RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova*, tome 88 (1992), p. 127-138
<http://www.numdam.org/item?id=RSMUP_1992__88__127_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 88 (1992)

# The Role of the Boundary in Some Semilinear Neumann Problems.

GIOVANNI MANCINI - ROBERTA MUSINA (*)

## Introduction.

In a series of papers (see [3], [4], [5], [7]) it has been shown that the topology and geometry of the domain enters the existence and/or multiplicity of solutions in nonlinear elliptic Dirichlet problems of the form

$$(0.1)_\lambda \qquad \begin{cases} -\Delta u + \lambda^2 u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth and bounded domain of $\boldsymbol{R}^N$, $N \geqslant 3$, and $p \in ]2, 2^*[$, $2^* = 2N/(N-2)$.

This phenomena has been first observed by [10] in the special case $p = 2^*$, $\lambda = 0$: while $(0.1)_\lambda$ has no solution if $\Omega$ is starshaped, it has trivially a solution if $\Omega$ is an annulus. A first explanation (and a striking result) has been given by J. M. Coron in [6]: low energy sublevels of the variational functional associated to $(0.1)_\lambda$ inherit the topology of $\Omega$. The basic tool to relate the change of topology of the sublevels to the topology of $\Omega$ is a concentration-compactness Lemma due to P. L. Lions in [12], and, more in general, the analysis of the concentration phenomena along gradient lines of the variational functional (cfr. [2]).

Subsequently, Benci-Cerami observed that concentration is not necessarily related to the critical growth of the nonlinearity, but it occurs

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Partially supported by Ministero dell'Università e della Ricerca Scientifica (fondi 60% e 40%).

whenever, after rescaling, problem $(0.1)_\lambda$ looks like an associated problem in the whole space. More precisely, if $\lambda$ is large, minimal energy solutions to $(0.1)_\lambda$ look like the ground state solutions in $\boldsymbol{R}^N$ with $\lambda = 1$, highly concentrated around some point of $\Omega$.

In this paper we analyze the same equation but with Neumann boundary conditions:

$$(0.2)_\lambda \qquad \begin{cases} -\Delta u + \lambda^2 u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

We observe that concentration around boundary points or in the interior play different roles. Ground state solutions concentrate around the boundary, while other solutions correspond to concentration in the interior. Our main result is the following theorem.

THEOREM 0.1. *Problem* $(0.2)_\lambda$ *has at least* $(\operatorname{cat}\partial\Omega) + 1$ *solutions, provided* $\lambda$ *is large enough.*

While writing this paper we learned that Wei-Ming Ni and Izumi Takagi have proved in [14] that ground state solutions concentrate at the boundary.

## 1. The concentration lemma.

First, we introduce some notations. Solutions to problem $(0.2)_\lambda$ correspond to positive critical points of the functional

$$E_{\lambda,\Omega}(u) = \int_\Omega [|\nabla u|^2 + \lambda^2 |u|^2],$$

on the manifold $S := \{u \in H^1(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$. We set

$$(1.1) \qquad I_{\lambda,\Omega} := \operatorname*{Inf}_S E_{\lambda,\Omega} = \operatorname*{Inf}_{(u \in H^1(\Omega),\, u \neq 0)} \frac{E_{\lambda,\Omega}(u)}{\|u\|^2_{L^p(\Omega)}}$$

and we define

$$I = I_{1,\boldsymbol{R}^N} = \operatorname*{Inf}_{(u \in H^1(\boldsymbol{R}^N),\, u \neq 0)} \frac{E_{1,\boldsymbol{R}^N}(u)}{\|u\|^2_{L^p(\boldsymbol{R}^N)}}.$$

When no confusion can arise, we write $E_\lambda(u)$, $I_\lambda$ and $\|u\|_p$ instead of $E_{\lambda,\Omega}(u)$, $I_{\lambda,\Omega}$ and $\|u\|_{L^p(\Omega)}$ respectively.

*Scaling properties.* Set

$$u^\lambda(x) := \lambda^{-N/p} u(x/\lambda),$$

for $u \in H^1(\Omega)$, $\lambda > 0$. Then $u^\lambda \in H^1(\lambda\Omega)$ and

$$E_{1,\lambda\Omega}(u^\lambda) = \lambda^{-\alpha} E_{\lambda,\Omega}(u), \qquad \int_{\lambda\Omega} |u^\lambda|^p = \int_\Omega |u|^p, \tag{1.2}$$

where $\alpha = 2N(1/p - 1/2^*) \in ]0, 2[$.

