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Normal Projective Surfaces with $\rho = 1$, $P_{-1} \geq 5$.

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0. Introduction.

One of the most interesting problems in the study of normal surfaces is to classify normal surfaces X with $\rho(X) = 1$ and $-K_X$ ample (cf. [Sa2], Problem 3.3).

The study of such surfaces arises naturally when we study normal degenerations of rational surfaces and in particular normal degenerations of $\mathbb{P}^2$ ([Ma], [Ba1], [Ba2]).

A first partial result for these surfaces is obtained by putting together a theorem of Sakai with one of Badescu.

THEOREM A (Sakai-Badescu). *Let X be a normal projective surface with $\rho = 1$, $P_n = 0$ $\forall n > 0$ and let $u: Y \to X$ be its minimal resolution. Then $H^1(\mathcal{O}_X) = 0$ ond one of the following possibilities holds:*

1) *Y is a rational surface and the singularities of X are rational.*

2) *Y is a ruled surface with irregularity $q > 0$, X contains exactly one nonrational singularity at x, the geometric genus of (X, x) is q, the exceptional divisor of u over x is given by a section of the canonical fibration $p: Y \to B$ (B smooth curve of genus q) plus possibly components of degenerate fibres of p.*

In both cases we have no information about the structure and number of rational singularities of X. Here we prove, using elementary algebraic geometry, a structure theorem for surfaces belonging to class 1) of Theorem A having $P_{-1} \geq 5$. Our results give in particular the following:

THEOREM B. *Let X be a normal projective surface with $\rho(X) = 1$,*

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$P_{-1}(X) \geqslant 5$ with at most rational singularities. Then X has at most one non cyclic singularity and, if all the singularities are cyclic then X has at most three singular points.

We remark that the result we prove also gives information about the minimal resolution of X. Finally in § 4 we study the particular case where the surface X is a normal degeneration of $\mathbb{P}^2$ (from [Ma] follows that $\rho(X) = 1$ and $P_{-1}(X) \geqslant 10$), and we prove in particular (Corollary 12) that if X has at most rational singularities then X has at most 4 singular points.

NOTATION. For every normal surface X and every Weil divisor D on X we denote:

$\mathcal{O}_X(D)$ = sheaf of meromorphic functions f such that $(f) + D \geqslant 0$.

$h^i(D) = \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(D)) \quad i \geqslant 0$.

K_X = canonical divisor for X.

θ_X = tangent sheaf of X, defined as the dual of the sheaf Ω^1_X of Kähler differentials.

$q(X) = h^1(\mathcal{O}_X)$ irregularity of X.

$p_g(X) = h^2(\mathcal{O}_X)$ geometric genus of X.

$P_n(X) = h^0(nK_X)$ n-th plurigenus of X.

$NS(X) = (\mathrm{Pic}(X)/\mathrm{Pic}^0(X)) \oplus \mathbb{Q}$ Neron-Severi group of X.

$\rho(X) = \dim_{\mathbb{Q}} NS(X)$ Picard number.

A (-1)-curve in a surface is a smooth rational curve E such that $E^2 = -1$.

If $\delta\colon Y \to X$ is a proper birational morphism from a smooth surface Y to a normal surface X we shall call exceptional divisor of δ the set $D \subset Y$ given by irreducible curves contracted by δ.

Aknowledgement. I would like to thank F. Catanese for drawing my attention to the surface which we consider here and for many valuable discussions.

1. Curves with negative self intersection in a rational surface.

Let S be a smooth rational surface, then S does not contain any irreducible curve with negative self intersection if and only if $S = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$. From now on, by abuse of notation we shall denote by a rational surface a rational surface different from $\mathbb{P}^2$, $\mathbb{P}^1 \times \times \mathbb{P}^1$.

Let S be a such a rational surface, then there exists an integer $d \geqslant 1$ and a birational morphism $\mu: S \to \mathbb{F}_d$ such that μ is an isomorphism in a neighbourhood of the section σ_∞ with self intersection $-d$ (cf. for example [Be]). We note that μ is the composition of $\rho(S) - 2$ blowings-up.

Let $p: S \to \mathbb{P}^1$ be the fibration obtained by composing μ with the natural projection $\pi: \mathbb{F}_d \to \mathbb{P}^1$.

(*) In order to simplify the presentation of next proofs we introduce some technical notation.

In the situation above let $r = \rho(S) - 2$, let h be the number of degenerate fibres of p and let e be the number of (-1)-curves contained in the fibres of p. We note that $e \geqslant h$ and $r = \sum_{\text{fibres of } p} (b_2(f) - 1)$.

