RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

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**$T_3$-systems of finite simple groups**

*Rendiconti del Seminario Matematico della Università di Padova*, tome 89 (1993), p. 19-27
<http://www.numdam.org/item?id=RSMUP_1993__89__19_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 89 (1993)

# $T_3$-Systems of Finite Simple Groups.

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## 1. Introduction.

We present here some further evidence in support of the following conjecture, first formulated by Wiegold in the seventies.

CONJECTURE. Every finite non-abelian simple group has exactly one $T_3$-system.

Gilman [5] has shown that the conjecture holds for the simple groups $PSL(2, p)$ with $p$ prime, (indeed it was this result that prompted the conjecture), while Evans [4] has done it for certain Suzuki groups. In both cases the action of the automorphism group on the $G$-defining subgroups is alternating or symmetric, and this too seems likely to reflect a general truth.

The Suzuki groups and the $PSL(2, p)$ are easier to cope with than the alternating groups, no doubt because of the much greater diversity of subgroups in alternating groups. Since $A_5 \simeq PSL(2, 5)$, Gilman's result provides the answer, while $A_6$ is so small that a simple calculation is sufficient. The aim of this note is to sketch a proof of the following result.

THEOREM. The alternating group $A_7$ has just one $T_3$-system, and the action of $\mathrm{Aut}\, F_3$ on the $A_7$ defining subgroups is alternating or symmetric.

The methods are elementary throughout. I see no way of establishing the conjecture for the general alternating group $A_n$.

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## 2. $T_3$-systems and a result of Evans.

Let $F_n$ be a free group of finite rank $n$, and let $G$ be any group. We say that $N$ is a $G$-defining subgroup of $F_n$ if $N \lhd F_n$ and $F_n/N \simeq G$. Denote the set of all $G$-defining subgroups of $F_n$ by $\Sigma(G, n)$ and notice that $\Sigma(G, n)$ is not empty if and only if $G$ can be generated by $n$ elements.

For each $\sigma \in \operatorname{Aut} F_n$ and $N \in \Sigma(G, n)$ we clearly have $F_n/N\sigma \simeq G$ so that $N\sigma \in \Sigma(G, n)$. In this way we obtain an action of $\operatorname{Aut} F_n$ on $\Sigma(G, n)$, the orbits of which are called the $T_n$-systems of $G$. ([5] and [3]).

When we investigate $T$-systems of a specific group $G$, it is found to be rather difficult to work directly with the action of $\operatorname{Aut} F_n$ on $\Sigma(G, n)$. B. H. Neumann and H. Neumann [7] introduced the notion of generating $G$-vectors which enabled them to define an equivalent action of $\operatorname{Aut} F_n$ which is more manageable. The details with respect to $T_3$ are now given following the argument indicated in [4].

Let $G$ be a 3-generator group. A generating $G$-vector of length 3 is defined to be an ordered triple $(g_1, g_2, g_3)$ where $\langle g_1, g_2, g_3\rangle = G$. The set of all generating $G$-vectors of length 3 is denoted by $V(G, 3)$.

Fix a set of free generators $x_1, x_2, x_3$ for $F_3$ and let $E$ be the set of epimorphisms from $F_3$ to $G$. Define an action of $\operatorname{Aut} F_3 \times \operatorname{Aut} G$ on $E$ by

$$\rho(\sigma, \alpha) = \sigma^{-1}\rho\alpha \tag{2.1}$$

where $\rho \in E$ and $(\sigma, \alpha) \in \operatorname{Aut} F_3 \times \operatorname{Aut} G$.

We can identify $\operatorname{Aut} F_3$ and $\operatorname{Aut} G$ with their copies in $\operatorname{Aut} F_3 \times \operatorname{Aut} G$ and speak of the action of $\operatorname{Aut} F_3$ or $\operatorname{Aut} G$ on $E$. We clearly have

(2.2) *$\rho_1$ and $\rho_2$ lie in the same* $\operatorname{Aut} G$*-orbit of $E$ if and only if* $\ker\rho_1 = \ker\rho_2$.