In particular, $u$ is a (constrained) minimizer for $E_{\lambda,\Omega}$ iff $u^\lambda$ is a (constrained) minimizer for $E_{1,\lambda\Omega}$, and

$$I_{1,\lambda\Omega} = \lambda^{-\alpha} I_{\lambda,\Omega}. \tag{1.3}$$

*Problem* $(0.2)_\lambda$ *on a half space.* – Here we take $\lambda = 1$ and we drop the subscript $\lambda$.

It has been recently proved by Kwong in [11] that equation (0.1) has a unique (up to translations) solution $V$ in $\boldsymbol{R}^N$, which is known to be radially symmetric (see [8]). As a consequence, (0.2) has, by a reflection argument, a unique solution on any half space $H$, given by $V|_H$. Setting

$$U := \frac{2^{1/p} V}{\|V\|_{L^p(\boldsymbol{R}^N)}},$$

we see that $U$ is the (unique) minimizer in (1.1) (with either $\Omega = \boldsymbol{R}^N$ or $\Omega = H$) satisfying

$$\begin{cases} \displaystyle\int_{\boldsymbol{R}^N} U^p = 2, & \displaystyle\int_{\boldsymbol{R}^N} |\nabla U|^2 + U^2 = 2^{2/p} I, \\ \displaystyle\int_H U^p = 1, & \displaystyle\int_H |\nabla U|^2 + U^2 = 2^{\frac{2}{p}-1} I, \end{cases} \tag{1.4}$$

where $H$ is any half space.

Acting by translations and dilations on $U$ will play a crucial role in the sequel. Set

$$U_{\lambda,y}(x) := \lambda^{N/p} U(\lambda(x-y)),$$

for $x, y \in \boldsymbol{R}^N$.

LEMMA 1.1. *We have, uniformly for* $y \in \partial V$,

$$E_{\lambda, \Omega}(U_{\lambda, y}) = \lambda^{\alpha}[2^{\frac{2}{p}-1} I + o(1)],$$

$$\|U_{\lambda, y}\|_{L^p(\Omega)} = 1 + o(1),$$

*where* $o(1) \to 0$ *as* $\lambda \to \infty$.

PROOF. By (1.2) we get

$$E_{\lambda, \Omega}(U_{\lambda, y}) = \lambda^{\alpha} E_{1, \lambda(\Omega - y)}(U), \qquad \int_{\Omega} U_{\lambda, y}^p = \int_{\lambda(\Omega - y)} U^p.$$

Now, fix $y_0 \in \partial\Omega$ and set $H = \{z \in \boldsymbol{R}^N \mid z \cdot \nu(y_0) \geq 0\}$, where $\nu(y_0)$ is the interior normal vector at $\partial\Omega$ in $y_0$. Notice that $\chi_{\lambda(\Omega - y_\lambda)} \to \chi_H$ if $y_\lambda$ belong to the boundary of $\Omega$ and approach the point $y_0$. Thus Lemma 1.1 follows by (1.4). ∎

REMARK 1.2. By definition (1.1), and by Lemma 1.1 and (1.3) we easily get

$$\limsup_{\lambda \to \infty} \lambda^{-\alpha} I_{\lambda, \Omega} = \limsup_{\lambda \to \infty} I_{1, \lambda\Omega} \leq 2^{\frac{2}{p}-1} I. \qquad \blacksquare$$

Now we describe the concentration behavior of $U_{\lambda, y}$ as $\lambda \to \infty$, by means of a «barycentre function»

$$\beta(u) = \frac{\int_{\Omega} x|u|^p}{\int_{\Omega} |u|^p}, \qquad \text{for } u \in H^1(\Omega) \setminus \{0\}.$$

LEMMA 1.3. *We have, uniformly for* $y \in \partial\Omega$,

$$\beta(U_{\lambda, y}) = y + o(1)$$

*where* $o(1) \to 0$ *as* $\lambda \to \infty$.

