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On Non-Supersoluble and Non-Polycyclic Normal Subgroups.

JAMES C. BEIDLEMAN - HOWARD SMITH (*)

ABSTRACT - This paper presents an investigation of groups G with the property that every non-polycyclic normal subgroup H has a finite G -invariant insoluble image and continues a previous investigation of groups with a similar property in relation to non-supersolubility.

0. Introduction.

This work is concerned mainly with groups having a certain property in relation to their non-polycyclic normal subgroups. As such, it represents a continuation of the investigations carried out in papers [1] and [2], where properties ν and σ (respectively) were introduced. We recall that a group G was said to have property ν (respectively σ) if, given any non-nilpotent (respectively non-supersoluble) normal subgroup H of G , there exists a G -invariant subgroup K of finite index in H such that H/K is non-nilpotent (respectively non-supersoluble). By Theorem 2 of [2], any group with σ also has ν .

The first section proper presents some further results on groups with property σ and, it is hoped, provides some motivation for the introduction of the property π which is defined in Section 2 and which is discussed throughout the remainder of the paper.

1. Non-supersoluble normal subgroups.

If the group G has σ , then we know from Proposition 1 and Theorem 1 of [2] that the union $H(G)$ of the series of iterated Hirsch-Plotkin

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radicals is polycyclic and that the intersection $\phi_f(G)$ of all maximal subgroups of finite index in G is contained in the Fitting radical $\text{Fit}(G)$ (which is finitely generated). These results draw attention to normal polycyclic subgroups, which are always contained in $H(G)$. Defining $P(G)$ to be product of all normal polycyclic subgroups of the group G , we have the following.

LEMMA 1. *Let G be a group with property σ . Then $\phi_f(G) \leq P(G)$, $P(G)$ is polycyclic and $P(G/\phi_f(G)) = P(G)/\phi_f(G)$.*

PROOF. In view of the preceding remarks, all that needs to be shown is that, in a group G with σ , $P(G/\phi_f(G)) \leq P(G)/\phi_f(G)$ (the reverse inclusion being obvious). But, from Theorem 1 of [2], we know that $\bar{G} = G/\phi_f(G)$ has σ and so, again from our remarks, $P(\bar{G})$ is polycyclic. The result follows.

Now Theorem 1 of [1] tells us that a group G has ν if and only if $\phi_f(G) \leq \text{Fit}(G)$, $\text{Fit}(G)$ is nilpotent and $\text{Fit}(G/\phi_f(G)) = \text{Fit}(G)/\phi_f(G)$. Further, we have seen that all normal locally nilpotent subgroups of a group G with σ are polycyclic. Accordingly, it would perhaps be reasonable to hope that the converse of Lemma 1 holds, but it is easy to show that this is not the case, for if G is the group with presentation $\langle x, y \mid y^{-1}xy = x^2 \rangle$, then G has no non-trivial normal polycyclic subgroups and $\phi_f(G) = 1$, but G does not have σ since $\text{Fit}(G)$ is not finitely generated. (This is the same example as that mentioned at the end of Section 3 of [2]). To find necessary and sufficient conditions for a group to satisfy σ , conditions which are in some way similar to those for ν -groups as given in Theorem 1 of [1], we alter our perspective a little and notice that, for groups G with ν , $\text{Fit}(G)$ coincides with the Baer radical of G (and, indeed, with the Hirsch-Plotkin radical). For any group G , let us denote by $P_1(G)$ the subgroup generated by all subnormal polycyclic subgroups of G . Then the following result holds.

