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The Triangle Groups.

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ABSTRACT - The aim of this paper is to consider the structure and other properties of some of the triangle groups $\Delta(l, m, n)$ for positive integers $l, m, n \geq 2$.

The triangle group $\Delta(l, m, n)$ is defined by the presentation

$$\Delta(l, m, n) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^l = (bc)^m = (ca)^n = e \rangle.$$

It is the group of tesselation of a space with a triangle [7]. The group $\Delta(l, m, n)$ is finite iff the corresponding space is compact. This implies that $|\Delta(l, m, n)| < \infty$ iff $1/l + 1/m + 1/n > 1$. [7]. We get the following three cases for $\Delta(l, m, n)$.

1) The Euclidean case if $1/l + 1/m + 1/n = 1$. This equation has the solution $(3, 3, 3)$, $(2, 3, 6)$ and $(2, 4, 4)$.

2) The elliptic case if $1/l + 1/m + 1/n > 1$. This inequality has the following solutions $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ for $n \geq 2$.

3) The hyperbolic case if $1/l + 1/m + 1/n < 1$. This inequality has an infinite number of solutions.

REMARK 1. $\Delta(-l, m, n) \cong \Delta(l, m, n) \cong \Delta(m, l, n)$. The group $\Delta(l, m, n)$ depends only on the absolute values of l, m, n and not on their order or sign.

THEOREM 1. *The group $\Delta(l, m, n)$ is finite iff* $1/l + 1/m + 1/n > 1$.

PROOF. We use the fact that $\Delta(l, m, n)$ is a Coxeter group. Its asso-

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ciated quadratic form has the matrix

$$Q = \begin{bmatrix} 1 & -\cos\frac{\pi}{l} & -\cos\frac{\pi}{n} \\ -\cos\frac{\pi}{l} & 1 & -\cos\frac{\pi}{m} \\ -\cos\frac{\pi}{n} & -\cos\frac{\pi}{m} & 1 \end{bmatrix}.$$

Therefore $\Delta(l, m, n)$ is finite iff Q is positive definite [12]. It is easy to see that Q is positive definite iff

$$|Q| = 1 - \left[\cos^2\frac{\pi}{l} + \cos^2\frac{\pi}{m} + \cos^2\frac{\pi}{n} + 2\cos\frac{\pi}{l}\cos\frac{\pi}{m}\cos\frac{\pi}{n}\right] = 1 - B$$

is positive. We consider now the three possible cases for l, m, n:

(i) If $1/l + 1/m + 1/n > 1$, then (l, m, n) is one of: $(2,3,3)$, $(2,3,4)$, $(2,3,5)$, $(2,2,n)$, $n \geqslant 2$. It is easy to see that $B < 1$ in every case and hence $|Q| > 0$. Therefore Q is positive definite and $\Delta(l, m, n)$ is finite.

(ii) If $1/l + 1/m + 1/n = 1$. The solutions of this equation are $(2,3,6)$, $(2,4,4)$ and $(3,3,3)$. In every case $B = 1$ and so Q is not positive definite and $\Delta(l, m, n)$ is infinite.

(iii) $1/l + 1/m + 1/n < 1$. The number of solutions of this inequality is infinite. We classify them as follows:

$$\{(2, 3, n) \,|\, n \geqslant 7\}, \quad \{(2, 4, n) \,|\, n \geqslant 5\}, \quad \{(2, m, n) \,|\, m \geqslant n \geqslant 5\},$$

$$\{(3, 3, n) \,|\, n \geqslant 4\}, \quad \{(3, n, n) \,|\, \geqslant m \geqslant n \geqslant 4\}, \quad \{(l, m, n) \,|\, l \geqslant m \geqslant 4\}.$$

It is easy to see that in every case $B > 1$ and hence Q is not positive definite. Therefore $\Delta(l, m, n)$ is infinite.

Notational conventions. We use the abbreviation RSRP for the Reidemeister-Schreier rewriting process. We use $\rtimes$ for the semi-direct product and ≀ for the wreath product and h.c.f. for the highest common factor.

General properties of the group $\Delta(l, m, n)$.

a) Let $x = ab$, $y = bc$ and $H = \langle x, y \rangle$. It is easy to see that $H \trianglelefteq \Delta(l, m, n)$ and $\Delta/H \cong Z_2$. Using the RSRP we find that H is isomorphic to the von-Dyck group $D(l, m, n) = \langle x, y \mid x^l = y^m = (xy)^n = e\rangle$. So we have the following theorem.

