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A ZJ-Theorem for $p^*$, $p$-Injectors in Finite Groups.

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1. Introduction and notation.

All groups considered in this paper are finite. A group $G$ is said to be $p$-stable, $p$ prime, if whenever $A$ is a $p$-subgroup of $G$ and $B$ is a $p$-subgroup of $\text{NG}(A)$ such that $[A, B, B] = 1$ then $B \leq \text{Op}(\text{NG}(A) \mod \text{CG}(A))$. In [5] Glauberman obtained the following theorem:

Let $p$ be an odd prime and let $P$ be a Sylow $p$-subgroup of a group $G$. Assume that $G$ is $p$-stable and that $\text{CG}(\text{Op}(G)) \leq \text{Op}(G)$. Then $ZJ(P)$ is a characteristic subgroup of $G$, where $ZJ(P) = Z(J(P))$ and $J(P)$ is the Thompson’s subgroup of $P$, that is, subgroup of $P$ generated by the set $\mathcal{A}(P)$ of all abelian subgroups of maximum order of $P$.

With the same conditions he also obtained a factorization of the group $G$.

In the same paper Glauberman introduces the characteristic subgroup $ZJ^*(P)$ and proves the following result:

Let $p$ be an odd prime and let $P$ be a Sylow $p$-subgroup of a group $G$. Suppose that $\text{CG}(\text{O}_p(G)) \leq O_p(G)$ and that $\text{SA}(2, p)$ is not involved in $G$. Then $ZJ^*(P)$ is a characteristic subgroup of $G$ and $\text{CG}(ZJ^*(P)) \leq \leq ZJ^*(P)$.

Some related results were obtained by Ezquerro [4] and Pérez Ramos [9, 10].

Given a fixed prime $p$, we shall denote by $\mathcal{S}_p$ the class of all $p$-

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groups, \( C_{p^*} \) that all \( p^* \)-groups [2] and \( C_{p^*} \) that of the \( p^* \), \( p \)-groups; the corresponding radicals in a group \( G \) are denoted, as usual, by \( O_{p^*}(G) \), \( O_{p^*}(G) \) and \( O_{p^*}(G) \) respectively. Every group \( G \) possesses \( p^* \), \( p \)-injectors which are the subgroups of the form \( O_{p^*}(G)P \) where \( P \) describes the set of Sylow \( p \)-subgroups of \( G \) [8].

The aim of this paper is to establish the analogous to Glauberman's results with the subgroups \( ZJ(K) \) and \( ZJ^*(K) \) where \( K \) is a \( p^* \), \( p \)-injector of a group \( G \).

2. The factorization.

**Lemma 2.1.** Let \( G \) be a group of \( F^*(G) \leq H \leq G \). Then it follows:

\[
\pi(ZJ(H)) \subseteq \pi(F(G)) = \pi(F(H)).
\]

**Proof.** Clearly \( \pi(ZJ(H)) \subseteq \pi(F(G)) \) and \( \pi(F(G)) \subseteq \pi(F(H)) \). On the other hand as \( \pi(F(H)) = \pi(Z(F(H))) \) and \( Z(F(H)) \leq C_G(F(G)) \cap \cap C_G(E(G)) = C_G(F^*(G)) \leq F(G) \), then the result follows.

**Corollary 2.2.** Let \( G \) be a group and \( K \) a \( p^* \), \( p \)-injector of \( G \). If \( p \in \pi(F(G)) \) then \( p \in \pi(ZJ(K)) \). Moreover if \( O_{p^*}(F(G)) \leq Z(K) \) then \( \pi(F(G)) = \pi(F(K)) = \pi(ZJ(K)) \) and in particular the following are equivalent: i) \( F(G) \neq 1 \), ii) \( F(K) \neq 1 \), iii) \( ZJ(K) \neq 1 \).