Suppose that $\ker\rho = N$. Then $\ker\rho\alpha = N$ too, and so we can associate $N$ with the $\operatorname{Aut} G$-orbit of $E$ that contains $\rho$, *viz.* $\{\rho\alpha\colon \alpha \in \operatorname{Aut} G\}$. Notice that for all $\sigma \in \operatorname{Aut} F_3$ we have $\ker(\rho(\sigma, 1)) = \ker(\sigma^{-1}\rho) = N\sigma$. Hence $N\sigma$ is associated with the $\operatorname{Aut} G$-orbit of $E$ containing $\rho(\sigma, 1)$. Moreover, $N \in \Sigma(G, 3)$ if and only if $N = \ker\rho$ for some $\rho \in E$. Therefore

(2.3) *The action of* $\operatorname{Aut} F_3$ *on $\Sigma(G, 3)$ is equivalent to its action on the* $\operatorname{Aut} G$*-orbits of $E$.*

The map $\pi: E \to V(G,3)$ given by

(2.4) $\rho\pi = (x_1\rho, x_2\rho, x_{3\rho})$ *is a bijection. Furthermore,* $\pi$ *enables us to carry over the action of* $\mathrm{Aut}\,F_3 \times \mathrm{Aut}\,G$ *on* $E$ *to an action on* $V(G,3)$.

This is given by

(2.5) $$\rho\pi(\sigma, \alpha) = \sigma^{-1}\rho\alpha\pi\,.$$

The action of $\mathrm{Aut}\,F_3 \times \mathrm{Aut}\,G$ on $V(G,3)$ given by (2.5) is equivalent to its action on $E$. Therefore the action of $\mathrm{Aut}\,F_3$ on the $\mathrm{Aut}\,G$-orbits of $V(G,3)$ is equivalent to its action on the $\mathrm{Aut}\,G$-orbits of $E$. Combining this last remark with (2.3) gives the following fundamental result.

(2.6) *The action of* $\mathrm{Aut}\,F_3$ *on the* $\mathrm{Aut}\,G$*-orbits of* $V(G,3)$ *is equivalent to its action on* $\Sigma(G,3)$.

Let us now examine in greater detail the actions of $\mathrm{Aut}\,F_3$ and $\mathrm{Aut}\,G$ on $V(G,3)$. Here we again identify $\mathrm{Aut}\,F_3$ and Aut G with their copies in $\mathrm{Aut}\,F_3 \times \mathrm{Aut}\,G$.

Suppose throughout that $(g_1, g_2, g_3)$ is a typical element of $V(G,3)$. By (2.4) there exists $\rho \in E$ with $(g_1, g_2, g_3) = \rho\pi = (x_1\rho, x_2\rho, x_3\rho)$. The action of $\mathrm{Aut}\,G$ on $V(G,3)$ is now easily given explicitly; by (2.5) we have $(g_1, g_2, g_3)(1, \alpha) = \rho\pi(1, \alpha) = (x_1\rho\alpha, x_2\rho\alpha, x_3\rho\alpha) = (g_1\alpha, g_2\alpha, g_3\alpha)$. Moreover since $\langle g_1, g_2, g_3\rangle = G$ we have $(g_1, g_2, g_3) = (g_1\alpha, g_2\alpha, g_3\alpha)$. if and only if $\alpha = 1$. Hence

(2.7) *The action of* $\mathrm{Aut}\,G$ *on* $V(G,3)$ *is given by* $\alpha: (g_1, g_2, g_3) \to \to (g_1\alpha, g_2\alpha, g_3\alpha)$ *for all* $\alpha \in \mathrm{Aut}\,G$ *and all* $(g_1, g_2, g_3) \in V(G, 3)$.

We next consider the action of $\mathrm{Aut}\,F_3$ on $V(G,3)$. For all $\sigma \in \mathrm{Aut}\,F_3$ we have $(g_1, g_2, g_3)(\sigma, 1) = \rho\pi(\sigma, 1) = \sigma^{-1}\rho\pi = (x_1\sigma\rho, x_2\sigma\rho, x_3\sigma\rho)$ from (2.5). Suppose that

(2.8) $$\begin{cases} x_1\sigma^{-1} = w_1(x_1, x_2, x_3)\,, \\ x_2\sigma^{-1} = w_2(x_1, x_2, x_3)\,, \\ x_3\sigma^{-1} = w_3(x_1, x_2, x_3)\,, \end{cases}$$

where $w_1(x_1, x_2, x_3)$ is a word in $(x_1, x_2, x_3)$. Now

$$(x_1\sigma^{-1}\rho, x_2\sigma^{-1}\rho, x_3\sigma^{-1}\rho) = (w_1\rho, w_2\rho, w_3\rho) =$$

$$= (w_1(g_1, g_2, g_3), w_2(g_1, g_2, g_3), w_3(g_1, g_2, g_3))$$

where $\sigma \in \operatorname{Aut} F_3$ and $w_1, w_2, w_3$ are given by (2.8). Therefore