THEOREM 2. *$D(l, m, n)$ isormal subgroup of $\Delta(l, m, n)$ of index 2.*

REMARK 2. We consider the map θ: $\Delta(l, m, n) \to Z_2 = \langle x \mid x^2 = e\rangle$ defined by $a \to x$, $b \to x$, $c \to x$. Then θ is a split extension. $\Delta/\ker\theta \cong Z_2$ and using the RSRP we get $\ker\theta \cong D(l, m, n)$. Hence $\Delta(l, m, n) \cong$ $\cong D(l, m, n) \rtimes Z_2$.

REMARK 3. *a)* $D(-l, m, n) \cong D(l, m, n) \cong D(m, l, n)$. The group $D(l, m, n)$ depends only on the absolute values of l, m, n and not on their order or sign.

b) The abelianized von-Dyck group is $D(l, m, n)/D'(l, m, n,) =$ $= \langle x, y \mid x^l = y^m = x^n y^n = e, xy = yx\rangle$. The following theorem determines the cases when this group is finite.

THEOREM 3. *The group $D(l, m, n)/D'(l, m, n,)$ is finite iff at most one of l, m, n is zero.*

PROOF. The relation matrix of $\dfrac{D(l, m, n)}{D'(l, m, n)}$ is $\begin{bmatrix} l & 0 \\ 0 & m \\ n & n \end{bmatrix}$. We consider the following cases:

(i) Let l, m, n be non-zero. Then $D(l, m, n)/D'(l, m, n,) \cong$ $\cong Z_{d_1} \times Z_{d_2}$ where

$$d_1 = hcf\{l, m, n\} \quad \text{and} \quad d_2 = \frac{hcf\{lm, mn, ln\}}{hcf\{l, m, n\}}.$$

Thus, $D/D,$ is a finite group of order $d_1 d_2 = hcf\{lm, mn, ln\}$.

(ii) Let one and only one of l, m, n be zero. WLOG we take $n = 0$. Then $D/D' = Z_l \times Z_m$ and so finite of order lm.

(iii) Let two of l, m, n be zeros. WLOG we take $m = n = 0$. Thus $D/D' = Z_l \times Z$ which is infinite.

(iv) Let $l = m = n = 0$. Thus $D/D' \cong Z \times Z$ which is infinite.

Therefore D/D' is finite iff at most one of l, m, n is zero.

Properties of some of the triangle groups.

1) *The Euclidean case.* The group $\Delta(3,3,3)$ is the affine Weyl group of type $\tilde{A}_2$. We showed in our paper [2] that $\Delta(3, 3, 3) \cong (Z \times$ $\times Z) \rtimes S_3$, $Z(\Delta(3, 3, 3))$ is trivial and $\Delta(3,3,3)$ is solvable of derived

length 3. In our paper [3] we showed that $\Delta(3,3,3)$ is a subgroup of the wreath product $Z \wr S_3$.

REMARK 4. To identify the structure of a group G we look for a known group H and a split extension $\theta: G \to H$. Then $G/\ker\theta \cong H$. If $|H|$ is small, then we can find ker θ using the RSRP. Hence we get $G \cong \ker\theta \rtimes H$. We use this method in several places of this paper.

We observe the following properties of $\Delta(3,3,3)$.

a) $\Delta'(3, 3, 3) = D(3, 3, 3)$, $\Delta''(3, 3, 3) = Z \times Z$ and hence $\Delta(3,3,3)$ is solvable of derived length 3. $D(3, 3, 3) \cong (Z \times Z) \rtimes Z_3$.

b) We define $\theta: \Delta(3, 3, 3) \to s_3 = \langle x, y \mid x^2 = y^2 = (xy)^3 = e\rangle$ by $a \to x$, $b \to x$, $c \to y$. θ is a split extension and $\ker\theta \cong D(3, 3, 3)$. Hence we get $\Delta(3, 3, 3) \cong D(3, 3, 3) \rtimes S_3$.

2) *The group* $\Delta(2,4,4)$. The group $\Delta(2,4,4)$ is $\tilde{C}_3$ which is one of the affine Weyl groups of type $\tilde{C}_l$. We showed in our paper [5] the following properties of $\Delta(2,4,4)$:

a) $\Delta'(2, 4, 4) = \langle x, y, z \mid x^2 = y^2 = z^2 = (xyz)^2 = e\rangle$ an $\Delta''(2, 4, 4) = Z \times Z$. Thus $\Delta(2,4,4)$ is solvable of derived length 3. We also showed that $\Delta'(2, 4, 4) \cong (Z \times Z) \rtimes Z_2$. and $D(2, 4, 4) \cong (Z \times Z) \rtimes Z_4$.

b) $\Delta(2, 4, 4) \cong D(2, 4, 4) \rtimes (Z_2 \times Z_2)$.

c) $\Delta(2, 4, 4) \cong \Delta'(2, 4, 4) \rtimes (D_4 \times Z_2)$.

d) $\Delta(2, 4, 4) \cong D(2, 4, 4) \rtimes D_4$.

e) $\Delta(2, 4, 4) \cong H \rtimes (Z_2 \times Z_2)$ where $H = \langle c, d \mid d^2cd^2 = c\rangle$.