DEFINITION 1. In the notation above, a smooth irreducible curve $C \subset S$ is said to be μ-transversal or simply transversal if $C \cdot f > 0$ where f is a fibres of p.

THEOREM 1. *Let S be a rational surface, $\mu: S \to \mathbb{F}_d$ a birational morphism which is an isomorphism in a neighbourhood of σ_∞ and $C \subset S$ a transversal curve $\neq \sigma_\infty$.*

If $h^0(-K_S) + \min\{d, 3\} \geqslant 8$ then $C^2 \geqslant -1$.

We prove this theorem later on. Let X be a smooth surface, $x \in X$ and $\tilde{X} \xrightarrow{f} X$ the blowing up of X at x. We have an exact sequence of sheaves on X

$$0 \to f_* \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}}) \to \mathcal{O}_X(-K_X) \to \Lambda^2 T_x X \to 0.$$

In particular the vector space $H^0(-K_{\tilde{X}})$ is naturally isomorphic to the space of sections of the anticanonical sheaf of X which vanish at x.

COROLLARY 2. *Let S be a rational surface: if, in the notation above, $h^0(-K_S) + \min\{d, 3\} \geqslant 9$ and $C \in S$ is a transversal curve $\neq \sigma_\infty$, then $C^2 \geqslant 0$.*

PROOF. The proof follows by considering the blowing up of S at a point of C. ■

Theorem 1 cannot be improved. Let in fact $S_d (d \geqslant 1)$ be a surface obtained by blowing up the surface $\mathbb{F}_d$ at $d+1$ generic points $p_0, \ldots, p_d$. These points lie on a section $\sigma_0 \subset \mathbb{F}_d$ such that $\sigma_0^2 = d$, let $C \subset S_d$ be the strict transform of σ_0: clearly $C^2 = -1$, and, recalling that

$$h^0(-K_{\mathbb{F}_d}) = \begin{cases} 9 & 1 \leqslant d \leqslant 3, \\ d+6 & d \geqslant 3, \end{cases}$$

it follows that $h^0(-K_S) + \min\{d, 3\} = 8$.

LEMMA 3. *In the previous notation let S be a rational surface and let f be a generic fibre of p. Then $h^0(-K_S - f - \sigma_\infty) \geqslant h^0(-K_S) + \min\{d, 3\} - 5$.*

PROOF. We have two exact sequences of sheaves

1) $0 \to \mathcal{O}_S(-K_S - \sigma_\infty) \to \mathcal{O}_S(-K_S) \to \mathcal{O}_{\sigma_\infty}(-K_S) \to 0$,

2) $0 \to \mathcal{O}_S(-K_S - f - \sigma_\infty) \to \mathcal{O}_S(-K_S - \sigma_\infty) \to \mathcal{O}_f(-K_S - \sigma_\infty) \to 0$.

By the genus formula $-K_S \cdot \sigma_\infty = 2 + \sigma_\infty^2 = 2 - d$, thus $h^0(\mathcal{O}_{\sigma_\infty}(-K_S)) = 3 - \min\{d, 3\}$.

The proof follows by considering cohomology exact sequences associated to 1) and 2). ■

PROOF OF THEOREM 1. If $S = \mathbb{F}_d$ we already know that σ_∞ is the only curve with negative self intersection, so we can assume that p has a degenerate fibre f_0.

If A is the irreducible component of f_0 which intersects σ_∞ then we have an exact sequence

$$0 \to \mathcal{O}_S(-K_S - f - \sigma_\infty - A) \to \mathcal{O}_S(-K_S - \sigma_\infty - f) \to \mathcal{O}_A(-K_S - \sigma_\infty - f) \to 0.$$

By the genus formula $(-K_S - f - \sigma_\infty) \cdot A = 2 + A^2 - 1 \leqslant 0$ and by Lemma 3 $h^0(-K_S - f - \sigma_\infty - A) \geqslant 2$. Let $C \subset S$ be a transversal curve different from σ_∞ with $C^2 \leqslant -2$; for every $D \in |-K_S - f - \sigma_\infty - A|$ we have

$$D \cdot C \leqslant 2 + C^2 - f \cdot C - \sigma_\infty \cdot C - A \cdot C < 0$$

thus $D = C + E$ for some effective divisor E.