(2.9) *The action of* $\operatorname{Aut} F_3$ *on* $V(G,3)$ *is given by*

$$\sigma: (g_1, g_2, g_3) \to (w_1(g_1, g_2, g_3), w_2(g_1, g_2, g_3), w_3(g_1, g_2, g_3))$$

*where* $\sigma \in \operatorname{Aut} F_3$ *and* $w_1, w_2, w_3$ *are given by* (2.8).

We continue, using the following result, a convenient reference for which is [6] Chapter 3.

(2.10) $\operatorname{Aut} F_3$ *is generated by the automorphisms given below, where* $1 \leqslant i, k \leqslant 3$, $i \neq k$ *and unmentioned generators of* $F_3$ *are fixed.*

$$P(i, k): x_i \to x_k, \qquad x_k \to x_i\,,$$

$$\sigma(i): x_i \to x_i^{-1}\,,$$

$$L(i, k): x_i \to x_k x_i\,,$$

$$R(i, k): x_i \to x_i x_k\,.$$

These are called the elementary automorphisms of $F_3$. Their effect on $(g_1, g_2, g_3) \in V(G, 3)$ is to interchange any two entries, invert any entry or multiply any entry by any other on the left or right. This is seen with the aid of (2.9).

As $\operatorname{Aut} F_3$ is generated by elementary automorphisms, the above remark has an important consequence, namely

(2.11) *Two elements of* $V(G,3)$ *lie in the same* $\operatorname{Aut} F_3$*-orbit if and only if one can be transformed into the other by a finite sequence of the following operations:*

— *Interchanging two entries*:

e.g. $(g_1, g_2, g_3) \to (g_1, g_2, g_3)$.

— *Inverting an entry*:

e.g. $(g_1, g_2, g_3) \to (g_1^{-1}, g_2, g_3)$.

— *Multiplying one entry on the left by another*:

e.g. $(g_1, g_2, g_3) \to (g_2 g_1, g_2, g_3)$.

— *Multiplying one entry on the right by another*:

e.g. $(g_1, g_2, g_3) \to (g_1 g_2, g_2, g_3)$.

We say that two elements of $V(G,3)$ are equivalent if they lie in the same $\operatorname{Aut} F_3$-orbit.

An important property of $A_7$ in our context is that it has *spread* 2 in the sense of Brenner and Wiegold ([1] and [2]). This means that for any pair $x$, $y$ of non-trivial elements of $A_7$, there is a third element $z$ such that $\langle x, z\rangle = \langle y, z\rangle = A_7$. The connection with $T_3$-systems is the following simple but important result of Evans [4].

(2.12) *Let G be any group of spread 2. Then all redundant generating triple are equivalent.*

A redundant generating triple $(g_1, g_2, g_3)$ is one where one of $g_1, g_2, g_3$ can be omitted and the remaining two elements still generate the group. Thus our strategy will be to show that every generating triple for $A_7$ is equivalent to a redundant triple.

## 3. $T_3$-systems of $A_7$.

The 2520 elements of $A_7$ are classified into distinct types of permutations. We shall use the representation of these permutations as products of disjoint cycles, omitting cycles of length one. If an element is a product of disjoint cycles of lengths $r_1, r_2, \ldots, r_k$ where $r_1 > 1$ the we say it is of *type* $r_1, r_2, \ldots, r_k$. The table below gives the number of elements of each type in $A_7$ and also in each of the maximal subgroups of $A_7$ which are isomorphic to $PSL(2,7)$.

| Type | 7 | 5 | 4, 2 | 3, 3 | 3, 2, 2 | 3 | 2, 2 | Ident. | Total |
|---|---|---|---|---|---|---|---|---|---|
| $A_7$ | 720 | 504 | 630 | 280 | 210 | 70 | 105 | 1 | 2520 |
| $PSL(2,7)$ | 48 | 0 | 42 | 56 | 0 | 0 | 21 | 1 | 168 |

There are 15 maximal subgroups of $A_7$ which are isomorphic to $PSL(2,7)$. Each element of type 7 of $A_7$ is in one and only one of these maximal subgroups. This property is also true for each element of type 4, 2 of $A_7$.