PROOF. We just observe that for a sequence $y_\lambda \in \partial\Omega$, $y_\lambda \to y_0$, we have

$$\int_{\Omega} x U_{\lambda, y_\lambda}^p = \int_{\lambda(\Omega - y_\lambda)} (x/\lambda + y_\lambda) U^p \, dx \to y_0 \int_H U^p = y_0. \qquad \blacksquare$$

From now on we fix any $\delta(\lambda) = o(1)$ as $\lambda \to \infty$, and we consider the energy sublevel

$$\Sigma_\lambda = \{v \in H^1(\Omega) \mid \|v\|_p = 1,\ E_\lambda(v) \leq I_\lambda + \delta(\lambda)\lambda^\alpha\}.$$

LEMMA 1.4. *As $\lambda \to \infty$, we have that*

$$\sup_{v \in \Sigma_\lambda} \inf_{y \in \partial\Omega} \|v - U_{\lambda, y}\|_{L^p(\Omega)} = o(1).$$

Lemma 1.4 is the crucial step in the proof of the existence theorem. Before proving it, we notice that Lemma 1.4, together with Lemma 1.3 and the continuity of the function $\beta$ on $L^p(\Omega) \setminus \{0\}$ implies the following result.

COROLLARY 1.5. *As $\lambda \to \infty$, we have that*

$$\sup_{v \in \Sigma_\lambda} d(\beta(v), \partial\Omega) = o(1).$$

PROOF OF LEMMA 1.4. Fix any sequence $\lambda_n \to \infty$. For every $n$, we fix a function $v_n$ in $\Sigma_{\lambda_n}$ and a point $y_n \in \partial\Omega$ such that

$$\|v_n - U_{\lambda_n, y_n}\|_p = \sup_{v \in \Sigma_{\lambda_n}} \inf_{y \in \partial\Omega} \|v - U_{\lambda_n, y}\|_p + o(1).$$

Now we consider the rescaled function $u_n = v_n^{\lambda_n}$, that is

$$u_n(x) = \lambda_n^{-N/p} v_n(x/\lambda_n),$$

so that, since $v_n \in \Sigma_{\lambda_n}$, by (1.2) and Remark 1.2 we have

$$(1.5)\quad E_{1, \lambda_n\Omega}(u_n) = \lambda_n^{-\alpha} E_{\lambda_n, \Omega}(v_n) = \lambda_n^{-\alpha} I_{\lambda_n} + o(1) \leq 2^{2/p - 1} I + o(1).$$

The last inequality in (1.5) follows from Remark 1.2, since $v_n \in \Sigma_{\lambda_n}$. We want to apply the concentration-compactness lemma by P. L. Lions, with concentration function

$$\rho_n = \chi_{\lambda_n\Omega} |u_n|^p \in L^1(\boldsymbol{R}^N), \qquad \int_{\boldsymbol{R}^N} \rho_n = 1,$$

to prove the following statement.

CLAIM. *The sequence $\rho_n$ is tight, that is, there exists a subsequence $\lambda_n$ and a sequence of points $z_n \in \boldsymbol{R}^N$, such that*

$$\forall \varepsilon > 0,\ \exists R > 0 \text{ with } \int_{B_R(z_n) \cap \lambda_n\Omega} |u_n|^p \geq 1 - \varepsilon.$$

We first show how tightness implies Lemma 1.4. Still denote by $u_n$ an uniformly bounded $H^1(\boldsymbol{R}^N)$-extension of $u_n$ (see the Appendix) and set $\widehat{u}_n = u_n(\cdot + z_n)$. We can assume $\widehat{u}_n \to V$ weakly in $H^1(\boldsymbol{R}^N)$ for some function $V \in H^1(\boldsymbol{R}^N)$. Since

$$\int_{B_R(0) \cap [\lambda_n \Omega - z_n]} |\widehat{u}_n|^p \geqslant 1 - \varepsilon \quad \text{and} \quad \int_{\lambda_n \Omega - z_n} |\widehat{u}_n|^p = 1 .$$