Moreover $E \cdot f = E \cdot \sigma_\infty = 0$, in fact, by genus formula $D \cdot f = 1$, $D \cdot \sigma_\infty = 0$ and by hypothesis $C \cdot f > 0$, $C \neq \sigma_\infty$. E is contained in the exceptional locus of μ but yhis is not possible because $\dim |D| = \dim |E| \geqslant 1$. ■

REMARK 1. Looking at the proof of Theorem 1 we note that if there exist a degenerate fibre f_0 such that the irreducible component A

which intersects σ_∞ has self intersection $A^2 \leq -2$ then Theorem 1 holds under the less restrictive assumption $h^0(-K_S) + \min\{d, 3\} \geq 7$. We also note that the condition $A^2 \leq -2$ holds in particular if f_0 contains exactly one (-1)-curve.

REMARK 2. One can prove that Cor. 2 is still valid if we change the condition $h^0(-K_S) + \min\{d, 3\} \geq 9$ with $h^0(\theta_S) \geq 4$. We don't need this result so we don't prove it here.

LEMMA 4. *In the same notation of Lemma* 3, *if* $h^0(-K_S) + \min\{d, 3\} \geq 6$ *then there exists at most one transversal curve* $C \neq \sigma_\infty$ *with* $C^2 \leq -2$. *If such a curve exists then* $C \cdot f = 1$.

PROOF. By Lemma 3 $h^0(-K_S - f - \sigma_\infty) \geq 1$, consider $D \in | -K_S - f - \sigma_\infty|$. By the genus formula

$$D \cdot C \leq 2 + C^2 - C \cdot f - C \cdot \sigma_\infty < 0$$

thus $D = C + B$ where B is an effective divisor. We note that $B \cdot f = 0$ and thus C is the only component of D such that $C \cdot f = D \cdot f = 1$. ■

2. The weight of a rational surface.

Let $p: X \to B$ a holomorphic map from a surface X to a smooth curve B. We shall say that p is a rational fibration with section (r.f.w.s. for short) if:

1) The generic fibre of p is a smooth rational curve.
2) It's given a section $s: B \to X$.

Without loss of generality we can obviously assume that $B \subset X$ and s is the embedding of B in X.

DEFINITION 2. A r.f.w.s. $p: X \to B$ is minimal if every fibre contains no (-1)-curves disjoint from B.

PROPOSITION 5. *In a minimal r.f.w.s* $p: X \to B$ *every fibre is smooth rational.*

PROOF. The proof is essentially the same as Lemma III.8 of [Be]. ■

DEFINITION 3. The *weight* $w(S)$ of a rational surface $S \neq \mathbb{P}^2$ is the greatest integer n such that there exists a birational morphism $\mu: S \to \mathbb{F}_n$.

We note that $w(S) \leqslant h^1(\theta_{\mathbb{F}_{w(S)}}) + 1 \leqslant h^1(\theta_S) + 1$.

Let $\mathcal{C}$ be the set of irreducible curves $C \subset S$ such that there exists a smooth rational curve $f \subset S$ with $f^2 = 0$, $C \cdot f = 1$.

THEOREM 6. *In the notation above* $w(S) = \max\{-C^2 \mid C \in \mathcal{C}\}$.

PROOF. $\leqslant$ is trivial.

Conversely let $C \in \mathcal{C}$ such that $C^2 < 0$, we have to show that $-C^2 \leqslant \leqslant w(S)$. Let f be a smooth rational curve such that $f^2 = 0, f \cdot C = 1$, then it's very easy to prove that the linear system $|f|$ is a base point free pencil. The associated morphism $p: S \to \mathbb{P}^1$ is a rational fibration with section C.

The conclusion follows from Proposition 5 by considering the surface S' obtained by contracting all (-1)-curves contained in the degenerate fibres of p which are disjoint from C. ■

3. Normal projective surfaces with $\rho = 1$, $P_{-1} \geqslant 5$.

We first observe that in this case, since X is normal projective, $P_n(X) = 0$ for every $n > 0$.

LEMMA 7 (Sakai). *Let X be a normal projective surface with $\rho(X) = 1$, $P_n(X) = 0$ for every $n > 0$. Then $q(X) = 0$.*

PROOF. A proof of this lemma follows from the results of [Sa1] § 4, for the reader's convenience we write here a direct proof. Let $\delta: Y \to X$ be the minimal resolution of X; since for every integer n the sheaf $\mathcal{O}_X(nK_X)$ is reflexive we have $P_n(Y) \leqslant P_n(X)$. In particular all the positive plurigenus of Y vanish and, by Enriques criterion, Y is a ruled surface.