In order to show that every generating $G$-vector $(g_1, g_2, g_3)$, is equivalent to a redundant vector we systematically look at all possible cases.

CASE 1. If one of the elements of the triple is of type 7, say $g_1$ then as we remarked above, it is one and only one of the $PSL(2,7)$ contained in $A_7$; call this group $B$.

If $g_2 \in B$ then $\langle g_1, g_2\rangle \subseteq B$ while if $g_2 \notin B$ then $\langle g_1, g_2) = A_7$ as $B$ is a maximal subgroup. The same holds for $g_3$.

As $(g_1, g_2, g_3)$ is a generating set for $A_7$, one of $g_2, g_3$ is not an element of $B$ and will generate $A_7$ with $g_1$. Thus any generating triple containing an element of type 7 is equivalent to a redundant triple.

CASE 2. Suppose that $g_1$ is of type 5, without loss of generality, (12345) say. If $\langle g_1, g_2\rangle$ is transitive over the set $\{1,2,3,4,5,6,7\}$ then $\langle g_1, g_2\rangle = A_7$.

So we look at the cases when $\langle g_1, g_2\rangle$ and $\langle g_1, g_3)$ are non transitive but of course, $g_2$ and $g_3$ between them must move 6 and 7. We need to consider two cases.

i) $g_1 = (12345)$, $g_2 = (\ldots)(67)$, $g_3 = (\ldots 6)(\ldots)(7)$. Then $6g_3 = i$ with $i \neq 6$ and $i \neq 7$ and $7g_3 = 7$ so $6g_2g_3 = 7$ and $7g_2g_3 = i$.

This means that $g_2g_3 = (\ldots 67i \ldots)(\ldots)$ and hence $\langle g_1, g_2g_3\rangle$ is transitive and so must be $A_7$.

ii) $g_1 = (12345)$, and let $g_2$ move 6 but not 7 and $g_3$ move 7 but not 6. Then $g_2g_3$ will move 6 and 7 and then $\langle g_1, g_2g_3\rangle$ is again transitive and so is $A_7$.

Thus if the generating triple contains an element of type 5 it is equivalent to a redundant triple.

The further cases, with $g_1$, $g_2$ and $g_3$ taking all possible types, are shown in the following table, which indicates the length of the calculation required.

We investigate the cases 3, 4, 5, 6 and 7, using the following consideration.

i) There is a need for transitivity over $\{1,2,3,4,5,6,7\}$.

ii) Any triple equivalent to a triple with an element of type 7 or of type 5 is no problem.

iii) Two elements generating a transitive subgroup of $A_7$, in which one is of type 3 will generate $A_7$ ([7], p. 34).

iv) Two elements generating a transitive subgroup of $A_7$ and each of type 4, 2 in different $PSL(2,7)$ subgroups will generated $A_7$.

The investigation leads to the conclusion that if the generating triple contains an element of type 4, 2 it is equivalent to a redundant triple.

| Case | $g_1$ type | $g_2$ type | $g_3$ type |
|---|---|---|---|
| 3 | 4, 2 | 4, 2 | 4, 2 or 3 or 3, 3 or 3, 2, 2 or 2, 2 |
| 4 | 4, 2 | 3 | 3 or 3, 3 or 3, 2, 2 or 2, 2 |
| 5 | 4, 2 | 3, 3 | 3, 3 or 3, 2, 2 or 2, 2 |
| 6 | 4, 2 | 3, 2, 2 | 3, 2, 2 or 2, 2 |
| 7 | 4, 2 | 2, 2 | 2, 2 |
| 8 | 3 | 3 | 3 or 3, 3 or 3, 2, 2 or 2, 2 |
| 9 | 3 | 3, 3 | 3, 3 or 3, 2, 2 or 2, 2 |
| 10 | 3 | 3, 2, 2 | 3, 2, 2 or 2, 2 |
| 11 | 3 | 2, 2 | 2, 2 |
| 12 | 3, 3 | 3, 3 | 3, 3 or 3, 2, 2 or 2, 2 |
| 13 | 3, 3 | 3, 2, 2 | 3, 2, 2 or 2, 2 |
| 14 | 3, 3 | 2, 2 | 2, 2 |
| 15 | 3, 2, 2 | 3, 2, 2 | 3, 2, 2 or 2, 2 |
| 16 | 3, 2, 2 | 2, 2 | 2, 2 |
| 17 | 2, 2 | 2, 2 | 2, 2 |

We provide here a proof of some of Case 3 to demonstrate the methods used. The complete proofs of the assertions made here involve a great deal of simple but tedious calculation.

