tau) \in B([0, T/2]; D_{B(\cdot)}(\varepsilon, \infty)),
\end{equation}

so that by (2.11), $Av \in B([\varepsilon, T/2]; D_{A(\varepsilon)}(\varepsilon, \infty))$, $0 < \varepsilon < T/2$.

If $T/2 < t \leq T$ and $\phi(T/2) < \sigma \leq \phi(T)$, by (1.1) and (1.3) it follows:

\begin{equation*}
\varphi(\phi^{-1}(\sigma)) \geq k[T - \phi^{-1}(\sigma)]^{\beta} \geq (k/k_1^{\beta/(\beta + 1)}) [\phi(T) - \sigma]^{\beta/(\beta + 1)};
\end{equation*}
hence

\begin{equation}
\sigma \to [\hat{\varphi}(T) - \sigma]^{\beta_1/(\beta_1 + 1)} f(\sigma) \in \\
eq B([\hat{\varphi}(T/2), \hat{\varphi}(T)]; X) \cap B([\hat{\varphi}(T/2), \hat{\varphi}(T)]; D_{B(\eta)}(x, \infty)).
\end{equation}

So that, plugging (2.14) in (2.11), we obtain that

\[ Av \in B([T/2, T - \varepsilon]; D_{A(\varphi)}(\varphi, \infty)), \quad 0 < \varepsilon < T/2. \]

Since \( u = w + v \), then (2.6) holds. \( \square \)

Now we consider the case where both \( f \) and \( \varphi \) are H"older continuous functions.

**Theorem 2.4.** Let \( \varphi \in C^\gamma([0, T]), 0 < \gamma < 1, \gamma < \beta \) if \( \beta > 0, \gamma < \beta_1 \) if \( \beta_1 > 0 \) (see (1.1)), and let \( f \in C^\alpha([0, T]; X), 0 < \alpha < 1, x \in D(A(0)) \). Then the function \( u \) defined in (0.3) is the unique classical solution of (0.1) and

\begin{equation}
\begin{align*}
(2.15) & \quad u'(t) = w'(t) + \varphi(t) A(t) \int_0^t G(\hat{\varphi}(t), \hat{\varphi}(s))(f(s) - f(t)) \, ds + \\
& \quad + \varphi(t) A(t) \int_0^t G(\hat{\varphi}(t), \hat{\varphi}(s))f(t) \, ds + f(t) = \varphi(t) A(t) u(t) + f(t).
\end{align*}
\end{equation}

Moreover for each \( \varepsilon \in ]0, T/2[, \gamma \leq \min \{ \alpha, \gamma \} \) and \( \gamma < \varepsilon \) we have:

\begin{equation}
\begin{align*}
(2.16) & \quad u' \in C^\gamma([\varepsilon, T - \varepsilon]; X) \text{ if } \beta_1 > 0, \quad u' \in C^\gamma([\varepsilon, T]; X) \text{ if } \beta_1 = 0; \\
(2.17) & \quad u' \in B([\varepsilon, T - \varepsilon]; D_{A(\eta)}(\eta, \infty)).
\end{align*}
\end{equation}

**Proof.** As before, it is sufficient to show that \( v \) is the classical solution of (0.1) with \( x = 0 \). From (2.2) and (2.1) it follows \( v \in C([0, T]; X) \) and \( v(0) = 0 \). Let us estimate \( A(t)v(t) \). By (1.23) and (1.46), for \( 0 < t < T \) we have (with either \( a = 0 \) and \( 0 \leq t < T/2 \), or \( a = T/2 \) and
\[ T/2 \leq t \leq T: \]
\[
\|A(t)v(t)\| = \left\| B(\phi(t)) \int_a^t G(\phi(t), \phi(s))(f(s) - f(t)) \, ds \right\| + 
\]
\[
+ \left\| B(\phi(t)) \int_a^t G(\phi(t), \phi(s)) f(t) \, ds \right\| \leq 
\]
\[
\leq \frac{C_0[f]_{C^*}}{g(t, t)} \int_a^t \frac{ds}{(t-s)^{1-\alpha}} + \frac{C(\beta, \beta_1, \gamma, \delta, \nu, T)}{g(t, t) \varphi(t)} \|f\|_{\infty}
\]

and \( v(t) \in D(A(t)) \) for \( 0 < t < T \). To show (2.16) we set, for \( 0 < t < T \):

\[
\begin{aligned}
A(t)v(t) &= h_1(t) + h_2(t)/\varphi(t), \\
h_1(t) &= A(t) \int_a^t G(\phi(t), \phi(s))(f(s) - f(t)) \, ds, \\
h_2(t) &= \varphi(t) A(t) \int_a^t G(\phi(t), \phi(s)) f(t) \, ds.
\end{aligned}
\]

(2.18)

Both for \( 0 \leq a < \tau < t < T/2 \) and for \( T/2 \leq a < \tau < t < T \) we have:

\[
\|h_1(t) - h_1(\tau)\| \leq \left\| B(\phi(t)) \int_\tau^t G(\phi(t), \phi(s))(f(s) - f(t)) \, ds \right\| + 
\]
\[
+ \left\| \int_\tau^a [B(\phi(t)) G(\phi(t), \phi(s)) - B(\phi(\tau)) G(\phi(\tau), \phi(s))](f(s) - f(t)) \, ds \right\| + 
\]
\[
+ \left\| B(\phi(t)) \int_\tau^a G(\phi(t), \phi(s))(f(\tau) - f(t)) \, ds \right\| = I_1 + I_2 + I_3.
\]

By (1.23) we get:

\[ I_1 \leq C_0 \frac{(t - \tau)^\alpha}{\alpha g(t, t) [f]_{C^*}}; \]
Hölder regularity in non autonomous degenerate etc.