By Serre duality $H^2(\mathcal{O}_X) = 0$ and by the Leray spectral sequence we get $q(Y) = q(X) + h(X)$ where, by definition, $h(X) = h^0(R^1\delta_*\mathcal{O}_Y)$. Let's assume $h(X) < q(Y)$ and let $p: Y \to B$ be the canonical ruled fibration onto a smooth curve B of genus $g = q(Y)$.

If D is an irreducible component of the exceptional divisor of δ then, by a general result (cf. [B-P-V], p. 74), $g(D) \leqslant h(X)$ and thus p is constant on D. We can thus factorize p to a ruled fibration $p': X \to B$, but this is impossible by the assumpion $\rho(X) = 1$. ■

THEOREM 8 (Badescu). *Let X be a normal projective surface such that $q(X) = P_n(X) = 0$ for every $n > 0$ and let $\delta: Y \to X$ be its minimal resolution. Then either*

1) *The singularities of X are rational and Y is a rational surface, or*

2) *Y is a ruled surface of irregularity $q > 0$, X has precisely one non-rational singularity x of geometric genus q, the fibre of δ over x is composed by a section of the canonical ruled fibration $p: Y \to B$ and (possibly) by components of the degenerate fibres of p, the fibre of δ over a rational singularity of X is contained in a degenerate fibre of p.* ■

PROOF ([Ba], Th. 2.3). ■

Our goal is to give a structure theorem for surfaces X belonging to class 1) of Theorem 8 under the more restrictive assumption that $\rho(X) = 1$, $P_{-1}(X) \geq 5$.

DEFINITION 4. A normal projective surface $X \neq \mathbb{P}^2$ belongs to class (A) if:

$A1)$ $\rho(X) = 1$, $P_n(X) = 0$ $\forall n \geq 1$ and X has at most rational singularities,

$A2)$ If $\delta: S \to X$ is the minimal resolution then S is a rational surface of weight $d \geq 2$.

$A3)$ There exists a birational morphism $\mu: S \to \mathbb{F}_d$ such that the irreducible curves contracted by δ are exactly σ_∞ and the components with self intersection ≤ -2 of degenerate fibres of $p = \pi \circ \mu: S \to \to \mathbb{P}^1$.

Let's denote, for every normal projective surface X with minimal resolution $\delta: Y \to X$, by $s(X)$ the number of singular points of X and by $b(X) = \max_{x \in X} \{b_2(\delta^{-1}(x))\}$.

PROPOSITION 9. *If X belongs to class (A) then:*

1) *$s(X) \leq b(X)$.*

2) *X has at most one non cyclic singularity.*

3) *If every singularity of X is cyclic then $s(X) \leq 3$.*

PROOF. Let $D \subset S$ be the exceptional divisor of δ, since the singularities of X are rational $\rho(S) = 1 + b_2(D)$, this forces every degenerate fibre of p to contain exactly one (-1)-curve, in fact by easy considerations about ρ we have, in the notation $(*)$ of Section 1, $r + h =$

$= b_2(\overline{D \setminus \sigma_\infty}) + e$ and then $e = h$. In particular the components of degenerate fibres which intersect σ_∞ belong to D.

It's easy to see that if f_0 is a degenerate fibre, $E \subset f_0$ the (-1)-curve and $A \subset f_0$ the component intersecting σ_∞ then $\overline{f_0 \setminus E}$ has at most two connected component and the possible component that doesn't contain A is a string.

Thus it holds $s(X) \leq h + 1 \leq b(X)$ and, if (X, x) is a noncyclic singularity, then $\delta^{-1}(x)$ must be the connected component D' of D which contains σ_∞. This prove 1) and 2).

3) follows from the fact that D' is a string iff $h \leq 2$. ∎

The main result that we prove is the following:

THEOREM 10. *Let X be a normal projective surface with $\rho(X) = 1$, $P_{-1}(X) \geq 5$ with at most rational singularities. Then X belong to class (A).*

PROOF. Let $\delta: S \to X$ be the minimal resolution and let $D \subset S$ be the exceptional curve of δ. S is a rational surface of weight $d \geq 1$ and, according to (3.9.2.) $P_{-1}(S) = P_{-1}(X) \geq 5$.

We first note that, by Lemma 4, for every $\mu: S \to \mathbb{F}_d$ there exists at most one transversal curve $C \subset D$ different from σ_∞ and then $e \leq h + 1$.