THEOREM 1 (c.f. Theorem 1 of [1] and Theorem 1 of [2]). *A group G has property σ if and only if*

$$(a) \phi_f(G) \leq P_1(G) \text{ and } P_1(G) \text{ is polycyclic.}$$

Further, if G has σ , then

$$(b) P_1(G/\phi_f(G)) = P_1(G)/\phi_f(G).$$

PROOF. If A is a subnormal polycyclic subgroup of the group G then A is contained in the iterated Hirsch-Plotkin radical of G (since it is contained in the iterated Baer radical). If G has σ , then this radical is

polycyclic. Arguing as in the proof of Lemma 1, we see that properties (a) and (b) hold. Conversely, assume that (a) holds. Then (b) also holds. Further, $\text{Fit}(G)$ is finitely generated (since it is contained in the Baer radical). We shall show that $\phi_f(G)$ is nilpotent and that $\text{Fit}(G)/\phi_f(G) = \text{Fit}(G/\phi_f(G))$. It will follow, by Theorem 1 of [2], that G has σ .

Suppose L is a G -invariant subgroup of finite index in $\phi_f(G)$. Then $\phi_f(G)/L = \text{Frat}(G/L)$ is finite and hence nilpotent, by a result of Gaschütz (see 5.2.15 (i) of [7]). Since $\phi_f(G)$ is polycyclic, it follows that all finite images of $\phi_f(G)$ are nilpotent and hence, by a theorem of Hirsch (see 5.4.18 of [7]), $\phi_f(G)$ is nilpotent. Now let $F/\phi_f(G)$ be a normal nilpotent subgroup of $G/\phi_f(G)$. We need to show that F is nilpotent. By (a) and (b), F is polycyclic and so, as above, if F is not nilpotent there is a G -invariant subgroup T of finite index in F such that $\overline{F} = F/T$ is non-nilpotent. Since $\overline{\phi_f(G)} = \phi_f(G)T/T$ is finite, it is contained in $\text{Frat}(G/T)$. Also, $\overline{F}/\overline{\phi_f(G)}$ is nilpotent. We can now apply the Frattini argument as in the proof of 5.2.15(i) of [7] to conclude that F/T is nilpotent, a contradiction which completes the proof of the theorem.

A further radical and one which is related to those discussed above, is $LP(G)$, the (unique) maximal normal locally polycyclic subgroup of the group G . Clearly $P(G) \leq P_1(G) \leq LP(G)$, and we have already mentioned an example of a group G in which $P(G) = 1 = \phi_f(G)$ and $P_1(G) = \text{Fit}(G)$ is not finitely generated. An example due to McLain and referred to in Section 4 of [2] provides us with a non-polycyclic, locally polycyclic group G with σ in which $\phi_f(G) = 1$. Thus $P_1(G)$ cannot be replaced by $LP(G)$ in the statement of Theorem 1. Another example worth mentioning (with regard to the relationship between the various radicals) is one due to Dark [3]. His group G is a (non-trivial) Baer group with no non-trivial normal abelian subgroup. In particular, $P_1(G) = G$ and $P(G) = 1$.

To illustrate further the significance of polycyclic subgroups with regard to σ , we present the following result, which follows easily from Theorem 1, and which should be compared with Corollary 5 of [1] and Theorem 4 of [2].

THEOREM 2. *A group G has σ if and only if*

(a) *whenever $H \triangleleft G$, $\phi_f(G) \leq H$ and $H/\phi_f(G)$ is polycyclic, then H is polycyclic and*

(b) *$P_1(G)$ is polycyclic.*

We note that condition (b) cannot be dispensed with, as is shown

by the example mentioned following the proof of Lemma 1 (or by any group G satisfying $\phi_f(G) = 1$ but not σ).

2. Non-polycyclic normal subgroups.

A group G shall be said to have property π if, given any non-polycyclic normal subgroup H of G , there exists a G -invariant subgroup K of finite index in H such that H/K is non-polycyclic (i.e. insoluble). Among other things, our intent is to establish, for the property π , results similar to those obtained for ν and σ . In some cases, the proofs are not substantially different from those of the corresponding results for ν and σ and are therefore sketched briefly or omitted altogether. We begin with a couple of rather straightforward results.

LEMMA 2 (c.f. Lemma 1 of [1] and Lemma 1 of [2]). *A group G has π if and only if, whenever H is a non-polycyclic normal subgroup of G , there is a normal subgroup N of finite index in G such that HN/N is insoluble.*

LEMMA 3. *If G has property π then G has σ .*

We note that McLain's (locally finite) group, mentioned earlier, has the property σ but not the property π , since a locally soluble group with π must be polycyclic.