by (1.25) for \( \eta < \delta \) we get:

\[
I_2 \leq \left\{ C_2 \frac{t - \tau}{g(\tau, \tau)} \int_a^t (t - s)^{-1} (\tau - s)^{\alpha - 1} ds + \right. \\
+ C_3 (\delta, \eta) \frac{g(t, \tau) \tau^{\alpha - \eta} \tau^{\tau - \eta}}{g(\tau, \tau)} (t - \tau)^{\eta} \int_a^\tau (\tau - s)^{\alpha + \tau - \eta - 1} ds \right\} [f]_{C^*} \leq \\
\leq \left\{ C_2 \frac{(t - \tau)^\alpha}{g(\tau, \tau)} \int_0^\infty (1 + y)^{-1} y^{\tau - 1} dy + \right. \\
+ C_3 (\delta, \eta) \frac{g(t, \tau) \tau^{\alpha - \eta} \tau^{\tau - \eta}}{(\alpha + \delta - \eta) g(\tau, \tau)} (t - \tau)^{\eta} (\tau - \alpha)^{\alpha + \tau - \eta - 1} \right\} [f]_{C^*};
\]

by (1.46) we have:

\[
I_3 \leq \frac{C(\beta, \beta_1, \gamma, \delta, \nu, T)}{g(t, t) \varphi(t)} (t - \tau)^\eta [f]_{C^*};
\]

and taking into account (1.4) and (1.1), for \( 0 < \varepsilon < T/2 \), we get:

(2.20) \( h_1 \in C^\gamma([\varepsilon, T - \varepsilon]; X) \) if \( \beta_1 > 0 \), \( h_1 \in C^\gamma([\varepsilon, T]; X) \) if \( \beta_1 = 0 \).

On the other hand we have:

(2.21) \( \| h_2(t) - h_2(\tau) \| \leq \left\| B(\phi(t)) \int_a^t (\phi(t) - \phi(s)) G(\phi(t), \phi(s)) f(t) ds \right\| + \right. \\
+ \left\| B(\phi(t)) \int_a^\tau (\phi(t) - \phi(\tau)) G(\phi(t), \phi(\tau)) f(t) ds \right\| + \\
+ \left\| \int_a^\tau (\phi(\tau) - \phi(s)) [B(\phi(t)) G(\phi(t), \phi(s)) f(t) - B(\phi(\tau)) G(\phi(\tau), \phi(s)) f(\tau)] ds \right\| + \\
+ \left\| \int_a^t \phi(s) B(\phi(t)) G(\phi(t), \phi(s)) f(t) ds - \int_a^{\tau} \phi(s) B(\phi(\tau)) G(\phi(\tau), \phi(s)) f(\tau) ds \right\| =: I_1 + I_2 + I_3 + I_4. \)
By (1.23) we have:

\[ I_1 \leq C_0 \frac{(t - \tau)^\gamma}{\gamma g(t, t)} \|f(t)\|_C^\gamma ; \]

by (1.46) we have:

\[ I_2 \leq \frac{C(\beta, \beta_1, \gamma, \delta, \nu, T)}{g(t, t) \varphi(t)} (t - \tau)^\gamma \|f(t)\|_C^\gamma ; \]

by (1.25) and (1.23) for \( \eta < \delta \) we have:

\[ I_3 \leq \left\{ \int_a^\tau (\tau - s)^\gamma \left[ \frac{C_2 (t - \tau)}{g(\tau, \tau)(\tau - s)(t - s)} + C_3 (\delta, \eta) \frac{g(t, \tau)^\gamma g(\tau, s)^\nu (t - \tau)^\gamma}{g(\tau, \tau)(t - s)^{1 - \delta + \gamma}} \right] ds \right\} \|f(\tau)\| + \int_a^\tau (\tau - s)^\gamma \|B(\phi(t)) G(\phi(t), \phi(s))(f(t) - f(\tau))\| ds \frac{[\varphi]}{C} \leq \]

\[ \leq \left\{ \left[ C_2 \frac{(t - \tau)^\gamma}{g(\tau, \tau)} \int_0^\infty (1 + y)^{-1} y^{\gamma - 1} dy + \right. \right. \]

\[ \left. \left. C_3 (\delta, \eta) \frac{g(t, \tau)^\gamma g(\tau, \alpha)^{\delta - \gamma} (\tau - a)^{\nu + \delta - \eta}}{(\gamma + \delta - \eta) g(\tau, \tau)} (t - \tau)^\gamma \right] \right\} \|f(\tau)\| + C_0 \frac{(t - \tau)^\gamma}{g(t, \tau)} \int_a^\tau (\tau - s)^{\gamma - 1} ds \frac{[f]}{C} \frac{[\varphi]}{C} \leq \]

\[ \leq \left\{ \left[ C(\gamma) C_2 \frac{(t - \tau)^\gamma}{g(\tau, \tau)} + C_3 (\delta, \eta) \frac{g(t, \tau)^\gamma g(\tau, \alpha)^{\delta - \gamma} (\tau - a)^{\nu + \delta - \eta}}{(\gamma + \delta - \eta) g(\tau, \tau)} (t - \tau)^\gamma \right] \right\} \|f(\tau)\| + C_0 \frac{(t - \tau)^\gamma (\tau - \alpha)^\gamma}{\gamma g(t, \tau)} \frac{[f]}{C} \frac{[\varphi]}{C} . \]
Concerning $I_4$, set $\phi(s) = \sigma$. Then we obtain:

$$I_4 = \left\| B(\phi(t)) \int_{\phi(a)}^{\phi(t)} G(\phi(t), \sigma)f(t) \, d\sigma - B(\phi(t)) \int_{\phi(a)}^{\phi(\tau)} G(\phi(\tau), \sigma)f(\tau) \, d\sigma \right\| =$$

$$= \left\| B(\phi(t)) \int_{\phi(a)}^{\phi(t)} G(\phi(t), \sigma)(f(t) - f(\tau)) \, d\sigma \right\| +$$

$$+ \left\| B(\phi(t)) \int_{\phi(a)}^{\phi(t)} G(\phi(t), \sigma)f(\tau) \, d\sigma - B(\phi(t)) \int_{\phi(a)}^{\phi(\tau)} G(\phi(\tau), \sigma)f(\tau) \, d\sigma \right\| .$$