**Proof.** We can assume that \( K = O_{p^*}(G)P \) where \( P \) is a Sylow \( p \)-subgroup of \( G \). If \( O_{p^*}(G) \neq 1 \) then we have:

\[
1 \neq Z(P) \cap O_{p^*}(G) \leq C_K(O_{p^*}(G)) \cap C_K(P) = Z(K) \leq ZJ(K).
\]

If moreover \( O_{p^*}(F(G)) \leq Z(K) \) then \( O_{p^*}(F(G)) \leq ZJ(K) \) and we can conclude that \( \pi(F(G)) \subseteq \pi(ZJ(K)) \). Now Lemma 2.1 applies.

**Lemma 2.3.** Let \( G \) be a group and \( K \) a \( p^* \), \( p \)-injector of \( G \). Assume that \( O_{p^*}(F(G)) \leq Z(K) \). Let \( B \) be a nilpotent normal subgroup of \( G \) and let \( A \) be a nilpotent subgroup of \( K \), then \( AB \) is a nilpotent subgroup of \( G \).

**Proof.** By a known result of Bialostocki [3] it is enough to prove that \( AO_{q^*}(B) \) is nilpotent for every \( q \in \pi(B) \).

Since \( K \) is a \( p^* \), \( p \)-injector of \( G \) we have \( B \leq F(G) \leq K \) and so \( O_{p^*}(B) \leq O_{p^*}(F(G)) \leq Z(K) \). Then \( AO_{q^*}(B) \) is nilpotent for every prime \( q \) with \( q \neq p \). Now assume that \( K = O_{p^*}(G)P \) where \( P \) is a Sylow \( p \)-sub-
group of $G$, then:

$$[O_p(B), A] \leq [O_p(B), O_{p^*}(G) P] = [O_p(B), P] \leq [P, P] = P'.$$

By induction on $n$ we can prove that for every $n \geq 1$:

$$[O_p(B), A; n] \leq [O_p(B), P; n] \leq \Gamma_{n+1}(P)$$

and so, for some positive integer $m$ is $[O_p(B), A; m] = 1$, thus $A$ acts nilpotently on $O_p(B)$ and then $AO_p(B)$ is nilpotent.

**PROPOSITION 2.4.** Let $G$ be a $p$-stable group. If $I_p$ is a $p^*, p$-injector of $G$ and $A$ is an abelian normal subgroup of $K$ then $A \trianglelefteq G$ and $A \leq F(G)$. In particular $ZJ(K) \leq F(G)$.

**PROOF.** Clearly $O_{p^*}(A) \leq O_{p^*}(K) = O_{p^*}(G) \leq O_{p^*}(G) \leq K$ thus $O_{p^*}(A) \trianglelefteq O_{p^*}(G)$. If $K = O_{p^*}(G) P$ then $O_p(A) \trianglelefteq P$. Set $Q = P \cap O_{p^*}(G)$ then $G = O_{p^*}(G) N_G(Q)$. Since $Q \leq N_G(O_p(A))$ and $A$ is abelian then $[Q, O_p(A), O_p(A)] = 1$. As $O_p(A) \trianglelefteq N_G(Q)$ and $G$ is $p$-stable we have:

$$O_p(A) C_G(Q)/C_G(Q) \leq O_p(N_G(Q)/C_G(Q)) = M/C_G(Q).$$

On the other hand $C_G(Q) \leq O_{p^*}(G)$ [7], so $MO_{p^*}(G)/O_{p^*}(G)$ is a normal $p$-subgroup of $G/O_{p^*}(G)$, hence is trivial. So $O_p(A) \trianglelefteq O_{p^*}(G)$ thus $A \trianglelefteq G$ and $A \leq F(G)$.

**THEOREM 2.5.** Let $G$ be a $p$-stable group, $p$ odd and $F(G) \neq 1$. If $K$ is a $p^*, p$-injector of $G$ and $O_{p^*}(F(G)) \leq Z(K)$ then $1 \neq ZJ(K) \leq G$.