3) *The group* $\Delta(2,3,6)$. We get the following properties of $\Delta(2,3,6)$:

a) $\Delta'(2, 3, 6) = D(2, 3, 6)$, $\Delta''(2, 3, 6) = Z \times Z$. Hence $\Delta(2,3,6)$ is solvable at derived length 3.

b) Let $\theta: D(2, 3, 6) \to Z_6 = \langle a \mid a^6 = c\rangle$ defined by: $x \to a^3$, $y \to a^2$. Then θ is a split extension and we find $D(2, 3, 6) = (Z \times Z) \rtimes Z_6$.

c) We define $\theta: \Delta(2, 3, 6) \to S_3 \times Z_2 = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^3 = (xz)^2 = (xy)^2 = e\rangle$ by $a \to z$, $b \to y$, $c \to x$. Then θ is a split extension and we get $\Delta(2, 3, 6) = D(3, 3, 3) \rtimes (S_3 \times Z_2)$.

d) We let $\theta: \Delta(2, 3, 6) \to S_3 = \langle x, y \mid x^2 = y^2 = (xy)^3 = e\rangle$ defined by $a \to x$, $b \to x$, $c \to y$. Then $\ker\theta \cong G = \langle p, q, r \mid p^2 = q^2 = r^2 = (pqr)^2 = e\rangle$ and $\Delta(2, 3, 6) = G \rtimes S_3$.

e) We let $\theta: \Delta(2, 3, 6) \to Z_2 \times Z_2 = \langle x, y \mid x^2 = y^2 = (xy)^2 = e\rangle$ defined by $a \to x$, $b \to y$, $c \to y$. Then $\Delta(2, 3, 6) \cong D(3, 3, 3) \rtimes (Z_2 \times Z_2)$.

4) *The elliptic case.* The groups in this case are $\Delta(l, m, n)$ where

$1/l + 1/m + 1/n > 1$. These groups are well-known [8]. They are as follows: $\Delta(2, 2, n) = D_n \times Z_2$, $\Delta(2, 3, 3) = S_4$, $D(2, 3, 3) = A_4$, $\Delta(2, 3, 4) = S_4 \rtimes Z_2$, $D(2, 3, 4) = S_4$, $\Delta(2, 3, 5) = A_5 \rtimes Z_2$, $D(2, 3, 5) = A_5$. We note here that $\Delta(2, 3, 4)$ is B_3 a special case of the Coxeter groups of type B_n. The structure of $\Delta(2, 3, 4)$ is $\Delta(2, 3, 4) \cong Z_2 \wr S_3$ [4].

5) *The hyperbolic case.* The groups in this case are $\Delta(l, m, n)$, where $1/l + 1/m + 1/n < 1$. The number of possible values of the ordered triple (l, m, n) satisfying the inequality is infinite. We classify these solutions of the inequality in the following categories:

(i) $(2, 3, n)$, $n \geqslant 7$,
(ii) $(2, 4, n)$, $n \geqslant 5$,
(iii) $(2, m, n)$, $n \geqslant m \geqslant 5$,
(iv) $(3, 3, n)$, $n \geqslant 4$,
(v) $(3, m, n)$, $n \geqslant m \geqslant 4$,
(vi) (l, m, n), $l \geqslant m \geqslant n \geqslant 4$.

We investigate some of the properties of some of the groups in these categories.

a) The groups $\Delta(2, 3, n)$, $n \geqslant 7$.

We obtain the following results about these groups:

(i) If $(n, 6) = 1$, then $\Delta'(2, 3, n) = D(2, 3, n) =$ and $D(2, 3, n)$ is perfect. Hence $\Delta(2, 3, n)$ is not solvable.

(ii) If $(n, 6) = 2$, $\Delta'(2, 3, n) = D(3, 3, n/2)$ and $\Delta''(2, 3, n) = D(n/2, n/2, n/2)$.

(iii) If $(n, 6) = 3$, $\Delta'(2, 3, n) = D(2, 3, n) = \langle r, s, t \mid r^{n/3} = s^2 = t^2 = (rst)^2 = e \rangle$ and $\Delta'''(2, 3, n) = \langle d, f, g \mid d^{n/3} = f^{n/3} = g^{n/3} = (dfg)^{n/3} = e \rangle$.