Thanks to (1.21), (1.22), (1.2), (1.6) and (1.4) we get:

$$I_4 \leq c(t - \tau)^2 [f]_{C^\alpha} + c(\tau) [\frac{\phi(t) - \phi(\tau)}{\phi(\tau) - \phi(a)}] [f(\tau)] \leq$$

$$\leq c(t - \tau)^2 [f]_{C^\alpha} + C(\alpha, K, k_1) \frac{g(t, \tau)^2(t - \tau)^2}{g(\tau, a)^2(\tau - a)^2} \| f(\tau) \| \leq$$

$$\leq \left[ c + C(\alpha, K, k_1, \beta, \beta_1, T) \frac{1}{\tau^{2\alpha} (\tau - a)^2} \right] (t - \tau)^2 \| f \|_{C^\alpha} .$$

Hence by (1.1) and (1.4), for $0 < \varepsilon < T/2$, we get:

(2.22) $Av \in C^\gamma(\varepsilon, T - \varepsilon]; X)$ if $\beta_1 > 0$, $Av \in C^\gamma(\varepsilon, T]; X)$ if $\beta_1 = 0$.

Let us show that $v$ is differentiable for $0 < t < T$. For $0 < \varepsilon < 1$, let $v_\varepsilon$ be defined by (2.10): then $v_\varepsilon$ is differentiable in $]0, T[$ with

$$v_\varepsilon'(t) = \varepsilon G(\phi(t), \phi(\varepsilon)) f(\varepsilon) + \phi(t) B(\phi(t)) \int_{\varepsilon}^{t} G(\phi(t), \phi(s)) (f(s) - f(\varepsilon)) \, ds +$$

$$+ B(\phi(t)) \int_{\varepsilon}^{t} (\phi(s) - \phi(\varepsilon)) G(\phi(t), \phi(s)) \, ds +$$

$$+ B(\phi(t)) \int_{\varepsilon}^{t} \phi(s) G(\phi(t), \phi(s)) f(s) \, ds - B(\phi(t)) \int_{t_\varepsilon}^{t} \phi(s) G(\phi(t), \phi(s)) f(s) \, ds.$$
By (1.45) we get:

\[
\lim_{\varepsilon \to 1} \left\| -B(\phi(t)) \int_0^t \varphi(s) G(\phi(t), \phi(s)) f(t) ds + e^{\varphi(t, t_0)B(\phi(t))} f(t) - f(t) \right\| = 0.
\]

By (1.11), (1.12), (1.15) and (1.16), thanks to continuity of \(f\), we have:

\[
\lim_{\varepsilon \to 1} \| eG(\phi(t), \phi(t\varepsilon)) f(t\varepsilon) - e^{\varphi(t, t\varepsilon)B(\phi(t))} f(t) \|
\]

\[
= \lim_{\varepsilon \to 1} \left\| e^{\varphi(t, t\varepsilon)B(\phi(t\varepsilon))} \varepsilon f(t\varepsilon) + \int_{\phi(t\varepsilon)}^{\phi(t)} Z(r, \phi(t\varepsilon)) \varepsilon f(t\varepsilon) dr - 
\right.
\]

\[
- e^{\varphi(t, t\varepsilon)B(\phi(t\varepsilon))} f(t) - e^{\varphi(t, t\varepsilon)B(\phi(t))} f(t) + e^{\varphi(t, t\varepsilon)B(\phi(t\varepsilon))} f(t) \right\| = 
\]

\[
= \lim_{\varepsilon \to 1} \left\| e^{\varphi(t, t\varepsilon)B(\phi(t\varepsilon))} (\varepsilon f(t\varepsilon) - f(t)) + [e^{\varphi(t, t\varepsilon)B(\phi(t\varepsilon))} - e^{\varphi(t, t\varepsilon)B(\phi(t))} ] f(t) \right\| \leq 
\]

\[
\leq \lim_{\varepsilon \to 1} hM_2 \varphi(t, t\varepsilon)^\nu \| f(t) \| = 0.
\]

Hence as \(\varepsilon \to 1\), \(v'(t)\) converges uniformly to the function

\[
z(t) = \varphi(t) B(\phi(t)) \int_a^t G(\phi(t), \phi(s)) (f(s) - f(t)) ds + 
\]

\[
+ \varphi(t) B(\phi(t)) \int_a^t G(\phi(t), \phi(s)) f(t) ds + f(t), \quad 0 < t < T,
\]

on each compact subset of \([0, T]\). Therefore \(v\) is differentiable in \([0, T]\) with \(v'(t) = z(t)\) for \(0 < t < T\) and (2.15) follows. Since \(\varphi \in C^\gamma ([0, T])\) and \(u'(t) = w'(t) + \varphi(t) A(t) v(t) + f(t), \) (2.16) follows from (1.39) and (2.22). Summing up, we find that \(u = w + v\) is a classical solution of (0.1). Finally from (1.32), (2.22) and (2.16) it follows that

\[
u \in C^{1, \gamma} ([\varepsilon, T - \varepsilon]; X) \quad \text{and} \quad t \to A(t) u(t) \in C^\gamma ([\varepsilon, T - \varepsilon]; X),
\]

so that (2.17) follows by interpolation (it is sufficient to argue as in [11, Lemma 1.1], with obvious modifications).
3. The strict solution.

Definition 3.1. If \( f: [0, T] \rightarrow X \) is continuous, any function \( u \in C^1([0, T]; X) \) is said to be a strict solution of (0.1) in \([0, T]\), if \( t \rightarrow A(t)u(t) \) belongs to \( C([0, T]; X) \) and (0.1) holds. ■

Since any strict solution is also a classical one, then the strict solution of (0.1) is unique, and it is given by the representation formula (0.3). If \( u \) is the strict solution of (0.1) then \( \varphi(\cdot)A(\cdot)u(\cdot) \) belongs to \( C([0, T]; X) \). We showed in Section 1 that in the case \( f \equiv 0 \), the function \( w \) defined in (1.28) is the strict solution of (0.1) if and only if either \( x \in D_A(0) \) (in the case \( \beta > 0 \)), or \( x \in D(A(0)) \). (1.36).

Let us give some sufficient conditions for the existence and uniqueness of the strict solution to problem (0.1). As for the classical solution, we begin with the case where \( f \) has values in an interpolation space.