**PROOF.** (This proof is based, in part, on Glauberman’s proof of his $ZJ$-theorem ([6], Th. 8.2.10)).

First note that as the $p^*, p$-injectors of $G$ are conjugated, the statements $ZJ(K) \trianglelefteq G$ and $ZJ(K) \operatorname{char} G$ are equivalent.

As a consequence of the above Proposition we know that $ZJ(K) \leq F(G)$ and by Corollary 2.2 $ZJ(K) \neq 1$. Now to obtain the theorem it is enough to prove that if $B$ is a nilpotent normal subgroup of $G$ then $B \trianglelefteq ZJ(K)$ is normal in $G$.

Let $G$ be a minimal counterexample and $B$ a nilpotent normal subgroup of $G$ of least order such that $B \trianglelefteq ZJ(K)$ is not normal in $G$.

Set $Z = ZJ(K)$ and let $B_1$ be the normal closure of $B \trianglelefteq Z$ in $G$, then $B \trianglelefteq Z = B_1 \trianglelefteq Z$ an by our minimal choice it follows $B = B_1$.

Now $B' < B$ then $B' \trianglelefteq Z \trianglelefteq G$, hence for every $g$ of $G$ is $[(B \cap Z), B] = [B \cap Z, B] \leq B' \cap Z$. Since $B$ is the normal closure of $B \cap Z$
in $G$ it follows that $B' \leq B' \cap Z$. In particular $B \cap Z$ centralizes $B'$ and by an analogue argument we obtain that $[B, B, B] = 1$.

Consider $A \in \mathfrak{A}(K)$. For Lemma 2.3 we know that $AB$ is nilpotent. Thus there exists a positive integer $n$ such that $[B, A; n] = 1$. Moreover as $O_p(B) \leq Z(K)$ and $p$ odd it follows that $[A, B'] = [A, O_p(B)]'$ has odd order.

By ([1], Cor. 2.8) there exists $A \in \mathfrak{A}(K)$ such that $B \leq N_G(A)$, therefore $[B, A, C] = 1$. In particular $[O_p(B), O_p(A), O_p(A)] = 1$. Since $G$ is $p$-stable we have:

$$O_p(A) C/C \leq O_p(G/C) = T/C \leq G/C$$

where $C = C_G(O_p(B))$. Now, since $O_p(A)$ centralizes $O_p(B)$ it follows that $A \leq T$.

If $T = G$ then $G/C$ is a $p$-group so $KC = G$. Moreover as $O_p(B) \leq Z(K) \leq Z = O_p(B)(O_p(B) \cap Z)$ a normal subgroup of $KC$, what is a contradiction. Thus $T < G$. Since $A \leq K \cap T$ it follows $\mathfrak{A}(K \cap T) \subseteq \mathfrak{A}(K), J(K \cap T) \leq J(K)$ and $ZJ(K) \leq ZJ(K \cap T)$. By our minimal choice of $GZJ(K \cap T)$ char $T$ and so it follows $ZJ(K \cap T)$ normal in $G$. Then $B \leq ZJ(K \cap T)$. In particular $B$ is abelian. If $J(K \cap T) < J(K)$ then there exists $A_1 \in \mathfrak{A}(K)$ such that $A_1$ there is not a subgroup of $T$, then we must have $[B, A_1, A_1] \neq 1$. Set $\mathfrak{D} = \{A \in \mathfrak{A}(K) | [B, A, A] \neq 1\}$, we choose $A_2 \in \mathfrak{D}$ such that $[A_2] = 1$ thus $A_2 \leq T$ and $ZJ(K \cap T) \leq A_2$. Hence:

$$[B, A_1, A_1] \leq [ZJ(K \cap T), A_1, A_1] \leq [A_2, A_1, A_1] = 1$$

what is a contradiction.

Consequently $J(K \cap T) = J(K)$ and $ZJ(K) = ZJ(K \cap T) \leq G$. This is the last contradiction.

