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$T_3$-systems of finite simple groups

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1. Introduction.

We present here some further evidence in support of the following conjecture, first formulated by Wiegold in the seventies.

**Conjecture.** Every finite non-abelian simple group has exactly one \( T_3 \)-system.

Gilman [5] has shown that the conjecture holds for the simple groups \( PSL(2, p) \) with \( p \) prime, (indeed it was this result that prompted the conjecture), while Evans [4] has done it for certain Suzuki groups. In both cases the action of the automorphism group on the \( G \)-defining subgroups is alternating or symmetric, and this too seems likely to reflect a general truth.

The Suzuki groups and the \( PSL(2, p) \) are easier to cope with than the alternating groups, no doubt because of the much greater diversity of subgroups in alternating groups. Since \( A_5 = PSL(2, 5) \), Gilman's result provides the answer, while \( A_6 \) is so small that a simple calculation is sufficient. The aim of this note is to sketch a proof of the following result.

**Theorem.** The alternating group \( A_7 \) has just one \( T_3 \)-system, and the action of \( \text{Aut } F_3 \) on the \( A_7 \) defining subgroups is alternating or symmetric.

The methods are elementary throughout. I see no way of establishing the conjecture for the general alternating group \( A_n \).

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2. $T_3$-systems and a result of Evans.

Let $F_n$ be a free group of finite rank $n$, and let $G$ be any group. We say that $N$ is a $G$-defining subgroup of $F_n$ if $N \triangleleft F_n$ and $F_n/N = G$. Denote the set of all $G$-defining subgroups of $F_n$ by $\Sigma(G, n)$ and notice that $\Sigma(G, n)$ is not empty if and only if $G$ can be generated by $n$ elements.

For each $\sigma \in \text{Aut } F_n$ and $N \in \Sigma(G, n)$ we clearly have $F_n/N \sigma = G$ so that $N \sigma \in \Sigma(G, n)$. In this way we obtain an action of $\text{Aut } F_n$ on $\Sigma(G, n)$, the orbits of which are called the $T_n$-systems of $G$. ([5] and [3]).

When we investigate $T$-systems of a specific group $G$, it is found to be rather difficult to work directly with the action of $\text{Aut } F_n$ on $\Sigma(G, n)$. B. H. Neumann and H. Neumann [7] introduced the notion of generating $G$-vectors which enabled them to define an equivalent action of $\text{Aut } F_n$ which is more manageable. The details with respect to $T_3$ are now given following the argument indicated in [4].

Let $G$ be a 3-generator group. A generating $G$-vector of length 3 is defined to be an ordered triple $(g_1, g_2, g_3)$ where $(g_1, g_2, g_3) = G$. The set of all generating $G$-vectors of length 3 is denoted by $V(G, 3)$.

Fix a set of free generators $x_1, x_2, x_3$ for $F_3$ and let $E$ be the set of epimorphisms from $F_3$ to $G$. Define an action of $\text{Aut } F_3 \times \text{Aut } G$ on $E$ by

\[
\rho(\sigma, \alpha) = \sigma^{-1} \rho \alpha
\]

where $\rho \in E$ and $(\sigma, \alpha) \in \text{Aut } F_3 \times \text{Aut } G$.

We can identify $\text{Aut } F_3$ and $\text{Aut } G$ with their copies in $\text{Aut } F_3 \times \text{Aut } G$ and speak of the action of $\text{Aut } F_3$ or $\text{Aut } G$ on $E$. We clearly have

\[
(2.2) \quad \rho_1 \text{ and } \rho_2 \text{ lie in the same } \text{Aut } G\text{-orbit of } E \text{ if and only if } \ker \rho_1 = \ker \rho_2.
\]

Suppose that $\ker \rho = N$. Then $\ker \rho \alpha = N$ too, and so we can associate $N$ with the $\text{Aut } G$-orbit of $E$ that contains $\rho$, viz. $\{\rho \alpha : \alpha \in \text{Aut } G\}$. Notice that for all $\sigma \in \text{Aut } F_3$ we have $\ker (\rho(\sigma, 1)) = \ker (\sigma^{-1} \rho) = N \sigma$. Hence $N \sigma$ is associated with the $\text{Aut } G$-orbit of $E$ containing $\rho(\sigma, 1)$. Moreover, $N \in \Sigma(G, 3)$ if and only if $N = \ker \rho$ for some $\rho \in E$. Therefore

\[
(2.3) \quad \text{The action of } \text{Aut } F_3 \text{ on } \Sigma(G, 3) \text{ is equivalent to its action on the } \text{Aut } G\text{-orbits of } E.
\]
The map \( \pi: E \to V(G, 3) \) given by
\[
\rho \sigma = (x_1 \rho, x_2 \rho, x_3 \rho)
\]
is a bijection. Furthermore, \( \pi \) enables us to carry over the action of \( \text{Aut} F_3 \times \text{Aut} G \) on \( E \) to an action on \( V(G, 3) \).