Proposition 3.2. Let \( f \in C([0, T]; X) \cap B([0, T]; D_A(\alpha, \infty)) \), \( 0 < \alpha < 1 \) and assume \( x \in D_A(0) \). Then the function \( u \) defined in (0.3) is the unique strict solution of (0.1) in \([0, T]\).

Proof. By Theorem 2.3 \( u \) is the classical solution of (0.1). Moreover from (1.36) \( w \) belongs to \( C^1([0, T]; X) \). Therefore it is sufficient to show that \( t \rightarrow v'(t) \) is continuous at \( t = 0 \) and \( t = T \).

Since \( v'(t) = \varphi(t)A(t)v(t) + f(t), 0 < t < T \); by (2.1), (2.7) and (1.4) we get \( \lim_{t \to 0} v'(t) = f(0) \) and, in the case \( \beta > 0 \), \( \lim_{t \to T} v'(t) = f(T) \).

In the case \( \beta_1 = 0 \), \( t \rightarrow v'(t) \) is continuous at \( t = T \) thanks to (2.9), so that the statement follows. ■

Now we consider the case where both \( f \) and \( \varphi \) are Hölder continuous.

Theorem 3.3. Let \( \varphi \in C^\gamma([0, T]), 0 < \gamma < 1, \gamma \leq \beta \) if \( \beta > 0 \), \( \gamma \leq \beta_1 \) if \( \beta_1 > 0 \) (see (1.1)), \( \max \{\beta - \gamma, \beta_1 - \gamma\} < 1 \) and let \( f \in C^\alpha([0, T]; X) \) with \( \max \{0, \beta - \gamma, \beta_1 - \gamma\} < \alpha < 1 \). Let moreover:

a) \( x \in D_A(0)(1/(\beta + 1)), f(0) \in D_A(0)(\delta, \infty), \delta > (\beta - \gamma)/(\beta + 1) \) if \( \beta > 0 \);

b) \( x \in D(A(0)), \varphi(0)A(0)x + f(0) \in D(A(0)) \) if \( \beta = 0 \); \( f(T) = 0 \) if \( \beta_1 > 0 \).
Then the function $u$ given by (0.3) is the unique strict solution of (0.1) in $[0, T]$.

**Proof.** From Theorem 2.4 we know that $u$ is the classical solution of (0.1). Therefore we have only to show that $t \to u'(t) = w'(t) + v'(t)$ is continuous at $t = 0$ and $t = T$. First we consider the behaviour near $t = T$. By (1.36) $w' \in C([0, T]; X)$. Moreover in the case $\beta_1 = 0$, $t \to u'(t)$ is continuous at $t = T$ thanks to (2.16). We consider now the case $\beta_1 > 0$. By (2.15) using notation (2.18) for $T/2 < t < T$ we have:

$$u'(t) = w'(t) + \varphi(t) h_1(t) + h_2(t) + f(t).$$

From (2.18), (1.23), (1.4) and (1.1) it follows:

$$\| \varphi(t) h_1(t) \| \leq KC_0(T - t)^{\beta_1} \int_{T/2}^t (T - s)^{-\beta_1} (t - s)^{\alpha - 1} ds[f]_{C^\gamma}. $$

Hence from (1.42), (1.43) and (1.44) it follows:

$$\lim_{t \to T^-} \| \varphi(t) h_1(t) \| = 0.$$

Moreover by (2.18) and (1.46) it follows:

$$\lim_{t \to T^-} \| h_2(t) \| \leq$$

$$\leq C_0(\beta_1, \gamma, \phi, \nu, T) \lim_{t \to T^-} \left[ 1/(T - t)^{\beta_1 - \gamma + \nu} + (T - t)^{\beta_1} (t - T/2)^{\phi} + 1 \right] \cdot (T - t)^{\nu}[f]_{C^\gamma} = 0 \quad (\text{choosing } 0 < \nu < \alpha - \beta_1 + \gamma).$$

Concerning the behaviour of $u'$ as $t \to 0$, we set for $0 < t \leq T/2$:

$$h_2(t) = \varphi(t) A(t) \int_0^t G(\phi(t), \phi(s))(f(t) - f(0)) ds,$$

$$h_3(t) = \varphi(t) A(t) \int_0^t G(\phi(t), \phi(s)) f(0) ds.$$

(3.1)

By (2.15), using notation (2.18) and (3.1) for $0 < t \leq T/2$ we have:

$$u'(t) = w'(t) + \varphi(t) h_1(t) + h_2(t) + h_3(t) + f(t).$$
From (2.18), (1.23), (1.4) and (1.1) it follows, for $\beta \geq 0$:
\[
\lim_{t \to 0^+} \| \varphi(t) h_1(t) \| \leq KC_0 \lim_{t \to 0^+} \int_0^t (t-s)^{x-1} ds [f]_{C^x} = 0.
\]

From (3.1) and (1.46) it follows, for $\beta \geq 0$:
\[
\lim_{t \to 0^+} \| \bar{h}_2(t) \| \leq C_7 (\gamma, \delta) \lim_{t \to 0^+} [t^{\gamma-\beta} + t^{\delta(\delta+1)} + 1] t^x [f]_{C^x} = 0.
\]

By (3.1) we get:
\[
\begin{equation}
(3.2) \quad h_3(t) = \int_0^t \varphi(s) B(\varphi(t)) G(\varphi(t), \varphi(s)) f(0) ds + \nonumber
\end{equation}
\]
\[
+ B(\varphi(t)) \int_0^t (\varphi(t) - \varphi(s)) G(\varphi(t), \varphi(s)) f(0) ds =: I_1 + I_2,
\]
and by (1.45) we get:
\[
(3.3) \quad \| I_1 - e^{\varphi(t) B(\varphi(t))} f(0) + f(0) \| = O(t^{\delta}).
\]

We distinguish now two cases: $\beta > 0$ and $\beta = 0$.