**Corollary 2.6.** Let $G$ be a $p$-stable group, $p$ odd, with $F(G) \neq 1$. If $K$ is a $p^*$, $p$-injector of $G$ and $O_{p^*}(F(G)) \leq Z(K)$ then:

$$G = N_G(J(K)) C_G(ZJ(K)) = N_G(J(K)) C_G(Z(K)).$$

**Proof.** Let us write $Z = ZJ(K)$ and $C = C_G(Z)$. As $Z \leq G$ is also $C \leq G$ and therefore $G = CN_G(K \cap C)$. Now, as $J(K \cap C)$ char $K \cap C$, it follows $G = CN_G(J(K \cap C))$. Since $J(K) \leq K \cap C$, $J(K) = J(K \cap C)$. Moreover as $Z(K) \leq Z$ we have $C \leq C_G(Z(K))$ and we obtain:

$$G = N_G(J(K)) C_G(ZJ(K)) = N_G(J(K)) C_G(Z(K)).$$
**Corollary 2.7** (Glauberman’s ZJ-Theorem). Let $G$ be a group with $O_p(G) \neq 1$, $O_p'(G) = 1$, which is $p$-constrained and $p$-stable, $p$ odd. If $P$ is a Sylow $p$-subgroup of $G$, then $ZJ(P) \leq G$.

**Proof.** Since $G$ is $p$-constrained then $O_p^*(G) = O_p'(G) = 1$ ([2], Lemma 6.12) and so $K = O_p^*(G)P = P$ is a $p^*$, $p$-injector of $G$ and Theorem 2.5 applies.


**Definition 3.1** [4]. For any group $K$ define two sequences of characteristic subgroups of $K$ as follows. Set $ZJ^0(K) = 1$ and $K_0 = K$. Given $ZJ^i(K)$ and $K_i$, $i \geq 0$ let $ZJ^{i+1}(K)$ and $K_{i+1}$ the subgroups of $K$ that contain $ZJ^i(K)$ and satisfy:

$$ZJ^{i+1}(K)/ZJ^i(K) = ZJ(K_i/ZJ^i(K)),$$

$$K_{i+1}/ZJ^i(K) = C_{K_i/ZJ^i(K)}(ZJ^{i+1}(K)/ZJ^i(K)).$$

Let $n$ be the smallest integer such that $ZJ^n(K) = ZJ^{n+1}(K)$, then $ZJ^n(K) = ZJ^{n+r}(K)$ and $K_n = K_{n+r}$ for every $r \geq 0$. Set $ZJ^*(K) = ZJ^n(K)$ and $K^* = K_n$.

**Remarks.** 1) Notice that if $C_G(ZJ^*(K)) \leq ZJ^*(K)$ then by ([4], Prop.II 3.7) $K^*/ZJ^*(K)$ is nilpotent, therefore $ZJ(K^*/ZJ^*(K)) = 1$ implies $K^*/ZJ^*(K) = 1$, that is, $K^* = ZJ^*(K)$.

2) If $K$ is a $p^*$, $p$-injector of $G$ then $ZJ^i(K)$ is $p$-nilpotent for every $i \geq 0$. Later we will improve this statement.

**Lemma 3.2.** Let $K$ be a $p^*$, $p$-injector of $G$ and $N \leq G$ such that $C_K(N) \leq N \leq K$ then $C_G(N) = Z(N)$.

**Proof.** Observe that $Z(N) = C_K(N) = C_G(N) \cap K$ is a $p^*$, $p$-injector of $C_G(N)$. On the other hand if $x \in C_G(N)$ then $\langle x, Z(N) \rangle$ is an abelian subgroup of $G$ and $Z(N) \leq \langle x, Z(N) \rangle \leq C_G(N)$, therefore $\langle x, Z(N) \rangle = Z(N)$ and $x \in Z(N)$.