PROOF OF LEMMA 3. If G has π and H is a normal, non-supersoluble subgroup of G , then either H is non-polycyclic, in which case there is clearly a G -invariant subgroup K of finite index in H with H/K not supersoluble, or else H is polycyclic, in which case there is, by a theorem of Baer (see [8], p. 162), a normal subgroup K of finite index in H such that H/K is non-supersoluble. Since H is finitely generated, we may take K to be normal in G , and the proof of Lemma 3 is complete.

Using these lemmas, the following theorem may be proved without difficulty.

THEOREM 3. *A group G has π if and only if $\phi_f(G)$ is polycyclic and $G/\phi_f(G)$ has π . Further, if G has π , then $P(G)$ (indeed $LP(G)$) is polycyclic and $P(G/\phi_f(G)) \leq P(G)/\phi_f(G)$.*

As a consequence of Theorem 3, we note that a group G has π if and only if G has σ and $G/\phi_f(G)$ and π .

Our sufficient conditions for a group G to have π include the hypoth-

esis that $G/\phi_f(G)$ have π . This is, perhaps, a little unsatisfactory. Accordingly, it would be interesting to know of a (reasonable) sufficient condition for a group G satisfying $\phi_f(G) = 1$ to have π . Also, there is perhaps a theorem with regard to the property π which is comparable to Theorem 1 of [1] and Theorem 1 of [2] (or Theorem 1 above) for υ and σ respectively. One conjecture might be that a group G has π if and only if $\phi_f(G) \leq LP(G)$, $LP(G)$ is polycyclic and $LP(G/\phi_f(G)) = LP(G)/\phi_f(G)$. However, in the final section we give an example which shows that this is not the case. (The given hypotheses are, nonetheless, seen to imply that G has σ and if, in addition, G is locally finite then G has π — this follows from Theorem 8 below).

3. Closure properties and finiteness conditions.

We begin this section with some results concerning π which recall similar results from [1] and [2]. (Part (iii) of course has no counterpart for the property υ).

THEOREM 4. (i) *Let G be a group with property π and let H be a non-polycyclic ascendant subgroup of G . Then there is a normal subgroup L of finite index in G such that HL/L is insoluble (see Corollary 2 of [1]).*

(ii) *Let H be an ascendant subgroup of the group G . If G has π then so does H (see Lemma 2 of [1] and Lemma 3 of [2]).*

(iii) *Let G be a group with a subgroup H of finite index. Then G has property π if and only if H does (see Theorem 10 of [2]).*

We now present a theorem which indicates one way of distinguishing the class of groups with π within the class of groups satisfying σ . To do this we require the following definition.

A group G shall be said to satisfy the property π^* if, for each normal subgroup N whose finite G -quotients are all soluble, there is a bound (depending in general on N) for the derived lengths of these quotients.

We are now ready to prove:

THEOREM 5. *Let G be a group with σ . Then G has π if and only if G satisfies π^* .*

PROOF. It is easily seen that π implies π^* . Suppose then that G satisfies π^* . Then $\phi_f(G)$ is polycyclic and $G/\phi_f(G)$ has σ ([2], Theorem 1). It is straightforward to verify that $G/\phi_f(G)$ has π^* and, in order to show that G has π , we can (in view of the remark following Theorem 3 above)

assume that $\phi_f(G) = 1$. Let N be a normal subgroup of G all of whose finite G -quotients are soluble and hence of derived length at most d , say. Let M be a maximal subgroup of finite index in G . It suffices to prove that $N^{(d)} \leq M$, for then $N^{(d)} \leq \phi_f(G) = 1$ and so N is soluble and hence, by Proposition 1 of [2], polycyclic. Clearly we may suppose $N \not\leq M$, so that $G = MN$. Let $T = M \cap N$. Then $|N:T|$ is finite and the conjugates of T in G are precisely the conjugates in N . Write $T_0 = \text{Core}_G T$. Then $|N:T_0|$ is finite and so $N^{(d)} \leq T_0 \leq M$, as required.