In the case $\beta > 0$, from [2, Thm. 4.1(iii)], (1.23) and (1.4) we get:
\[
(3.4) \quad \| I_2 \| \leq \left\| B(\varphi(t)) \int_0^t (\varphi(t) - \varphi(s)) G(\varphi(t), \varphi(s)) [1 - G(\varphi(s), 0)] f(0) ds \right\| + \nonumber
\]
\[
+ \left\| B(\varphi(t)) \int_0^t (\varphi(t) - \varphi(s)) G(\varphi(t), \varphi(s)) G(\varphi(s), 0) f(0) ds \right\| \leq 
\]
\[
\leq \left[ C(\varphi) t^{-\beta} \int_0^t (t-s)^{\gamma-1} \varphi(s)^{\delta} ds + C_0 (\varphi) t^{-\gamma + (1-\delta)} \int_0^t (t-s)^\gamma ds \right].
\]

\[
\cdot [\varphi]_{C^\gamma} \| f(0) \|_{D_{A^{(\delta, \infty)}}} \leq \left[ C(K, \varphi) t^{-\beta} \int_0^t s^{(\delta+1)(\gamma-1)} ds + C(\gamma, \varphi) t^{\gamma-\beta + (\delta+1)\delta} \right].
\]

\[
\cdot [\varphi]_{C^\gamma} \| f(0) \|_{D_{A^{(\delta, \infty)}}} = C(K, \gamma, \varphi) t^{\gamma-\beta + (\delta+1)\delta} [\varphi]_{C^\gamma} \| f(0) \|_{D_{A^{(\delta, \infty)}}}.
\]

Therefore by (3.2), (3.3) and (3.4), if $\beta > 0$, $\lim_{t \to 0^+} h_3(t) = 0$, since $f(0) \in$
Then, from (3.2), (3.3), (3.5), (1.34), [2, (2.10)], (1.12) and (1.16) it follows:

\[
\lim_{t \to 0^+} \| I_2 \| \leq \lim_{t \to 0^+} C_0 \int_0^t (t - s)^{\gamma - 1} ds \| f \|_\infty = 0.
\]

Then, from (3.2), (3.3), (3.5), (1.34), [2, (2.10)], (1.12) and (1.16) it follows:

\[
\lim_{t \to 0^+} [w'(t) + h_\beta(t) + f(t)] = \lim_{t \to 0^+} [w'(t) + e^{\beta(t)B(t)} f(0) - f(0) + f(t)] = \\
= \lim_{t \to 0^+} \{ (\varphi(t) - \varphi(0)) B(\varphi(t)) G(\varphi(t), 0) x + \varphi(0)[B(0) e^{\beta(0)B(0)} x + \mathcal{Z}(\varphi(t), 0) x] + \\
+ [e^{\beta(t)B(t)} - e^{\beta(0)B(0)}] f(0) + e^{\beta(0)B(0)} f(0) \} = \\
= \lim_{t \to 0^+} e^{\beta(0)A(0)} [\varphi(0) A(0) x + f(0)] = \varphi(0) A(0) x + f(0),
\]

since \( \varphi(0) A(0) x + f(0) \in \overline{D(A(0))} \). Therefore both for \( \beta > 0 \) and for \( \beta = 0 \) there exist \( \lim_{t \to 0^+} u'(t) \).

Summarizing, under the previous assumptions \( u' \) is continuous up to \( t = 0 \) and \( t = T \), with \( u'(0) = f(0) \) if \( \beta > 0 \), \( u'(0) = \varphi(0) A(0) x + f(0) \) if \( \beta = 0 \), and \( u'(T) = 0 \) if \( \beta_1 > 0 \), \( u'(T) = \varphi(T) A(T) u(T) + f(T) \) if \( \beta_1 = 0 \).

In Theorem 3.3 we assumed \( \beta - \gamma < 1 \) and \( \beta_1 - \gamma < 1 \) for simplicity.

In fact, we could study the existence of a strict solution to (0.1) for any value of \( \beta - \gamma \) and \( \beta_1 - \gamma \); obviously, we should made much more regularity assumptions on \( f \).

Now we show some further regularity properties of the strict solution of problem (0.1): roughly speaking, the regularity of \( u \) up to \( t = 0 \) increases as the regularity of the initial value \( x \) increases.

**Proposition 3.4.** Let \( f \in C([0, T]; X) \cap B([0, T]; D_{A(\alpha)}(\infty, \infty)) \), \( 0 < \alpha \leq 1 \), and let \( x \in D_{A(0)}(\alpha + 1, \infty) \) if \( \beta = 0 \), \( x \in D_{A(0)}(\alpha + 1/(\beta + 1), \infty) \),
Let $u$ be the strict solution of (0.1). Then

$$
\begin{align*}
(3.6) \quad & \begin{cases}
\text{if } \beta = \beta_1 = 0 & u' \in B([0, T]; D_{A(\cdot)}(\alpha, \infty)), \\
\text{if } \beta = 0, \beta_1 > 0, 0 < \varepsilon < T/2 & u' \in B([0, T - \varepsilon]; D_{A(\cdot)}(\alpha, \infty))
\end{cases},
\end{align*}
$$

$$
\begin{align*}
(3.7) \quad & \begin{cases}
\text{if } \beta > 0 \text{ and } \beta_1 = 0 & t^{\alpha(\beta + 1)}u' \in B([0, T]; D_{A(\cdot)}(\alpha, \infty)), \\
\text{if } \beta > 0, \beta_1 > 0, 0 < \varepsilon < T/2 & t^{\alpha(\beta + 1)}u' \in B([0, T - \varepsilon]; D_{A(\cdot)}(\alpha, \infty))
\end{cases},
\end{align*}
$$

$$
\begin{align*}
(3.8) \quad & u' \in B([0, T]; D_{A(\cdot)}(\alpha, \infty)) \quad \text{for every } \sigma \in [0, \alpha].
\end{align*}
$$