Rellich theorem implies

$$\int_{\lambda_n \Omega - z_n} |\widehat{u}_n - V|^p = o(1), \tag{1.6}$$

and

$$\int_{\lambda_n \Omega - z_n} |V|^p = 1 + o(1). \tag{1.7}$$

Now we want to prove that the sequence $d(z_n, \partial(\lambda_n \Omega))$ is bounded. First, notice that $d(z_n, \lambda_n \Omega) \to \infty$ would imply $B_R(0) \cap [\lambda_n \Omega - z_n] = \emptyset$ for $n$ large, contradicting the very definitions of $R$ and $z_n$. It remains to prove that the case $d(z_n, \boldsymbol{R}^N \setminus \lambda_n \Omega) \to \infty$ cannot occur. If this were the case, we would get

$$\chi_n := \chi_{\lambda_n \Omega - z_n} \to 1 \quad \text{and} \quad \int_{\boldsymbol{R}^N} |V|^p = 1,$$

by (1.7). Let us prove that this implies

$$\liminf E_{1, \lambda_n \Omega - z_n}(\widehat{u}_n) \geqslant E_{1, \boldsymbol{R}^N}(V). \tag{1.8}$$

It is sufficient to notice that

$$\chi_n |\widehat{u}_n|^2 = \chi_n |\widehat{u}_n - V|^2 + \chi_n |V|^2 + 2(\chi_n V)(\widehat{u}_n - V) \geqslant o(1) + |V|^2,$$

and that a similar chain of inequalities holds true for the $L^2$-norms of the gradients. Hence, using (1.7) and (1.8) we would obtain

$$I \leqslant \frac{E_{1, \boldsymbol{R}^N}(V)}{\|V\|_p^2} \leqslant \liminf E_{1, \lambda_n \Omega - z_n}(\widehat{u}_n) = \lim E_{1, \lambda_n \Omega}(u_n) \leqslant 2^{2/p - 1} I,$$

by (1.5): a contradiction.

Since $d(z_n, \partial(\lambda_n \Omega))$ is bounded, we can actually assume that $z_n = \lambda_n y_n$, where $y_n$ is a sequence in the boundary of $\Omega$ such that $y_n \to y_0 \in$

$\in \partial\Omega$. Hence, it is easy to prove that

$$\chi_n = \chi_{\lambda_n \Omega - z_n} \to \chi_H,$$

where, as above, $H = \{z \in \boldsymbol{R}^N \mid z \cdot \nu(y_0) \geq 0\}$, and $\nu(y_0)$ is the interior normal vector at $\partial\Omega$ in $y_0$. We are ready to prove that $V = U$ on the half space $H$. Arguing as in (1.8) and using (1.7) and (1.5), we get

$$(1.9) \qquad 2^{\frac{2}{p}-1} I \geq \limsup E_{1, \lambda_n \Omega - z_n}(\hat{u}_n) \geq$$
$$\liminf E_{1, \lambda_n \Omega - z_n}(\hat{u}_n) \geq E_{1, H}(V) \geq 2^{\frac{2}{p}-1} I.$$

Hence $E_{1, H}(V) = 2^{\frac{2}{p}-1} I$, which implies that $V$ is the ground state solution on the half space $H$, and the conclusion follows.

Since (1.6) is equivalent to

$$\|v_n - U_{\lambda_n, y_n}\|_{L^p(\Omega)} \to 0.$$

Lemma 1.4 is completely proved. ■

PROOF OF THE CLAIM. We have to exclude both vanishing and dichotomy. We still denote by $u_n$ the $H^1(\boldsymbol{R}^N)$-extension of $u_n$ given in the Appendix. It satisfies the inequalities

$$(1.10) \qquad E_{1, \boldsymbol{R}^N}(u_n) \leq c E_{1, \lambda_n \Omega}(u_n),$$

$$(1.11) \qquad \sup_{y \in \boldsymbol{R}^N} \int_{B_t(y)} |u_n|^p \leq c \operatorname{Sup}_{y \in \boldsymbol{R}^N} \int_{B_{ct}(y) \cap \lambda_n \Omega} |u_n|^p,$$

for every $t \leq 1$ and $n$ large enough, where $c$ does not depend on $n$.