We first show by contradiction that $d \geq 2$. In fact if we assume $d = 1$ and $\mu: S \to \mathbb{F}_d$ is a birational morphism then $\rho(S) = 1 + b_2(D)$ and there exists a transversal curve $C \subset D$, $C \neq \sigma_\infty$ with $C^2 \leq -2$. By Lemma 4 $C \cdot f = 1$ and by Theorem 6, $d \geq -C^2 \geq 2$.

If $P_{-1}(S) + \min\{d, 3\} \geq 8$ then for every birational morphism $\mu: S \to \mathbb{F}_d$ the curves on S with self intersection ≤ -2 are σ_∞ and some components of degenerate fibres. In this case the conclusion follows from easy considerations about the Picard number of S. This proves the theorem if $d \geq 3$ or $P_{-1} \geq 6$. It remain to consider the case $d = 2$, $P_{-1}(S) = 5$. If, for some $\mu: S \to \mathbb{F}_d$ S contains a degenerate fibre f_0 such that $A^2 \leq -2$ where $A \subset f_0$ is the irreducible component which intersects σ_∞ then the proof follows by Remark 1.

The remaining case is the following: $d = 2$, $P_{-1}(S) = 5$, for every birational morphism $\mu: S \to \mathbb{F}_2$ the composite fibration $p = \pi \circ \mu$ has only one degenerate fibre f_0 and $A^2 = -1$ where $A \subset f_0$ is the component which intersects σ_∞. We prove that this case doesn't occur.

Let $\mu: S \to \mathbb{F}_2$ be a fixed morphism and write μ as a composition of blowings-up

$$S = S_r \xrightarrow{\mu_r} S_{r-1} \to \dots S_2 \xrightarrow{\mu_2} S_1 \xrightarrow{\mu_1} S_0 = \mathbb{F}_2 .$$

We note that $P_{-1}(S) = P_{-1}(\mathbb{F}_2) - 4$ thus $r \geq 4$. Let $p_i \in S_{i-1}$ be the base point of the blow up μ_i. p_i is exactly the image of the critical set of composite map $S \to S_{i-1}$. If $i \leq j$ let $E_i \subset S_j$ be the strict transform of the exceptional curve of μ_i. We have $E_r^2 = -1$ and $E_i^2 \leq -2$ on S if $i < r$, in particular $p_i \in E_{i-1} \setminus A$ $\forall i > 1$.

Let's consider the surface Y obtained by contracting the curve σ_∞ in $\mathbb{F}_2$. It is a well known fact that $Y \subset \mathbb{P}^3$ is the cone over a smooth conic in $\mathbb{P}^2$.

We can consider the point $p_2 \in E_1 \setminus A$ as a tangent vector $v \in T_{p_1} Y$, let $\psi \colon Y - - - \to \mathbb{P}^1$ be the projection of center the projective line L generated by v. Observe that L does not contain the vertex of Y and then the generic fibre of ψ is a smooth hyperplane section of Y.

By elimination of indeterminacy we get a fibration $S_2 \to E_2$ which has $\sigma_\infty \cup A \cup E_1$ as unique degenerate fibre and then a fibration $\tau \colon S \to E_2$. The inclusion of E_2 in S gives a section for τ, in particular $E_2^2 \geq -w(S)$ which implies $E_2^2 = -2$.

By hypothesis τ has at most one degenerate fibre, then $p_3 \in E_1 \cap E_2$, in particular $E_2^2 = -2$ in S_3 and $p_4 \in E_3 \setminus E_2$ otherwise $E_2^2 < -2$ in S, therefore E_3 is the component of the degenerate fibre that intersects E_2 and $E_3^2 \leq -2$ contrary to the assumption. ∎

Remark 3. It's no difficult to construct a normal projective surface X with $\rho = 1$, $P_{-1} = 4$ and with three rational double points of type A_2, hence by Proposition 9 X doesn't belong to class A.

4. The case of normal degenerations of $\mathbb{P}^2$.

Let $X \subset \mathbb{P}^n$ be a normal projective surface with $q = p_g = 0$, $P_{-1} > 0$ with at most rational singularities.

Lemma 11. *In the notation above* $H^2(\theta_X) = H^2(\mathcal{O}_X) = H^1(\mathcal{O}_X(1)) = 0$.

Proof. The minimal resolution $\delta \colon S \to X$ is a rational surface, in particular $H^2(\mathcal{O}_S) = H^1(\mathcal{O}_S) = H^0(\Omega_S^1) = 0$. Let $C \subset X$ be a smooth hyperplane section, then $C \cdot K_X < 0$ and from exact cohomology sequence associated to

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to \mathcal{O}_C(1) \to 0$$

we get immediately $H^1(\mathcal{O}_X(1)) = 0$.