PROOF. Since $u'(t) = w'(t) + v'(t)$, $0 \leq t \leq T$, and by (1.37) $w' \in B([0, T]; D_{A(\cdot)}(\alpha, \infty))$, it is sufficient to show that $v'(t) = \varphi(t)A(t)v(t) + f(t)$ satisfies (3.6), (3.7) and (3.8). Arguing as in Theorem 2.3, in the case $\beta = 0$, thanks to (2.11), (2.12) and (2.14), (3.6) follows from Prop. 3.1(iii) of [3]. In the case $\beta > 0$, by (1.3) we get:

$$
\begin{align*}
\tau = \varphi(t) & \geq k_1 t^{\alpha(\beta + 1)}, \quad 0 \leq t \leq T/2 \\
\text{and} & \\
\tau^{\alpha + \beta/(\beta + 1)} & \geq k_1^{\alpha + \beta/(\beta + 1)} t^{\beta + \alpha(\beta + 1)}.
\end{align*}
$$

Therefore from (2.11), (2.13) and (3.9) it follows that $t^{\alpha(\beta + 1)}v'(t)$ is bounded with values in $D_{A(\cdot)}(\alpha, \infty)$ for $0 \leq t \leq T/2$, so that (3.7) holds thanks to (3.6). Moreover for each $\beta$ and $\beta_1$, by (1.23) for $0 < \delta < \alpha, 0 \leq t \leq T$ we have:

$$
\varphi(t)\|A(t)v(t)\|_{D_{A(\cdot)}(\delta, \infty)} \leq \\
\leq KC_0(\alpha, \delta) g(t, t)^{\alpha - \delta} \int_0^t \! (t - s)^{\alpha - \delta - 1} ds \sup_{0 \leq s \leq T} \|f(s)\|_{D_{A(\cdot)}(\alpha, \infty)} \leq \\
\leq KC_0(\alpha, \delta) g(t, t)^{\alpha - \delta} (t - \alpha)^{\alpha - \delta} \sup_{0 \leq s \leq T} \|f(s)\|_{D_{A(\cdot)}(\alpha, \infty)} < + \infty.
$$

Hence $v'(t)$ belongs to $B([0, T]; D_{A(\cdot)}(\delta, \infty))$ and (3.8) holds.

PROPOSITION 3.5. Let $\varphi \in C^r([0, T]), 0 < \gamma < 1, \gamma \leq \beta$ if $\beta > 0, \gamma \leq \beta_1$, if $\beta_1 > 0$, and let $f \in C([0, T]; X) \cap B([0, T]; D_{A(\cdot)}(\alpha, \infty))$, max $\{0, (\beta - \gamma)/(\beta + 1), \beta_1/(\beta_1 + 1)\} < \alpha < 1, x \in D_{A(0)}(\alpha + 1/(\beta + 1), \infty)$. 

\begin{align*}
0 < \alpha < \beta/(\beta + 1) & \text{ if } \beta > 0. \text{ Let } u \text{ be the strict solution of (0.1). Then }
\end{align*}
Let \( u \) be the strict solution of (0.1). Then

\[
(3.10) \quad \varphi Au \in C^\gamma([0, T]; X)
\]

with \( \eta < \delta, \eta \leq \min\{\alpha, \gamma, \alpha(\beta + 1) - (\beta - \gamma)\} \).

**Proof.** By Prop. 3.2, \( \varphi(t)A(t)u(t) = w'(t) + \varphi(t)A(t)v(t), \quad 0 \leq t \leq T. \)

By (1.40) and (1.41) \( w' \in C^\gamma([0, T]; X) \). Moreover from (2.1), (1.23), (1.4), (2.7) and (2.8) it follows for \( 0 \leq \tau < t \leq T/2 \):

\[
\| \varphi(t)A(t)v(t) - \varphi(\tau)A(\tau)v(\tau) \| \\
\leq \| \varphi(t) - \varphi(\tau) \| \| A(t)v(t) \| + \varphi(\tau)\| A(t)v(t) - A(\tau)v(\tau) \|
\leq \left\{ \begin{array}{l}
\frac{C_0(\alpha)}{\alpha} t^{\alpha(\beta + 1) - \beta}(t - \tau)^\gamma \| \varphi \| C_r + \frac{KC_0(\alpha)}{\alpha} t^{\alpha\beta_1}(t - \tau)^\gamma + \frac{2KC_2(\alpha)}{\alpha} \tau^{\alpha\beta_1} \\
\cdot (t - \tau)^\gamma + \frac{KC_3(\alpha, \delta, \eta)}{\alpha + \delta - \eta} t^{\delta_1}(t - \tau)^{\alpha + \delta - \eta} (t - \tau)^\gamma
\end{array} \right\}
\sup_{0 \leq s \leq T/2} \| f(s) \|_{D_{\text{Adm}}(\alpha, \infty)};
\]

and for \( T/2 < \tau < t \leq T \):

\[
\| \varphi(t)A(t)v(t) - \varphi(\tau)A(\tau)v(\tau) \| \\
\leq \| \varphi(t) - \varphi(\tau) \| \| A(\tau)v(\tau) \| + \varphi(\tau)\| A(t)v(t) - A(\tau)v(\tau) \|
\leq \left\{ \begin{array}{l}
\frac{C_0(\alpha)}{\alpha} \int_{T/2}^t (T - s)^{\beta_1}(\alpha + 1) (\tau - s)^{\alpha - 1} ds [\varphi]_{C_r} + \\
+ \frac{KC_0(\alpha)}{\alpha} (T - t)^{\alpha\beta_1}(t - \tau)^\gamma + \frac{2KC_2(\alpha)}{\alpha} \\
\cdot (T - \tau)^{\alpha\beta_1}(t - \tau)^\gamma + \frac{KC_3(\alpha, \delta, \eta)}{\alpha + \delta - \eta} (T - \tau)^{\alpha + \delta - \eta} (T - \tau)^\gamma
\end{array} \right\}
\sup_{T/2 < s \leq T} \| f(s) \|_{D_{\text{Adm}}(\alpha, \infty)}.
\]

Hence (3.7) follows, thanks to (1.42).