Using results of Mal'cev and Zassenhaus (4.2 and 3.7 of [8]), the above theorem and Theorem 7 of [2], we obtain the following.

THEOREM 6. *Let G be a subgroup of $GL(n, R)$, where R is a finitely generated domain. Then the following statements are equivalent:*

- (a) G has π .
- (b) G has σ .
- (c) The unipotent radical of G is finitely generated.

COROLLARY 1. Let G be a subgroup of $GL(n, \mathbb{Z})$. Then G has π .

Returning now to closure properties, we shall establish a result which depends partly on the above corollary.

THEOREM 7. *Let G be a group with normal polycyclic-by-finite subgroup S . Then G has π if and only if G/S has π .*

PROOF. Suppose G/S has property π and let N be a normal subgroup of G with all of its finite G -quotients soluble. Then NS/S is polycyclic and so N is polycyclic-by-finite-by-polycyclic and hence polycyclic-by-finite and thus polycyclic. So G has π . Conversely, suppose that G has π and let $C = C_G(S)$. Assume that S is finite. Then, by Theorem 4, both C and CS have π . Since $C \cap S \leq Z(C)$, it follows easily that $C/C \cap S$ has π and thus, by Theorem 4 again, that G/S has π . By induction on the Hirsch length, we may thus suppose that S is finitely generated and free abelian. Then G/C is \mathbb{Z} -linear and thus has π , by Corollary 1.

Now let N be a normal subgroup of G such that $S \leq N$ and all finite G -quotients of N/S are soluble. Since $S \leq C$, we have NC/C polycyclic. If $N_1 = N \cap C$ is not polycyclic then there is a G -invariant subgroup T of finite index in N_1 with N_1/T insoluble. Let $D = C_N(N_1/T)$, which is normal in G and of finite index in N . Since $S \leq D$ we have N/D soluble and so N_1/T is centre-by-soluble and thus soluble. This contradiction completes the proof of the theorem.

We note the following consequence, whose proof requires (besides Theorem 7) Theorem 6 and Lemma 3 above and Proposition 1 of [2].

COROLLARY 2. *Let S be a soluble-by-finite normal subgroup of the group G . Then G has π if and only if G/S and S both have π .*

It is easy to see that any polycyclic-by-finite group G has π , and we know from Theorem 6 of [2] that a group G with finite rank has σ if and only if G is polycyclic-by-finite. Thus a group G of finite rank has π if and only if it has σ (if and only if it is polycyclic-by-finite).

Next, we give a couple of results on locally finite groups.

LEMMA 4. *Let G be a locally finite group which is also residually finite. Then G has π if and only if every locally soluble normal subgroup of G is finite.*

PROOF. Suppose G has π and let N be a locally soluble, normal subgroup of G . Since every finite image of N is soluble, N is polycyclic and therefore finite. Conversely, assume G satisfies the given hypothesis on locally soluble normal subgroups and let N be a normal subgroup of G all of whose finite G -quotients are soluble. Then N is residually soluble and hence locally soluble. This gives N finite and thus polycyclic. Hence G has π .

THEOREM 8. *A locally finite group G has property π if and only if the following conditions hold:*

- (a) *Every locally soluble, normal subgroup of G is finite.*
- (b) *G has σ .*

PROOF. If the locally finite group G has σ , then the finite residual R of G is finite (and nilpotent), by Theorem 8 of [2]. Suppose G has π and let N be a locally soluble, normal subgroup of G . Then G/R has π and so, by Lemma 4, NR/R is finite. Hence, by Lemma 3, N is finite. Now assume G satisfies (a) and (b) and let N/R be a locally soluble, normal subgroup of G . Then N is locally soluble and therefore finite. Thus N/R is finite and it follows from Lemma 4 that G/R has π . Let H be a normal subgroup of G all of whose finite G -quotients are soluble. Then HR/R is polycyclic and therefore H is polycyclic. Thus G has π .