*Step* 1. $\liminf I_{1, \lambda_n \Omega} = \liminf \lambda_n^{-\alpha} I_{\lambda_n, \Omega} > 0$. It is a consequence of the inequality

$$I \leq I \left( \int_{\boldsymbol{R}^N} |u|^p \right)^{2/p} \leq E_{1, \boldsymbol{R}^N}(u) \leq c E_{1, \lambda_n \Omega}(u) = c I_{1, \lambda_n \Omega},$$

where $u$ is a minimizer in (1.1), extended, as above, to $\boldsymbol{R}^N$.

*Step* 2. (*vanishing cannot occur*). First notice that if vanishing occurs for the density $\rho_n = \chi_{\lambda_n \Omega} |u_n|^p$, that is

$$\lim_n \sup_{y \in \boldsymbol{R}^N} \int_{B_R(y) \cap \lambda_n \Omega} |u_n|^p = 0 \quad \forall R > 0,$$

then it also occurs for the extended density $|u_n|^p$, in view of (1.11). According to a Lemma by P. L. Lions ([13], part 2, Lemma I.1), we get

$$\int_{R^N} |u_n|^q \to 0, \tag{1.12}$$

for every $q \in ]p, 2^*[$. Since, by (1.10), the sequence $u_n$ is bounded in $H^1(\boldsymbol{R}^N)$, then (1.12) holds for $q = p$ as well, a contradiction.

*Step* 3 (*dichotomy cannot occur*). The argument is nowadays standard, so we sketch it. Dichotomy means that, for $\varepsilon$ small, the sequence $u_n$ splits into

$$u_n^1 = \chi_{B_R(y_n) \cap \lambda_n \Omega} \quad \text{and} \quad u_n^2 = \chi_{\lambda_n \Omega \setminus B_{R_n}(y_n)},$$

for some $R$, $R_n$ tending to infinity, with the properties that

$$\int_{\lambda_n \Omega} |u_n^1|^p \geqslant a_0 - \varepsilon, \qquad \int_{\lambda_n \Omega} |u_n^2|^p \geqslant 1 - a_0 - \varepsilon,$$

for some $a_0 \in ]0, 1[$. After smoothing $u_n^1$ and $u_n^2$, we can assume they belong to $H^1(\lambda_n \Omega)$, so that the energy splits as well:

$$E_{1, \lambda_n \Omega}(u_n) \geqslant E_{1, \lambda_n \Omega}(u_n^1) + E_{1, \lambda_n \Omega}(u_n^2) - \varepsilon \geqslant$$

$$\geqslant [(a_0 - \varepsilon)^{2/p} + (1 - a_0 - \varepsilon)^{2/p}] \lambda_n^{-\alpha} I_{\lambda_n, \Omega} - \varepsilon + o(1),$$

by (1.3). Since $E_{1, \lambda_n \Omega}(u_n) = \lambda_n^{-\alpha} I_{\lambda_n, \Omega} + o(1)$ by (1.5), we get, sending $\lambda_n \to \infty$,

$$1 \geqslant [(a_0 - \varepsilon)^{2/p} + (1 - a_0 - \varepsilon)^{2/p}] + O(\varepsilon),$$

because $E_{1, \lambda_n \Omega}(u_n)$ is bounded away from zero in view of Step 1: a contradiction. ■

From (1.9) we also get a corollary on the ground state energy $I_{1, \lambda \Omega}$ for solutions to $(0.2)_\lambda$ (with $\lambda = 1$), on the «dilated» domains $\lambda \Omega$.

COROLLARY 1.6. *Let* $\lambda \to \infty$. *Then*

$$\lambda^{-\alpha} I_{\lambda, \Omega} = I_{1, \lambda \Omega} \to 2^{\frac{2}{p} - 1} I.$$

## 2. Proof of the main result.

In order to compare the topology of $\partial\Omega$ and the topology of some energy sublevels, we will make use of the functions (see [3]):

$$\Phi_\lambda : \partial\Omega \to H^1(\Omega), \qquad \Phi_\lambda(y) = \frac{U_{\lambda, y}\,|_\Omega}{(\int\limits_\Omega U_{\lambda, y}^p\,dx)^{1/p}},$$

$$\beta : H^1(\Omega)\setminus\{0\} \to \boldsymbol{R}^N, \qquad \beta(u) = \frac{\int\limits_\Omega x|u|^p}{\int\limits_\Omega |u|^p}.$$