If $H^2(\theta_X)^\vee = \mathrm{Hom}(\theta_X, K_X) \neq 0$ then, since both θ and K are reflexive sheaves $\mathrm{Hom}(\theta_X, K_X) = \mathrm{Hom}(\theta_U, K_U)$ where $U \subset X$ is the open set of regular points. Moreover K_U is an invertible sheaf and the composition bilinear map

$$\mathrm{Hom}(\theta_U, K_U) \times \mathrm{Hom}(K_U, \mathcal{O}_U) \to \mathrm{Hom}(\theta_U, \mathcal{O}_U)$$

is nondegenerate, thus $\mathrm{Hom}(\theta_U, \mathcal{O}_U) \neq 0$. This is a contradiction since, according to ([Pi], p. 176) $\mathrm{Hom}(\theta_U, \mathcal{O}_U) = H^0(\Omega_U^1) = H^0(\Omega_S^1) = 0$. ∎

Let $x_1, \ldots, x_r$ be the singular points of X: then there exist by restriction the following natural morphisms of germs of analytic spaces

$$\mathrm{Hilb}_{\mathbb{P}^n}(X) \xrightarrow{\alpha} \mathrm{Def}_X \xrightarrow{\beta} \prod_{i=1}^{r} \mathrm{Def}_{(X, x_1)}$$

where $\mathrm{Hilb}_{\mathbb{P}^n}(X)$ is the Hilbert scheme of $\mathbb{P}^n$ at X, Def_X(resp.: $\mathrm{Def}_{(X, x_i)}$) is the base of the semiuniversal deformation of X (resp.: (X, x_i)).

By Lemma 11 and standard deformation theory, the morphisms α and β are smooth, in particular every deformation of the singularities of X can be globalized to an embedded deformation of X. We note that the dimension of the fibres of β is precisely $h^1(\theta_X)$.

Given smoothing of the singularities (X, x_i) they can be globalized to a global smoothing of X, since rational singularities are smoothable, then X is smoothable to a rational surface.

This applies in particular to surfaces with $\rho = 1$, $P_{-1} \geqslant 5$ with at most rational singularities, for these surfaces is not difficult to prove that if the singularities admits a $\mathbb{Q}$-Gorenstein smoothing then they are degenerations of Del Pezzo surfaces.

We don't kwow any normal projective surface with $\rho = 1$, $P_{-1} \geqslant 5$ with at most rational singularities such that $h^1(\theta) \neq 0$, we think that such a surface doesn't exist.

From now on we shall restrict for simplicity to normal projective degenerations of $\mathbb{P}^2$. Let $f: Y \to \Delta$ be a flat projective family of normal surfaces such that $Y_t \simeq \mathbb{P}^2$ for every $t \neq 0$.

In [Ma] is proved that $\rho(Y_0) = 1$, $P_{-1}(X) \geqslant 10$, $q(Y_0) = 0$, $h^0(\theta_{Y_0}) \geqslant 8$ and if the singularities are quotient then $h^1(\theta) = 0$.

Let's suppose that Y_0 has at most rational singularities and let $y_1, \ldots, y_s \in Y_0$ be its singular points. We note that f is a smoothing of each (Y_0, y_i). Denote by $D \subset \prod_{i=1}^{s} \mathrm{Def}_{(Y_0, y_i)}$ the product of smoothing components which contain f and write $H = \alpha^{-1}\beta^{-1}D$.

The projective plane is rigid, thus every smooth surface corresponding to a point of H is isomorphic to $\mathbb{P}^2$. In particular for every $k \leqslant s$ if Y_0^k is the surface obtained from Y_0 by smoothing only the singularities (Y_0, y_i) for $i = 1, ..., k$ then Y_0^k is a normal projective degeneration of $\mathbb{P}^2$.

COROLLARY 12. *Let Y_0 be a normal projective degeneration of $\mathbb{P}^2$ with at most rational singularities, then Y_0 belong to class (A) and has at most four singular points.*

PROOF. The proof follows from the previous results by considering the surface obtained from Y_0 by smoothing only the possible noncyclic singular point. ■

For a deeper study of normal degenerations of $\mathbb{P}^2$ see [Ma].

REMARK 4. If Y_0 has a nonrational singularity then in general $H^2(\theta_{Y_0}) \neq 0$ and the above construction doesn't work.

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