This is given by
\[
\rho \pi = \pi^{-1} \rho \pi.
\]

The action of \( \text{Aut} F_3 \times \text{Aut} G \) on \( V(G, 3) \) given by (2.5) is equivalent to its action on \( E \). Therefore the action of \( \text{Aut} F_3 \) on the \( \text{Aut} G \)-orbits of \( V(G, 3) \) is equivalent to its action on the \( \text{Aut} G \)-orbits of \( E \). Combining this last remark with (2.3) gives the following fundamental result.

(2.6) The action of \( \text{Aut} F_3 \) on the \( \text{Aut} G \)-orbits of \( V(G, 3) \) is equivalent to its action on \( \Sigma(G, 3) \).

Let us now examine in greater detail the actions of \( \text{Aut} F_3 \) and \( \text{Aut} G \) on \( V(G, 3) \). Here we again identify \( \text{Aut} F_3 \) and \( \text{Aut} G \) with their copies in \( \text{Aut} F_3 \times \text{Aut} G \).

Suppose throughout that \( (g_1, g_2, g_3) \) is a typical element of \( V(G, 3) \). By (2.4) there exists \( \rho \in E \) with \( (g_1, g_2, g_3) \rho = (x_1 \rho, x_2 \rho, x_3 \rho) \). The action of \( \text{Aut} G \) on \( V(G, 3) \) is now easily given explicitly; by (2.5) we have \( (g_1, g_2, g_3) (1, \alpha) = \rho \pi (1, \alpha) = (x_1 \rho \alpha, x_2 \rho \alpha, x_3 \rho \alpha) = (g_1 \alpha, g_2 \alpha, g_3 \alpha) \). Moreover since \( (g_1, g_2, g_3) \) is a representative of \( G \) we have \( (g_1, g_2, g_3) = (g_1 \alpha, g_2 \alpha, g_3 \alpha) \) if and only if \( \alpha = 1 \). Hence

(2.7) The action of \( \text{Aut} G \) on \( V(G, 3) \) is given by \( \alpha: (g_1, g_2, g_3) \to (g_1 \alpha, g_2 \alpha, g_3 \alpha) \) for all \( \alpha \in \text{Aut} G \) and all \( (g_1, g_2, g_3) \in V(G, 3) \).

We next consider the action of \( \text{Aut} F_3 \) on \( V(G, 3) \). For all \( \sigma \in \text{Aut} F_3 \) we have \( (g_1, g_2, g_3) \sigma = \rho \pi (\sigma, 1) = \pi^{-1} \rho \pi = (x_1 \sigma \rho, x_2 \sigma \rho, x_3 \sigma \rho) \) from (2.5). Suppose that
\[
\begin{align*}
x_1 \sigma^{-1} &= w_1(x_1, x_2, x_3), \\
x_2 \sigma^{-1} &= w_2(x_1, x_2, x_3), \\
x_3 \sigma^{-1} &= w_3(x_1, x_2, x_3),
\end{align*}
\]
where \( w_1(x_1, x_2, x_3) \) is a word in \( (x_1, x_2, x_3) \). Now
\[
(x_1 \sigma^{-1} \rho, x_2 \sigma^{-1} \rho, x_3 \sigma^{-1} \rho) = (w_1 \rho, w_2 \rho, w_3 \rho) =
\]
\[
= (w_1(g_1, g_2, g_3), w_2(g_1, g_2, g_3), w_3(g_1, g_2, g_3))
\]
where \( \sigma \in \text{Aut} F_3 \) and \( w_1, w_2, w_3 \) are given by (2.8). Therefore

\[
(2.9) \quad \text{The action of Aut} F_3 \text{ on } V(G, 3) \text{ is given by}
\]

\[\sigma(g_1, g_2, g_3) \rightarrow (w_1(g_1, g_2, g_3), w_2(g_1, g_2, g_3), w_3(g_1, g_2, g_3))\]

where \( \sigma \in \text{Aut} F_3 \) and \( w_1, w_2, w_3 \) are given by (2.8).