(iv) If

$$(n, 6) = 6, \qquad \Delta'(2, 3, n) = D(3, 3, n/2),$$

$$\Delta''(2, 3, n) = \langle p, q \mid (pqp^{-1}q^{-1})^{n/6} = e \rangle.$$

Since the number if relations is less than the number of generators, we deduce that $\Delta''(2, 3, n)$ is infinite. Hence $\Delta(2, 3, n)$ is infinite.

b) The case $\Delta(2, 4, n)$ $n \geqslant 5$.

We obtain the following results about these groups:

(i) If

$$(n, 4) = 1, \qquad \Delta'(2, 4, n) = D(2, n, n) = D'(2, 4, n),$$

$$\Delta''(2, 4, n) = \langle p_1, p_2, \dots, p_{n-1} \mid p_1^2 = p_2^2 = \dots$$
$$\dots = p_{n-1}^2 = (p_1 p_2, \dots, p_{n-1})^2 = e \rangle = D''(2, 4, n).$$

(ii) If

$$(n, 4) = 2, \qquad \Delta'(2, 4, n) = \langle x, y, z \mid x^{n/2} = y^{n/2} = (xy)^2 = (yz)^2 = e \rangle,$$
$$D'(2, 4, n) = \langle a, b, c \mid a^{n/2} = b^2 = (bc)^{n/2} = (ca)^2 = e \rangle.$$

(iii) If

$$(n, 4) = 4, \qquad \Delta'(2, 4, n) = \langle x, y, z \mid x^{n/2} = y^{n/2} = (xy)^2 = (yz)^2 = e \rangle,$$

$$\Delta''(2, 4, n) = \langle p_i, p_j, 1 \leqslant i \leqslant k-1, 0 \leqslant j \leqslant k-2,$$
$$k = \frac{n}{2} \,\Big|\, p_{k-1} q_{k-2} p_{k-3}, \dots, p_1 q_0 = q_0 p_1 q_2, \dots, p_{k-1} \rangle,$$

Since the number of generators is greater than the number of relations, the group $\Delta''(2, 4, n)$ is infinite and hence the group $\Delta(2, 4, n)$ is infinite.

We also find in this case that $D'(2, 4, n) = \langle a, b, c \mid a^{n/4} = (abcb^{-1}a^{-1})^{n/4} = e \rangle$ which implies that $D'(2, 4, n)$ is infinite by the same argument as in the previous paragraph. Therefore $D(2, 4, n)$ and $\Delta(2, 4, n)$ are also infinite.

c) The groups $\Delta(2, 5, n)$, $n \geqslant 4$.

We find the following results about these groups:

(i) If $(n, 10) = 1$, $\Delta'(2, 5, n) = D(2, 5, n)$ and $D'(2, 5, n) = D(2, 5, n)$. Therefore $\Delta(2, 5, n)$ is not solvable.

(ii) If $(n, 10) = 2$, $\Delta'(2, 5, n) = D(5, 5, n/2)$, $D'(2, 5, n) = D(5, 5, n/2)$..

(iii) If $(n, 10) = 5$, $\Delta'(2, 5, n) = D(2, 5, n)$ and

$D'(2, 5, n) =$

$$= \langle p_0, p_1, p_2, p_3, p_4, \mid p_0^2 = p_1^2 = p_2^2 = p_3^2 = p_4^2 = (p_0 p_1 p_2 p_3 p_4)^{n/5} = e \rangle.$$

(iv) If $(n, 10) = 10$, $\quad \Delta'(2, 5, n) = D\left(5, 5, \frac{n}{2}\right)$ and

$$D'(2, 5, n) = \langle s_1, s_2, s_3, s_4 \mid (s_1 s_2 s_3 s_4 s_1^{-1} s_2^{-1} s_3^{-1} s_4^{-1})^k = e\rangle$$

where $k = n/10$. Thus $D'(2,5,n)$ is infinite and so $\Delta(2,5,n)$ is also infinite.

d) The groups $\Delta(3,3,n)$, $n \geqslant 4$.

(i) If

$$(n, 3) = 3, \qquad \Delta'(3, 3, n) = D(3, 3, n),$$

$$\Delta''(3, 3, n) = \langle a, b, c, d \mid a^{n/3} = (bcd)^{n/3} = (cabd)^{n/3} = e\rangle.$$

(ii) If $(n,3) = 1$, $\Delta'(3, 3, n) = D(3, 3, n)$ and $\Delta''(3, 3, n) = D(n, n, n)$.

e) The groups $\Delta(2,m,n)$, $n \geqslant m \geqslant 5$.