Set $M_\lambda := \max\limits_{y \in \partial\Omega} E_\lambda(\Phi_\lambda(y))$ and $\delta(\lambda) = \lambda^{-\alpha}(M_\lambda - I_\lambda)$. We have $\lambda^{-\alpha}M_\lambda = 2^{\frac{2}{p}-1} I + o(1) = \lambda^{-\alpha} I_\lambda + o(1)$ by Lemma 1.1 and Corollary 1.6. Thus $\delta(\lambda) \to 0$ as $\lambda \to \infty$. Clearly

$$\Phi_\lambda(\partial\Omega) \subseteq \Sigma_\lambda = \{u \in H^1(\Omega)\,|\, \|u\|_p = 1,\, E_\lambda(u) \leqslant I_\lambda + \delta(\lambda)\lambda^\alpha\}.$$

Also, by Corollary 1.5, for $\lambda$ large $\beta$ sends $\Sigma_\lambda$ into a small neighborhood of $\partial\Omega$ and hence $\pi_\partial \circ \beta$ is well defined, $\pi_\partial$ being the minimal distance projection of a neighborhood of $\partial\Omega$ onto $\partial\Omega$. So we have the following situation: $\Phi_\lambda : \partial\Omega \to \Sigma_\lambda$, $\pi_\partial \circ \beta$ is well defined on $\Sigma_\lambda$ and $\pi_\partial \circ \beta \circ \Phi_\lambda$ is close to the identity for $\lambda$ large (by Lemma 1.3). Like in [3] (see also Theorem 3.1 in [4]), this implies that, for such $\lambda$, $\operatorname{cat} \Sigma_\lambda \geqslant \operatorname{cat} \partial\Omega$. Standard Lusternik-Schnirelman arguments give the existence of at least cat $\partial\Omega$ solutions, which are certainly non constant, because $\lambda$ is taken large (and for $\lambda$ large, constant maps have high energy).

Finally, positivity of solutions follows as in [3], because their energy is below the level $2^{1-2/p} I$ (see also [5]).

The existence of one more solution easily follows, by observing that the map $\Phi_\lambda$ can be extended to $\overline{\Omega}$, simply by setting

$$\Phi_\lambda : \overline{\Omega} \to H^1(\Omega), \qquad \Phi_\lambda(y) = U_{\lambda, y}\,|_\Omega / \|U_{\lambda, y}\|_{\Omega, \beta}.$$

If there were no critical levels above $E_\lambda + \delta(\lambda)\lambda^\alpha$, the map $\Phi_\lambda$ could be deformed to a continuous function $\widetilde{\Phi}_\lambda$ from $\overline{\Omega}$ into $\Sigma_\lambda$. But in this case $\beta \circ \widetilde{\Phi}_\lambda$ would take values in a neighborhood of $\partial\Omega$, and hence $\pi_\partial \circ \beta \circ \widetilde{\Phi}_\lambda$ would be a continuous map from $\overline{\Omega}$ into $\partial\Omega$ with boundary values homotopic to the identity on $\partial\Omega$. This is clearly impossible, because $\deg(\pi_\partial \circ \beta \circ \widetilde{\Phi}_\lambda, \partial\Omega) = 1$ implies that the image of $\widetilde{\Phi}_\lambda$ has interior points.

The above argument gives a stationary point for the energy functional on $H^1(\Omega)$ which possibly changes sign. Actually by working in the positive cone of $H^1(\Omega)$, and following the Cerami-Passaseo arguments in [5], the positivity of this solution follows.

REMARK 2.1. Let $v_n$ be a minimum for (1.1), with $\lambda = \lambda_n \to \infty$, and $U_{\lambda_n, y_n}$ be the «rescaled» ground state solution which is closest to $v_n$ (see Lemma 1.4). It would be of interest to characterize the limit point of $y_n$. In view of known estimates involving the mean curvature of $\partial\Omega$ (see [14] and [1]), it seems reasonable that such limit points should maximize the mean curvature of $\partial\Omega$. From this, more informations on the number of positive solutions could be derived. Partial results in this direction have been communicated to the authors by F. Pacella and S. L. Yadava. ∎

## Appendix: The extension operator.

We conclude this paper with a remark on the existence of suitable extension operators on Sobolev spaces (see also [9]). The next lemma was fundamental in the proof of the existence theorem.