**Definition 3.6.** Let

\[
f \in C([0, T]; X), \quad x \in X.
\]

A function \( u \in C([0, T]; X) \) is said to be a strong solution of (0.1) in the interval \([0, T]\) if there is a sequence \( u_n \in C^1([0, T]; X) \) with \( \varphi(\cdot)A(\cdot)u_n(\cdot) \in

...
\[ u_n \to u \quad \text{uniformly in } [0, T] \text{ as } n \to \infty, \]

\[ u'_n - \varphi(\cdot) A(\cdot) u_n(\cdot) \to u \quad \text{uniformly in } [0, T] \text{ as } n \to \infty. \]

**Proposition 3.7.** Let \( f \in C([0, T]; X) \), \( x \in D(A(0)) \). Moreover, in the case \( \beta > 0 \), we assume also \( f(0) \in D(A(0)) \). Then the function defined in (0.3) is the unique strong solution of (0.1).

**Proof.** Let us consider first the case \( \beta > 0 \). Let \( \varphi_n \in C^\gamma([0, T]; X) \), \( \gamma > 0 \), be such that \( \varphi_n \to \varphi \) in \( C([0, T]) \) as \( n \to \infty \), and \( \varphi_n(t) > 0 \) for every \( t \). Let \( f_n \in C^{\alpha}([0, T]; X) \), \( 0 < \alpha < 1 \), be such that \( f_n \to f \) in \( C([0, T]; X) \) as \( n \to \infty \), and \( f_n(0) = f(0) \). Let finally \( x_n \in D(A^2(0)) \) be such that \( x_n \to x \) in \( X \). We can apply Theorem 3.3 to problem

\[
\begin{aligned}
&u_n'(t) = \varphi_n(t) A(t) u_n(t) + f_n(t), \\
&u_n(0) = x_n.
\end{aligned}
\]

By Theorem 3.3b), the function

\[
(3.12) \quad u_n(t) = G(\varphi_n(t), 0) x_n + \int_0^t G(\varphi_n(t), \varphi_n(s)) f_n(s) \, ds,
\]

where \( \varphi_n(t) = \int_0^t \varphi_n(s) \, ds \), is the unique strict solution to (3.11). Recalling that \( x \in D(A(0)) \) and letting \( n \to \infty \) in (3.12), we obtain that \( u_n \to u \) uniformly in \([0, T]\), where \( u \) is defined in (0.3). Let us consider now the case \( \beta = 0 \). Let \( \varphi_n, f_n \) be as before, and choose \( \lambda \in \varphi(A(0)) \).

Let \( z_n \in D(A^2(0)) \) be such that

\[
\lim_{n \to \infty} z_n = x - (\lambda - A(0))^{-1} f(0)/\varphi(0)
\]

and set

\[ x_n = z_n + (\lambda - A(0))^{-1} f(0)/\varphi(0). \]

Then \( x_n \) belongs to \( D(A(0)) \) and \( \lim_{n \to \infty} x_n = x \); moreover

\[ A(0) \varphi(0) x_n + f_n(0) = A(0) \varphi(0) z_n + \lambda(\lambda - A(0))^{-1} f(0) \in D(A(0)). \]

By Theorem 3.3b), the function \( u_n \) defined in (3.12) is the strict solution of (3.11) and the conclusion is the same as in the case \( \beta > 0 \).
4. An application.

We apply here some results of the previous sections to a degenerate parabolic initial boundary value problem:

\[
\begin{aligned}
\begin{cases}
    u_t(t, x) = \varphi(t) \sum_{i,j=1}^{n} a_{ij}(t, x) u_{x_i x_j}(t, x) + f(t, x), & 0 < t < T, \ x \in \Omega, \\
    u(0, x) = u_0(x), & x \in \Omega, \\
    \sum_{i=1}^{n} b_i(t, x) u_{x_i}(t, x) + c(t, x) u(t, x) = 0, & 0 < t < T, \ x \in \partial \Omega,
\end{cases}
\end{aligned}
\]

where \( \Omega \) is a bounded open set in \( \mathbb{R}^n \) with regular boundary \( \partial \Omega \):

\[
\begin{aligned}
    &a_{ij} \in C([0, T] \times \overline{\Omega}), \quad i, j = 1, \ldots, n \text{ and } \\
    &\Re \sum_{i,j=1}^{n} a_{ij}(t, x) \xi_i \xi_j \geq \nu |\xi|^2, \\
    &\nu > 0, \ \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \ \forall t \in [0, T], \ \forall x \in \overline{\Omega}, \\
    &\sum_{i=1}^{n} b_i(x) \nu_i(x) \neq 0, \ \forall x \in \partial \Omega, \ \nu(t) = (\nu_1(x), \ldots, \nu_n(x)) = \text{unit exterior normal vector to } \partial \Omega \text{ at } x, \\
    &t \to b_i(t, \cdot), \ c(t, \cdot) \in C^{\mu_1}(0, T; C^1(\partial \Omega)), \ \mu_1 > 1/2, \\
    &t \to a_{ij}(t, \cdot) \in C^{\mu}(0, T; C(\overline{\Omega})), \quad 0 < \mu < 1, \\
    &\delta = \min \{\mu - \varepsilon, \mu_1 - 1/2\}, \quad \varepsilon > 0.
\end{aligned}
\]

Setting \( u(t, \cdot) = u(t), \ f(t, \cdot) = f(t) \) we can write problem (4.1) as an abstract Cauchy problem of the type (0.1) choosing:

\[
\begin{aligned}
    &X = C(\overline{\Omega}), \\
    &D(A(t)) = \left\{ g \in \bigcap_{p > 1} W^{2, p}(\Omega): A(t) g \in C(\overline{\Omega}), \ C(t) g \equiv \right. \\
    &\left. \sum_{i=1}^{n} b_i(t, x) g_{x_i} + c(t, x) g|_{\partial \Omega} = 0 \right\}, \\
    &A(t) g = \sum_{i, j=1}^{n} a_{ij}(t, \cdot) g_{x_i x_j}, \quad \forall g \in D(A(t)).
\end{aligned}
\]