We continue, using the following result, a convenient reference for which is [6] Chapter 3.

\[
(2.10) \quad \text{Aut} F_3 \text{ is generated by the automorphisms given below, where } 1 \leq i, k \leq 3, \ i \neq k \text{ and unmentioned generators of } F_3 \text{ are fixed.}
\]

\[P(i, k): x_i \rightarrow x_k, \quad x_k \rightarrow x_i,\]

\[\sigma(i): x_i \rightarrow x_i^{-1},\]

\[L(i, k): x_i \rightarrow x_k x_i,\]

\[R(i, k): x_i \rightarrow x_i x_k.\]

These are called the elementary automorphisms of \( F_3 \). Their effect on \((g_1, g_2, g_3) \in V(G, 3)\) is to interchange any two entries, invert any entry or multiply any entry by any other on the left or right. This is seen with the aid of (2.9).

As \( \text{Aut} F_3 \) is generated by elementary automorphisms, the above remark has an important consequence, namely

\[
(2.11) \quad \text{Two elements of } V(G, 3) \text{ lie in the same } \text{Aut} F_3\text{-orbit if and only if one can be transformed into the other by a finite sequence of the following operations:}
\]

<table>
<thead>
<tr>
<th>Operation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interchanging two entries:</td>
<td>((g_1, g_2, g_3) \rightarrow (g_1, g_2, g_3)).</td>
</tr>
<tr>
<td>Inverting an entry:</td>
<td>((g_1, g_2, g_3) \rightarrow (g_1^{-1}, g_2, g_3)).</td>
</tr>
<tr>
<td>Multiplying one entry on the left by another:</td>
<td>((g_1, g_2, g_3) \rightarrow (g_2 g_1, g_2, g_3)).</td>
</tr>
<tr>
<td>Multiplying one entry on the right by another:</td>
<td>((g_1, g_2, g_3) \rightarrow (g_1 g_2, g_2, g_3)).</td>
</tr>
</tbody>
</table>
We say that two elements of \( V(G, 3) \) are equivalent if they lie in the same \( \text{Aut} F_3 \)-orbit.

An important property of \( A_7 \) in our context is that it has spread 2 in the sense of Brenner and Wiegold ([1] and [2]). This means that for any pair \( x, y \) of non-trivial elements of \( A_7 \), there is a third element \( z \) such that \( \langle x, z \rangle = \langle y, z \rangle = A_7 \). The connection with \( T_3 \)-systems is the following simple but important result of Evans [4].

(2.12) Let \( G \) be any group of spread 2. Then all redundant generating triple are equivalent.

A redundant generating triple \((g_1, g_2, g_3)\) is one where one of \( g_1, g_2, g_3 \) can be omitted and the remaining two elements still generate the group. Thus our strategy will be to show that every generating triple for \( A_7 \) is equivalent to a redundant triple.

3. \( T_3 \)-systems of \( A_7 \).

The 2520 elements of \( A_7 \) are classified into distinct types of permutations. We shall use the representation of these permutations as products of disjoint cycles, omitting cycles of length one. If an element is a product of disjoint cycles of lengths \( r_1, r_2, ..., r_k \) where \( r_1 > 1 \) the we say it is of type \( r_1, r_2, ..., r_k \). The table below gives the number of elements of each type in \( A_7 \) and also in each of the maximal subgroups of \( A_7 \) which are isomorphic to \( \text{PSL}(2, 7) \).

<table>
<thead>
<tr>
<th>Type</th>
<th>7</th>
<th>5</th>
<th>4,2</th>
<th>3,3</th>
<th>3,2,2</th>
<th>3</th>
<th>2,2</th>
<th>Ident.</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_7 )</td>
<td>720</td>
<td>504</td>
<td>630</td>
<td>280</td>
<td>210</td>
<td>70</td>
<td>105</td>
<td>1</td>
<td>2520</td>
</tr>
<tr>
<td>( \text{PSL}(2, 7) )</td>
<td>48</td>
<td>0</td>
<td>42</td>
<td>56</td>
<td>0</td>
<td>0</td>
<td>21</td>
<td>1</td>
<td>168</td>
</tr>
</tbody>
</table>

There are 15 maximal subgroups of \( A_7 \) which are isomorphic to \( \text{PSL}(2, 7) \). Each element of type 7 of \( A_7 \) is in one and only one of these maximal subgroups. This property is also true for each element of type 4, 2 of \( A_7 \).