We conclude this section with some consequences of Theorem 8, Theorem 6 of [1] and Theorem 4.32 of [6] on groups with finiteness conditions on conjugates. For the second theorem here, we recall that a

group G is a CC -group if $G/C_G(\langle x \rangle^G)$ is a Černikov group, for all x in G . By Lemma 2.1 of [5], $R' \leq Z(G)$ and G/R is an FC -group, where R denotes the radicable part of G .

THEOREM 9. *Let G be an FC -group. Then G has π if and only if, whenever H is a locally soluble, normal subgroup of G , then H has a subgroup $K \leq Z(G)$ such that H/K is finite.*

THEOREM 10. *Let G be a CC -group. Then*

- (i) G has υ if and only if $\text{Fit}(G)$ is nilpotent and $\text{Fit}(G/R) = (\text{Fit}(G))/R$.
- (ii) G has σ if and only if $\text{Fit}(G)$ is finitely generated and $\text{Fit}(G/R) = (\text{Fit}(G))/R$.
- (iii) G has π if and only if G has σ and G/R has π .

4. An example and an algorithm.

In this final section we present an algorithm which determines whether certain normal subgroups of a group with π are polycyclic, but first we give an example which disposes of the conjecture mentioned at the end of Section 2. In fact, we present a (non-trivial) locally soluble group G with trivial locally polycyclic radical in which $\phi_f(G) = 1$. Clearly this group cannot satisfy π . The group is easily described.

For each integer $i \geq 0$, let H_i be an infinite cyclic group. Define G_1 to be H_0 and, inductively, $G_{i+1} = H_i \text{ wr } G_i$ for each $i \geq 1$, where the wreath product in each case is the standard one. Then let G be the ascending union of the G_i , $i = 1, 2, \dots$. Clearly G is locally soluble and torsion-free.

CLAIM. $LP(G) = 1 = \phi_f(G)$.

In order to establish this claim, we require two lemmas, the first of which is almost obvious. Incidentally, we remark that the group G is not a «wreath power» of infinite cyclic groups in the sense of P. Hall (see 6.2 of [6]).

LEMMA 5. *With G defined as above, let i be an arbitrary positive integer and let N be the normal closure in G of $\langle H_i, H_{i+1}, \dots \rangle$. Then $G = G_i N$ and $G_i \cap N = 1$.*

LEMMA 6. *Suppose $G = \langle a \rangle \text{ wr } H$, the standard wreath product*

of the infinite cyclic group $\langle a \rangle$ with a group H satisfying $\phi_f(H) = 1$. Then $\phi_f(G) = 1$.

Before sketching the proofs of these lemmas, let us see how they suffice to establish our claim.

Firstly, suppose K is a non-trivial, locally polycyclic, normal subgroup of our group G and let g be a non-trivial element of K . Then $g \in G_i$, for some i , and $\langle g \rangle^{H_i}$ is locally polycyclic. Suppose $H_i = \langle h \rangle$. Then $[h, g] \neq 1$ and, since g has infinite order, $\langle [h, g] \rangle^{(g)}$ is not finitely generated. Thus $\langle [g, h], g \rangle$ is not polycyclic, a contradiction.

Now let x be any non-trivial element of G . Again $x \in G_i$, for some i . By Lemma 6 (and induction), $\phi_f(G_i) = 1$. By Lemma 5, therefore, we can find a maximal subgroup M of G which does not contain x . Thus $\phi_f(G) = 1$.

PROOF OF LEMMA 5. Clearly we have $G = G_i N$. Now, for each $j \geq i$, let $L_j = \langle H_i, \dots, H_j \rangle$, a subgroup of G_{j+1} . Suppose $x \in N \cap G_i$. Then $x \in L_k^{G_i} \cap G_i$, for some $k \geq i$. If $k > i$, then $L_k^{G_i} \leq L_{k-1}^{G_i} H_k^{G_{k+1}} = L_{k-1}^{G_i} H_k^{G_k}$ and so $L_k^{G_i} \cap G_i \leq L_{k-1}^{G_i} (H_k^{G_k} \cap G_k) \cap G_i = L_{k-1}^{G_i} \cap G_i$. Repeating this argument (if necessary) we obtain (eventually) $x \in H_i^{G_i} \cap G_i = 1$. The result follows.