Then \( D(A(t)) = X \) and \( A(t) \) generates, for \( t \in [0, T] \), an analytic semigroup in \( X \) thanks to [14].
The interpolation spaces $D_{A(0)}(\delta, \infty)$ are given by (see \cite{1}, \cite{4}):

$$
D_{A(0)}(\delta, \infty) = \begin{cases} 
C^{2\delta}(\overline{\Omega}) : C(t)g|_{\partial\Omega} = 0 & \text{if } \delta > 1/2, \\
C^{2\delta}(\overline{\Omega}) & \text{if } \delta < 1/2,
\end{cases}
$$

$$
D_{A(0)}(1/2, \infty) = \begin{cases} 
\exists K \in \mathbb{R} \text{ such that:} \\
\left| g(x) + g(y) - 2g \left( \frac{x + y}{2} \right) \right| \leq K|x - y|, \quad \forall x, y \in \overline{\Omega} \text{ with } \frac{x + y}{2} \in \overline{\Omega}, \\
\left| g(x - \sigma b(x)) - g(x) \right| \leq K\sigma \forall x \in \partial\Omega, \quad \forall \sigma > 0 \text{ with } x - \sigma b(x) \in \overline{\Omega}.
\end{cases}
$$

If in addition $a, b_1(t, \cdot), c(t, \cdot) \in C^{2\delta}(\overline{\Omega}), b_1(t, \cdot), c(t, \cdot) \in C^{2\delta + 1}(\partial\Omega)$, then $D_{A(0)}(\delta + 1, \infty)$ is given by

$$
D_{A(0)}(\delta + 1, \infty) = \begin{cases} 
C^{2\delta + 2}(\overline{\Omega}) : C(t)g|_{\partial\Omega} = C(t)A(t)g|_{\partial\Omega} = 0 & \text{if } \delta > 1/2, \\
C^{2\delta + 2}(\overline{\Omega}) : C(t)g|_{\partial\Omega} = 0 & \text{if } \delta < 1/2.
\end{cases}
$$

We state now two existence theorems for the classical and strict solution to (4.1) in the case where $f$ is Hölder continuous either with respect to $x$ or with respect to time.

**Proposition 4.1.** Let $\varphi$ satisfy (1.1), let $u_0 \in C(\overline{\Omega})$ and let $f \in C([0, T'] \times \overline{\Omega})$ be such that

$$
f(t, \cdot) \in C^{2\alpha}(\overline{\Omega}) \quad \forall t \in [0, T], \quad \sup_{0 \leq t \leq T} \| f(t, \cdot) \|_{C^{2\alpha}(\overline{\Omega})} < +\infty
$$

with $0 < 2\alpha < 1$. Then there is a unique function $u \in C([0, T'] \times \overline{\Omega})$ such that there exist $u_t, A(\cdot)u \in C[0, T' \times \overline{\Omega})$ and $u$ satisfies (4.1).

Moreover, let $\delta$ be defined by (4.2). The following statements hold true:

1) if either $\beta > 0$ and $u_0 \in D_{A(0)}(1/(\beta + 1))$, or $\beta = 0$ and $u_0 \in D(A(0))$, then

$$
u_t, A(\cdot)u \in C([0, T] \times \overline{\Omega});
$$

2) if either $\beta > 0$ and $u_0 \in D_{A(0)}(\alpha + 1/(\beta + 1), \infty)$, $0 < \alpha < \beta/(\beta + 1)$,
and $\varphi(Au)$ enjoy the same regularity properties of $f$.

**PROOF.** Under our assumptions, $(t \to f(t, \cdot))$ belongs to $C([0, T]; X) \cap B([0, T]; D_{A(1)}(\alpha, \infty))$, and $u_0 \in D(A(0))$. Then Theorem 2.3 is applicable and yields the first part of the statement. (4.5) and (4.6) follow from Propositions 3.2 and 3.4 respectively; (4.7) follows from Proposition 3.4. 

**PROPOSITION 4.2.** Let $\varphi \in C^\gamma([0, T])$, $0 < \gamma < 1$ ($\gamma \leq \beta > 0$, $\gamma \leq \beta_1$ if $\beta_1 > 0$), satisfy (1.1); let $u_0 \in C(\overline{\Omega})$ and let $f \in C([0, T] \times \overline{\Omega})$ be such that

$$f(\cdot, x) \in C^\alpha([0, T]) \quad \forall x \in \overline{\Omega}; \quad \sup_{x \in \overline{\Omega}} \|f(\cdot, x)\|_{C^\gamma([0, T])} < +\infty. \tag{4.8}$$

Then there is a unique function $u \in C([0, T] \times \overline{\Omega})$ such that there exist $u_t, A(\cdot)u \in C([0, T] \times \overline{\Omega})$ and $u$ satisfies (4.1).

Moreover for each $\varepsilon \in ]0, T/2[$,

$$u_\varepsilon(\cdot, x), A(\cdot)u_\varepsilon(\cdot, x) \text{ belong to } C^\gamma([\varepsilon, T - \varepsilon]), \tag{4.9}$$

uniformly with respect to $x \in \overline{\Omega}$,

$$u_\varepsilon(t, \cdot) \in C^{2\alpha}(\overline{\Omega}) \quad \forall t \in [\varepsilon, T - \varepsilon] \quad \text{and} \quad \sup_{x \in \overline{\Omega}} \|u_\varepsilon(t, \cdot)\|_{C^{2\alpha}(\overline{\Omega})} < +\infty, \tag{4.10}$$

with $\gamma \leq \min \{\alpha, \gamma\}$, $\gamma < \delta$.

In addition if $\max \{\beta - \gamma, \beta_1 - \gamma\} < 1$, $\max \{0, \beta - \gamma, \beta_1 - \gamma\} < \alpha < 1$ and

- $u_0 \in D_{A(0)}(1/(\beta + 1))$, $f(0) \in D_{A(0)}(\delta, \infty)$, $\delta > (\beta - \gamma)/(\beta + 1)$ if $\beta > 0$,
- $u_0 \in D(A(0))$ if $\beta = 0$,
- $f(T) = 0$ if $\beta_1 > 0$,

then

$$u_\varepsilon, A(\cdot)u \in C([0, T] \times \overline{\Omega}). \tag{4.11}$$
PROOF. Assumption (4.8) implies that $t \to f(t, \cdot)$ belongs to $C^\infty([0, T]; X)$. The first part of the statement, (4.9) and (4.10) follow applying Theorem 2.4. Finally (4.11) follows from Theorem 3.3.

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