In order to show that every generating \( G \)-vector \((g_1, g_2, g_3)\), is equivalent to a redundant vector we systematically look at all possible cases.
CASE 1. If one of the elements of the triple is of type 7, say $g_1$ then as we remarked above, it is one and only one of the $PSL(2, 7)$ contained in $A_7$; call this group $B$.

If $g_2 \in B$ then $\langle g_1, g_2 \rangle \subset B$ while if $g_2 \notin B$ then $\langle g_1, g_2 \rangle = A_7$ as $B$ is a maximal subgroup. The same holds for $g_3$.

As $(g_1, g_2, g_3)$ is a generating set for $A_7$, one of $g_2, g_3$ is not an element of $B$ and will generate $A_7$ with $g_1$. Thus any generating triple containing an element of type 7 is equivalent to a redundant triple.

CASE 2. Suppose that $g_1$ is of type 5, without loss of generality, $(12345)$ say. If $\langle g_1, g_2 \rangle$ is transitive over the set $\{1, 2, 3, 4, 5, 6, 7\}$ then $\langle g_1, g_2 \rangle = A_7$.

So we look at the cases when $\langle g_1, g_2 \rangle$ and $\langle g_1, g_3 \rangle$ are non transitive but of course, $g_2$ and $g_3$ between them must move 6 and 7. We need to consider two cases.

i) $g_1 = (12345), g_2 = (\ldots)(67), g_3 = (\ldots 6)(\ldots)(7)$. Then $6g_3 = i$ with $i \neq 6$ and $i \neq 7$ and $7g_3 = 7$ so $6g_2g_3 = 7$ and $7g_2g_3 = i$.

This means that $g_2g_3 = (\ldots 67i \ldots)(\ldots)$ and hence $\langle g_1, g_2g_3 \rangle$ is transitive and so must be $A_7$.

ii) $g_1 = (12345)$, and let $g_2$ move 6 but not 7 and $g_3$ move 7 but not 6.

Then $g_2g_3$ will move 6 and 7 and then $\langle g_1, g_2g_3 \rangle$ is again transitive and so is $A_7$.

Thus if the generating triple contains an element of type 5 it is equivalent to a redundant triple.

The further cases, with $g_1, g_2$ and $g_3$ taking all possible types, are shown in the following table, which indicates the length of the calculation required.

We investigate the cases 3, 4, 5, 6 and 7, using the following consideration.

i) There is a need for transitivity over $\{1, 2, 3, 4, 5, 6, 7\}$.

ii) Any triple equivalent to a triple with an element of type 7 or of type 5 is no problem.

iii) Two elements generating a transitive subgroup of $A_7$, in which one is of type 3 will generate $A_7$ ([7], p. 34).

iv) Two elements generating a transitive subgroup of $A_7$ and each of type 4,2 in different $PSL(2, 7)$ subgroups will generated $A_7$. 
The investigation leads to the conclusion that if the generating triple contains an element of type 4, 2 it is equivalent to a redundant triple.

<table>
<thead>
<tr>
<th>Case</th>
<th>$g_1$ type</th>
<th>$g_2$ type</th>
<th>$g_3$ type</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4, 2</td>
<td>4, 2</td>
<td>4, 2 or 3 or 3, 3 or 3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>4</td>
<td>4, 2</td>
<td>3</td>
<td>3 or 3, 3 or 3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>5</td>
<td>4, 2</td>
<td>3, 3</td>
<td>3, 3 or 3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>6</td>
<td>4, 2</td>
<td>3, 2, 2</td>
<td>3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>7</td>
<td>4, 2</td>
<td>2</td>
<td>2, 2</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>3</td>
<td>3 or 3, 3 or 3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>3, 3</td>
<td>3, 3 or 3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>3, 2, 2</td>
<td>3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>2</td>
<td>2, 2</td>
</tr>
<tr>
<td>12</td>
<td>3, 3</td>
<td>3</td>
<td>3, 3 or 3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>13</td>
<td>3, 3</td>
<td>3, 2, 2</td>
<td>3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>14</td>
<td>3, 3</td>
<td>2</td>
<td>2, 2</td>
</tr>
<tr>
<td>15</td>
<td>3, 2, 2</td>
<td>3, 2, 2</td>
<td>3, 2, 2 or 2, 2</td>
</tr>
<tr>
<td>16</td>
<td>3, 2, 2</td>
<td>2</td>
<td>2, 2</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>2</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

We provide here a proof of some of Case 3 to demonstrate the methods used. The complete proofs of the assertions made here involve a great deal of simple but tedious calculation.