**Remark.** Notice that if $K$ is a $p^*$, $p$-injector of $G$ then $K$ is also $p^*$, $p$-injector of $N_G(K^*)$. Moreover by ([4], Prop.II 3.7) $C_K(K^*) \leq C_K(ZJ^*(K)) \leq K^*$. Thus, by the above Lemma, is $C_{N_G(K^*)}(K^*) = C_G(K^*) \leq K^*$.

**Theorem 3.3.** Let $p$ be an odd prime and $K$ a $p^*$, $p$-injector of a group $G$. Assume that $SA(2, p)$ is not involved in $G$ and that
$O_p'(F(G)) \leq Z(K)$. Then $ZJ^i(K)$ is a characteristic subgroup of $G$ for every $i \geq 0$.

**Proof.** Let $G$ be a minimal counterexample. As $SA(2,p)$ is not involved in $G$, $G$ is $p$-stable. Thus by Theorem 2.5 we have $ZJ(K)$ char $G$.

If $ZJ(K) = 1$ then $ZJ^i(K) = 1$ contrary to the choice of $G$. So we can assume $1 \neq ZJ(K) \leq C_G(ZJ(K)) \leq G$. Set $C = C_G(ZJ(K))$. Assume $C < G$ then for every $i \geq 0$ it follows $ZJ^i(K \cap C)$ char $C$. Since $J(K) \leq K \cap C$ we have $J(K) = J(K \cap C)$ and $ZJ(K) = ZJ(K \cap C)$. Moreover if $K_1 = C_K(ZJ(K)) = C_K \cap C(ZJ(K \cap C))$ and by induction on $i$ it follows $ZJ^i(K) = ZJ^i(K \cap C)$. Thus $ZJ^i(K)$ char $C \leq G$ for every $i \geq 0$ and by the conjugacy of $p^*$, $p$-injectors of $G$, we obtain $ZJ^i(K)$ char $G$ for every $i \geq 0$ contrary to our choice of $G$. Therefore $C = G$, and so $ZJ(K) = Z(G)$. Since $1$ we have $|G/Z(G)| < |G|$ and we can conclude that $ZJ^i(K/Z(G))$ char $G/Z(G)$ for every $i \geq 0$. Now, since $K_1 = K$, using ([4], Prop.II 3.6) it follows $ZJ^i+1(K)/Z(G) = ZJ^i(K/Z(G))$, thus for every $i \geq 0$, $ZJ^i(K)$ char $G$. That is the last contradiction.

**Proposition 3.4.** Let $K$ be a $p^*$, $p$-injector of a group $G$. Assume $O_p'(F(G)) \leq Z(K)$, then for every $i \geq 0$:

i) $ZJ^i(K)$ is nilpotent

ii) $F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$ is a $p$-group

iii) $O_p^*(G) \leq K_i$.

**Proof.** Simultaneously we will prove i) and ii) by induction on $i$. First notice that $ZJ(K) \leq Z(C_K(ZJ(K))) = Z(K_1)$ thus $F(K_1/ZJ(K)) = F(K_1)/ZJ(K)$ and $O_p'(F(K_1/ZJ(K))) = O_p'(F(K_1)/ZJ(K)) = 1$. Clearly $ZJ^1(K) = Z(J(K))$ is nilpotent.

Assume that $F(K_1/ZJ^i(K)) = F(K_1)/ZJ^i(K)$ is a $p$-group, then $O_p'(ZJ^i+1(K)) \leq ZJ^i(K)$, therefore $O_p'(ZJ^i+1(K))$ is a nilpotent normal subgroup of $K$ and we have $O_p'(ZJ^i+1(K)) \leq O_p'(F(K)) \leq Z(K)$. Now, since $ZJ^i+1(K)$ is $p$-nilpotent, we have that $ZJ^i+1(K)$ is nilpotent.