(i) If m and n are even,

$$\Delta'(2, m, n) = \langle x, y, z \mid x^{n/2} = y^{n/2} = (xz)^{m/2} = (yz)^{m/2} = e\rangle.$$

(ii) If m is even and n is odd, $\Delta'(2, m, n) = D(n, n, m/2)$.

(iii) If m and n are both odd $\Delta'(2, m, n) = D(2, m, n)$. We let $k = (m.n)$ where $m = sk$ and $n = rk$. Then

$$D'(2, m, n) = \langle p_0, p_1, \ldots, p_{k-1}, q \mid p_0^2 = p_1^2 = \ldots$$
$$\ldots = p_{k-1}^2 = (p_0 p_1, \ldots, p_{k-1} q)^r = q^s = e\rangle.$$

f) The groups $\Delta(3,m,n)$, $n \geqslant m \geqslant 4$.

(i) If m and n are even,

$$\Delta'(3, m, n) = \langle x, y, z \mid x^{n/2} = y^3 = (yz)^{m/2} = (zx)^3 = e\rangle.$$

(ii) If m or n is odd, $\Delta'(3, m, n) = (3, m, n)$.

General properties of the groups $\Delta(l,m,n)$

a) The commutator subgroup of $\Delta(l,m,n)$ is:

(i) If l, m, n are even,

$$\Delta'(l, m, n) = \langle x_1 x_2, x_3, x_4, x_5 \mid x_1^{l/2} =$$
$$= x_2^{m/2} = x_3^{n/2} = (x_2 x_4 x_5)^{l/2} = (x_3 x_5)^{m/2} = (x_1 x_4)^{n/2} = e\rangle.$$

(ii) If two of l, m, n are even and one is odd, WLOG let n be odd and l, m be even

$$\Delta'(l, m, n) = \langle x_1, x_2, x_3 \mid x_1^{l/2} = x_2^n = (x_2 x_3)^{m/2} = (x_3 x_1)^n = e \rangle .$$

(iii) If at most one of l, m, n is even,

$$\Delta'(l, m, n) = D(l, m, n) .$$

b) We give a necessary and sufficient condition that makes $D(l, m, n)$ perfect.

THEOREM. *$D(l, m, n)$ is perfect iff l, m, n are mutually relatively prime.*

PROOF. The relation matrix for $\frac{D}{D'}$ is $\begin{bmatrix} l & 0 \\ 0 & m \\ n & n \end{bmatrix}$. Hence $\frac{D}{D'} = Z_{d_1} \times Z_{d_2}$ where

$$d_1 = hcf\{l, m, n\} \quad \text{and} \quad d_2 = \frac{hcf\{lm, mn, nl\}}{d_1} .$$

Let D be perfect, i.e., $D/D' \cong E \Rightarrow d_1 = d_2 = 1 \Rightarrow hcf\{l, m, n\}$ and $hcf\{lm, mn, nl\} = 1$. This easily implies that l, m, n are mutually relatively prime. Let l, m, n be mutually relative prime $\Rightarrow hcf\{l, m, n\} = 1 \Rightarrow d_1 = 1$. It is easy to see that $hcf\{lm, mn, nl\} = 1$ and hence $D/D' \cong E$ and

c) The derived subgroup of the group $D(n, n, n)$, $n \geqslant 3$.

$$D'(n, n, n) = \langle X \mid R, S, T \rangle \text{ where}$$

$$X = \{B_{i,j} \mid 0 \leqslant i \leqslant n-1, \quad 1 \leqslant j \leqslant n-1\},$$

$$R = \{B_{0,j} B_{1,j} \dots B_{n-1,j} = e \mid 1 \leqslant j \leqslant n-1\},$$

$$S = \{B_{0,1} B_{1,2} \dots B_{n-2} B_{n-1} = e\},$$

$$T = \{B_{i,1} B_{i+1,2} \dots B_{0,q+1} B_{1,q+2} \dots, B_{p,n-1} = e \mid 1 \leqslant i \leqslant n-1\}.$$

THEOREM. *The group $D(n, n, n)$ is infinite iff $n \geqslant 3$.*

PROOF. If $n = 1 \Rightarrow D = E$. If $n = 2 \Rightarrow D = Z_2 \times Z_2$. Let $n \geqslant 3$. The number of generators of D' is $n(n-1)$. The number of relations is $|R| + |S| + |T| = 2n - 1$. Now the number of generators is greater than the number of relations iff $n \geqslant 3$. Hence D' is infinite iff $n \geqslant 3$ and so D is infinite iff $n \geqslant 3$.

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