**CASE 3.** Let $g_1$, $g_2$ and $g_3$ be each of type 4, 2 and each in a different $\text{PSL}(2, 7)$-subgroup of $A_7$. As an example we consider the following case.

\[
g_1 = (3567)(12) \in \langle (1234567), (23)(47) \rangle,
\]

\[
f_2 \in \langle (2314567), (13)(47) \rangle,
\]

\[
g_3 \in \langle (2431567), (43)(17) \rangle.
\]

If $\langle g_1, g_2 \rangle$ is transitive over \{1, 2, 3, 4, 5, 6, 7\} there is no problem. We also find for the remaining elements $g_2$ that $g_1 g_2$ or $g_1 g_2^{-1}$ or $g_1 g_2^2$ is of type 7 or type 5 except for $g_2 = (2537)(16)$ or (1567)(23) and their inverses.

If $\langle g_1, g_3 \rangle$ is transitive over \{1, 2, 3, 4, 5, 6, 7\} there is no problem. We also find for the remaining elements $g_3$ that $g_1 g_3$ or $g_1 g_3^{-1}$ is of type 7
or type 5 except for $g_3 = (3657)(12)$ or $(3567)(14)$ or $(3576)(24)$ and their inverses.

For these elements or their inverses, $g_2 g_3$ or $g_2 g_3^{-1}$ is of type 7 or type 5.

We see that for the selected $g_1$ and the subgroups concerned, all the triples are equivalent to redundant triples. This is found to be true whichever of the 14 maximal subgroups are chosen to contain elements $g_2$ and $g_3$. Thus any generating triple containing three elements of type 4, 2 each in a different $PSL(2, 7)$ maximal subgroup is equivalent to a redundant triple.

We now consider the case with $g_1, g_2$ each of type 4, 2 and each in a different $PSL(2, 7)$-subgroup of $A_7$ with $g_3$ any element of type 3. We consider the following case.

$$g_3 = \text{any element of type 3 in } A_7,$$

When we consider the products of $g_1 g_2$ and $g_1 g_3$ we find problems only occur when $g_2 = (2537)(16)$ or $(1567)(23)$ and $g_3 = (124)$ or $(345)$ or $(346)$ or $(347)$ or $(456)$ or $(457)$.

For these elements we find that either an equivalent triple can be obtained with one element, a product of $g_1, g_2$ and $g_3$, which is of type 7 or of type 5, or the triple is not a generating triple.

We again see that for the selected $g_1$ and the subgroups concerned, all the triples are equivalent to redundant triples. This is found to be true whichever of the maximal subgroups are chosen to contain element $g_2$. Thus any generating triple containing two elements of type 4, 2 each in a different $PSL(2, 7)$ maximal subgroup with the third element of type 3 is equivalent to a redundant triple.

Case 3, when completed, and then cases 4, 5, 6 and 7 all lead to the same conclusion that the generating triples concerned are all equivalent to a redundant triple.

The information obtained from cases 1 to 7 is used in the other cases in the order as shown in the table and with each case leading to a redundant triple.

The final conclusion is that all the generating $G$-vectors are equivalent to redundant vectors and consequently $A_7$ has only one $T_3$-system.

A further result of Evans [4] can now be used to complete the proof.
Let $G$ be a nonabelian finite simple group with $d(G) = k$. Suppose that $G = \langle g_1, g_2, \ldots, g_k \rangle$ where $g_k^2 = 1$. Then $\text{Aut} F_{k+1}$ acts as a symmetric or alternating group on at least one of its orbits on $\Sigma(G, k+1)$.

The alternating group $A_7$ may be generated by $\langle g_1, g_2 \rangle$ where $g_1$ is an element of type 7 and $g_2$ is an element of type 2, 2 which is not in the $\text{PSL}(2, 7)$ maximal sub-group containing $g_1$. For example we have $A_7 = \langle (1234567), (12)(45) \rangle$. We conclude that the action of $\text{Aut} F_3$ on the $A_7$ defining subgroups is alternating or symmetric.

REFERENCES


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