CASE 3. Let $g_1$, $g_2$ and $g_3$ be each of type 4, 2 and each in a different $PSL(2,7)$-subgroup of $A_7$. As an example we consider the following case.

$$g_1 = (3567)(12) \in \langle (1234567), (23)(47) \rangle,$$

$$f_2 \in \langle (2314567), (13)(47) \rangle,$$

$$g_3 \in \langle (2431567), (43)(17) \rangle.$$

If $\langle g_1, g_2 \rangle$ is transitive over $\{1,2,3,4,5,6,7\}$ there is no problem. We also find for the remaining elements $g_2$ that $g_1 g_2$ or $g_1 g_2^{-1}$ or $g_1 g_2^2$ is of type 7 or type 5 except for $g_2 = (2537)(16)$ or $(1567)(23)$ and their inverses.

If $\langle g_1, g_3 \rangle$ is transitive over $\{1,2,3,4,5,6,7\}$ there is no problem. We also find for the remaining elements $g_3$ that $g_1 g_3$ or $g_1 g_3^{-1}$ is of type 7 or type 5 expect for $g_3 = (3657)(12)$ or $(3567)(14)$ or $(3576)(24)$ and their inverses.

For these elements or their inverses, $g_2 g_3$ or $g_2 g_3^{-1}$ is of type 7 or type 5.

We see that for the selected $g_1$ and the subgroups concerned, all the triples are equivalent to redundant triples. This is found to be true whichever of the 14 maximal subgroups are chosen to contain elements $g_2$ and $g_3$. Thus any generating triple containing three elements of type 4, 2 each in a different $PSL(2,7)$ maximal subgroup is equivalent to a redundant triple.

We now consider the case with $g_1$, $g_2$ each of type 4, 2 and each in a different $PSL(2,7)$-subgroup of $A_7$ with $g_3$ any element of type 3. We consider the following case.

$$g_1 = (3567)(12) \in (1234567), (23)(47)\rangle,$$

$$g_2 \in \langle (2314567), (13)(47)\rangle.$$

$$g_3 = \text{any element of type 3 in } A_7,$$

When we consider the products of $g_1 g_2$ and $g_1 g_3$ we find problems only occur when $g_2 = (2537)(16)$ or $(1567)(23)$ and $g_3 = (124)$ or $(345)$ or $(346)$ or $(347)$ or $(456)$ or $(457)$.

For these elements we find that either an equivalent triple can be obtained with one element, a product of $g_1$, $g_2$ and $g_3$, which is of type 7 or of type 5, or the triple is not a generating triple.

We again see that for the selected $g_1$ and the subgroups concerned, all the triples are equivalent to redundant triples. This is found to be true whichever of the maximal subgroups are chosen to contain element $g_2$. Thus any generating triple containing two elements of type 4, 2 each in a different $PSL(2,7)$ maximal subgroup with the third element of type 3 is equivalent to a redundant triple.

Case 3, when completed, and then cases 4, 5, 6 and 7 all lead to the same conclusion that the generating triples concerned are all equivalent to a redundant triple.

The information obtained from cases 1 to 7 is used in the other cases in the order as shown in the table and with each case leading to a redundant triple.

The final conclusion is that all the generating $G$-vectors are equivalent to redundant vectors and consequently $A_7$ has only one $T_3$-system.

A further result of Evans [4] can now be used to complete the proof.

(3.1) *Let $G$ be a nonabelian finite simple group with $d(G) = k$. Suppose that $G = \langle g_1, g_2, \ldots, g_k\rangle$ where $g_k^2 = 1$. Then* $\mathrm{Aut}\, F_{k+1}$ *acts as a symmetric or alternating group on at least one of its orbits on $\Sigma(G, k+1)$.*

The alternating group $A_7$ may be generated by $\langle g_1, g_2\rangle$ where $g_1$ is an element of type 7 and $g_2$ is an element of type 2, 2 which is not in the $PSL(2,7)$ maximal sub-group containing $g_1$. For example we have $A_7 = \langle (1234567), (12)(45)\rangle$. We conclude that the action of $\mathrm{Aut}\, F_3$ on the $A_7$ defining subgroups is alternating or symmetric.

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Manoscritto pervenuto in redazione il 19 luglio 1991.