On the other hand we have:

$$F(K_{i+1}/ZJ^{i+1}(K)) \equiv F(K_{i+1}/ZJ^i(K)/ZJ^{i+1}(K)/ZJ^i(K))$$

and as $ZJ^{i+1}(K)/ZJ^i(K) = Z(K_{i+1}/ZJ^i(K))$, it follows:

$$F(K_{i+1}/ZJ^i(K)/ZJ^{i+1}(K)/ZJ^i(K)) =$$

$$= F(K_{i+1}/ZJ^i(K))/ZJ^{i+1}(K)/ZJ^i(K).$$
But

\[ F(K_{i+1}/ZJ^i(K)) = F(K_i/ZJ^i(K)) \cap K_{i+1}/ZJ^i(K) = \]
\[ = F(K_i)/ZJ^i(K) \cap K_{i+1}/ZJ^i(K) = F(K_{i+1})/ZJ^i(K) \]

and we can conclude:

\[ F(K_{i+1}/ZJ^{i+1}(K)) = F(K_{i+1})/ZJ^{i+1}(K). \]

Moreover as \( O_p(F(K_{i+1})) \leq O_p(F(K)) \leq Z(K) \leq (ZJ^{i+1}(K)) \) it follows that \( F(K_{i+1}/ZJ^{i+1}(K)) \) is a p-group.

iii) Clearly \( O_p^*(G) = O_p^*(K) \leq C_K(O_p(ZJ^*(K))). \) Moreover as \( ZJ^*(K) \) is nilpotent it follows: \( O_p^*(ZJ^*(K)) \leq O_p^*(F(K)) \leq Z(K). \) Therefore \( O_p^*(G) \leq C_K(ZJ^*(K)) \leq K_i \) for every \( i \geq 0. \)

**Proposition 3.5.** Let \( G \) be a group with \( O_p^*(F(G)) \leq Z(K) \), where \( K \) is a \( p^* \)-injector of \( G \), then the following are equivalent:

i) \( G \) is an \( \mathfrak{A} \)-constrained group

ii) \( K^* = ZJ^*(K) \)

iii) \( C_G(ZJ^*(K)) \leq ZJ^*(K). \)

**Proof.** i) \( \Rightarrow \) ii) Set \( K = O_p^*(G)P \) where \( P \) is a Sylow \( p \)-subgroup of \( G \), then by ([2], Lemma 6.11) it follows \([P, O_p^*(G)] = 1\) and as \( O_p^*(G) \leq C_G(F(G)) \leq F(G) \) we have \( K \) is nilpotent, thus \( K^* \) is nilpotent, hence \( K^* = ZJ^*(K). \)

ii) \( \Rightarrow \) iii) By the remark of Lemma 3.2.

iii) \( \Rightarrow \) i) We know that \([O_p^*(G), ZJ^*(K)] = 1\) therefore \( E(G) \leq O_p^*(G) \leq C_G(ZJ^*(K)) \leq ZJ^*(K) \), then \( E(G) = 1 \), i.e. \( G \) is an \( \mathfrak{A} \)-constrained group.

**Corollary 3.6.** Let \( p \) be an odd prime and \( K \) a \( p^* \), \( p \)-injector of a \( \mathfrak{A} \)-constrained group \( G \). Assume that \( SA(2, p) \) is not involved in \( G \) and \( O_p^*(F(G)) \leq Z(K) \). Then \( ZJ^*(K) \) is a characteristic subgroup of \( G \) and \( C_G(ZJ^*(K)) \leq ZJ^*(K) \).

As a consequence of the above Corollary we can obtain the well-known result of Glauberman:

**Corollary 3.7.** Let \( p \) be an odd prime and \( P \) a Sylow \( p \)-subgroup of a group \( G \). Suppose that \( C_G(O_p(G)) \leq O_p(G) \) and that \( SA(2, p) \) is not involved in \( G \). Then \( ZJ^*(P) \) is a characteristic subgroup of \( G \) and \( C_G(ZJ^*(P)) \leq ZJ^*(P) \).

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