LEMMA A.1. *Let $\Omega$ be a bounded smooth domain of $\mathbf{R}^N$. Then, for every $\lambda > 1$, there exists an extension operator $P_\lambda : H^1(\lambda\Omega) \to H^1(\mathbf{R}^N)$ satisfying:*

(i) $P_\lambda u|_{\lambda\Omega} = u$ *on* $\lambda\Omega$,

(ii) $$\int_{\mathbf{R}^N} |\nabla P_\lambda u|^2 \leq c\left[\int_{\lambda\Omega} |\nabla u|^2 + \frac{1}{\lambda}\int_{\lambda\Omega} u^2\right],$$

(iii) $$\int_{\mathbf{R}^N} |P_\lambda u|^q \leq c\int_{\lambda\Omega} |u|^q \quad \forall q \in [2, 2^*],$$

iv) *there exists a constant $\varepsilon > 0$ which depends only on $\Omega$, such that*

$$\operatorname*{Sup}_{y \in \mathbf{R}^N} \int_{B_t(y)} |P_\lambda u|^q \leq c \operatorname*{Sup}_{y \in \mathbf{R}^N} \int_{B_{ct}(y) \cap (\lambda\Omega)} |u|^q, \qquad \forall q \in [2, 2^*], \ \ \forall t < \lambda\varepsilon,$$

*for every $u \in H^1(\lambda\Omega)$, where the constants $c$ does not depend on $\lambda$.*

PROOF. Let $\pi$ be the minimal distance projection of a neighborhood of $\Omega$ into $\overline{\Omega}$. We fix a cut-off function $\psi \in C_0^\infty(\mathbf{R}^N)$ such that $\psi = 1$ on $\overline{\Omega}$

and such that supp $\psi$ is contained in a small neighborhood of $\overline{\Omega}$, and we define

$$(P_\lambda u)(x) := \begin{cases} \psi\left(\frac{x}{\lambda}\right) u\left(2\lambda\pi\left(\frac{x}{\lambda}\right) - x\right) & \text{for } x \text{ close to } \lambda\Omega, \\ 0 & \text{otherwise in } \boldsymbol{R}^N. \end{cases}$$

It is clear that condition (i) is satisfied. Conditions (ii) and (iii) follow by simple computation, since the map

$$T_\lambda x := 2\lambda\pi\left(\frac{x}{\lambda}\right) - x$$

is Lipschitz continuous in a neighborhood of $\lambda\Omega$, with Lipschitz constant $c$ which depends only on $\partial\Omega$. In proving condition (iv), it is convenient to choose $\varepsilon > 0$ such that the projection $\pi$ is well defined on $\overline{N_{3\varepsilon}(\Omega)}$ and $\operatorname{supp}\psi \subseteq N_\varepsilon(\Omega)$ (here, $N_\varepsilon(\Omega)$ is the set of all $z \in \boldsymbol{R}^N$ such that $d(z, \Omega) < \varepsilon$). Thus the transformation $T_\lambda$ is well defined on $\overline{N_{3\lambda\varepsilon}(\lambda\Omega)}$ and $\operatorname{supp}\psi(\cdot/\lambda) \subseteq N_{\lambda\varepsilon}(\lambda\Omega)$. In particular, for $t < \lambda\varepsilon$ and for every map $u$ defined on $\lambda\Omega$ such that $P_\lambda u$ is not identically zero on a ball $B_t(y)$, we have that $B_t(y) \subseteq N_{3\lambda\varepsilon}(\lambda\Omega)$. Hence $T_\lambda$ is well defined on $B_t(y)$, and since it is Lipschitz continuous we also get that $T_\lambda(B_t(y) \setminus \Omega) \subseteq B_{ct}(T_\lambda(y)) \cap \Omega$. Finally, we just observe that

$$\int_{B_t(y)\setminus\lambda\Omega} |P_\lambda u|^q \leq c \int_{B_t(y)\setminus\lambda\Omega} |u(T_\lambda x)|^q \leq c \int_{B_{ct}(T_\lambda y)\cap\lambda\Omega} |u|_q,$$

and the conclusion follows immediately. ■

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Manoscritto pervenuto in redazione il 4 luglio 1991.