PROOF OF LEMMA 6. The proof here imitates (so far as possible) a couple of the proofs from [4]. Let G be as stated, and suppose first that H is finite. Let x be a non-trivial element of the base group $A = \langle a \rangle^H$ and choose a prime p not dividing the order of H such that $x \notin A^p$. Using Maschke's Theorem, one then easily obtains a G -invariant subgroup D of A such that $x \notin D$ and A/D is a simple H -module. Then $x \notin HD$, a maximal subgroup of finite index in G . It follows that $\phi_f(G) = 1$. In the general case, again assume x is a non-trivial element of $A = \langle a \rangle^H$. As in the proof of Lemmas 3.2 and 3.3 of [4], but using the stronger condition $\phi_f(H) = 1$, we can obtain a normal subgroup K of finite index in H such that $\text{Frat}(H/K) = 1$ and a homomorphism θ from G onto $\langle a \rangle$ wr H/K such that $x\theta \neq 1$. This allows us to reduce to the case where H is finite and the result follows.

Corollary 1 of each of papers [1] and [2] describes an algorithm for detecting nilpotency or supersolubility of certain normal subgroups. We conclude with a similar algorithm for the property π .

PROPOSITION. *Let G be a finitely presented group with the property π . Then there is an algorithm which, when a finite subset $\{x_1, \dots, x_m\}$*

of G is given, together with the information that $N = \langle x_1, \dots, x_m \rangle$ is normal in G , decides whether N is polycyclic.

PROOF. As in the proof of Corollary 1 of [1], we introduce two recursive procedures, one of which must terminate, by Lemma 2. The result will follow immediately.

The first procedure is the obvious analogue of that described in the above-mentioned corollary.

Now let $(\{w_{1,1}, \dots, w_{1,m_1}\}, \{w_{2,1}, \dots, w_{2,m_2}\}, \dots, \{w_{n,1}, \dots, w_{n,m_n}\})$ be an n -tuple of subsets of words $w_{i,j}$ in the generators x_1, \dots, x_m . Further, for each $i = 1, \dots, n$, let H_i be the subgroup (of N) generated by the elements $w_{1,1}, \dots, w_{1,m_1}, \dots, w_{i,1}, \dots, w_{i,m_i}$. The second procedure attempts to show that the subgroups H_i form a polycyclic series for N . Again referring to the proof of Corollary 1 of [1], it is not difficult to see how to proceed. The details are omitted here.

REFERENCES

- [1] J. C. BEIDLEMAN - D. J. S. ROBINSON, *On the structure of the normal subgroups of a group: nilpotency*, Forum Math., 3 (1991), pp. 581-593.
- [2] J. C. BEIDLEMAN - D. J. S. ROBINSON, *On the structure of the normal subgroups of a group: supersolubility*, Rend. Sem. Mat. Univ. Padova, 87 (1992), pp. 139-149.
- [3] R. S. DARK, *A prime Baer group*, Math. Z., 105 (1968), pp. 294-298.
- [4] K. W. GRUENBERG, *Residual properties of infinite soluble groups*, Proc. London Math. Soc., (3), 7 (1957), pp. 29-62.
- [5] J. OTAL - J. M. PEÑA, *Characterizations of the conjugacy of Sylow p -subgroups of CC-groups*, Proc. Amer. Math. Soc., 106 (1989), pp. 605-610.
- [6] D. J. S. ROBINSON, *Finiteness Conditions and Generalized Soluble Groups* (2 vols.), Springer, Berlin (1972).
- [7] D. J. S. ROBINSON, *A Course in the Theory of Groups*, Springer, New York (1982).
- [8] B. A. F. WEHRFRITZ, *Infinite Linear Groups*, Springer, Berlin (1973).

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