

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova,
tome 89 (1993), p. 83-86

http://www.numdam.org/item?id=RSMUP_1993__89__83_0

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On a Question of Deaconescu About Automorphisms.

JOHN C. LENNOX - JAMES WIEGOLD (*)

1. Introduction.

At a recent meeting in Barnaul, Siberia, Marian Deaconescu posed the following problem.

1.1. *Do there exist infinite groups G such that $\text{Aut } H \cong \cong N_G(H)/C_G(H)$ for all subgroups H of G ?*

Let us call (finite or infinite) groups with this property MD-groups. We note at the outset that the infinite dihedral group D has the MD-property, and this is fairly easy to prove. Theorem 2.1 characterizes the infinite metabelian MD-groups; there are just 2^{\aleph_0} of them, and they all have very simple structure. A corollary is that D is the only finitely generated infinite metabelian MD-group.

It is quite probable that the groups covered in Theorem 2.1 are the only infinite MD-groups, because automorphism groups tend to be insoluble and big. For example, every free group F_m of infinite rank m has F_{2^m} in its automorphism group; thus, by the splitting property of free groups, any MD-group containing a subgroup whose automorphism group contain a non-abelian free subgroup would have to be of order bigger than all the cardinals obtained from \aleph_0 by related exponentiation ω times; that is, bigger than $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots$.

On the other hand, there are very few finite MD-groups:

1.2. *The only non-trivial finite MD-groups are Z_2 and S_3 .*

PROOF. Let G be a finite non-trivial MD-group. We claim first that every non-trivial Sylow subgroup is of prime order. If not, G has a p -subgroup P of order bigger than p , and the celebrated Gaschütz theo-

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rem [2] shows that $\text{Out } P$ has an element of order p . Let $X/C_G(P)$ be a Sylow p -subgroup of $N_G(P)/C_G(P)$. Then $X/C_G(P)$ is of order more than $P/Z(P)$, so that X contains a p -subgroup strictly bigger than P . This process continues and we can construct p -subgroups of arbitrarily large order. This is impossible since G is finite.

So G is metacyclic, and we can choose a Carter subgroup H of G , that is, a self-normalizing nilpotent subgroup. Then $\text{Aut } H \cong N_G(H)/C_G(H) = H/Z(H) = 1$, so again by Gaschütz's theorem, H must be cyclic of order 2, with generator h , say. But then $C_G(h) = H$, and h has $|G|/2$ conjugates, so that $|G'| = |G|/2$. Let K be any subgroup of G' ; then $C_G(K) \geq G'$ and so $\text{Aut } K$ has order at most 2. This means that G' is of order 1, 2, 4 or 6, and a routine calculation completes the proof.

The only general property of MD-groups we shall require in the sequel is this:

1.3. *For every MD-group, G/G' is of exponent dividing 2.*

PROOF. For every x in G , $N_G(\langle x \rangle)/C_G(\langle x \rangle) \cong \text{Aut } \langle x \rangle$ so there must exist $g \in G$ such that $x^g = x^{-1}$. Thus $[x, g] = x^{-2}$, so every square is a commutator.

2. Infinite metabelian ND-groups.

Here is the characterization promised in the introduction.

THEOREM 2.1. *An infinite metabelian group G is an MD-group if and only if G can be generated by a non-trivial torsion-free locally cyclic group A and an element x of order 2 such that*

- (i) $a^x = a^{-1}$ for all a in A ,
- (ii) *the type of A is finite at every prime.*

COROLLARY 2.2. *There are precisely 2^{\aleph_0} infinite metabelian MD-groups.*

The corollary follows since $G' = A^2$ and there are 2^{\aleph_0} different structures for A ; for any locally cyclic group A of the type under consideration, $A \cong A^2$.

Before embarking on the proof of Theorem 2.1, we establish the following simple lemma, which is probably known:

LEMMA 2.3. *Every subgroup H of a torsion-free abelian group A*

has automorphism group of exponent at most 2 if and only if A is locally cyclic with type that is finite at each prime. In fact, Aut H is of order at most 2 in all such cases.

PROOF. Let A be a torsion-free abelian group such that Aut H is of exponent at most 2 for all subgroups H. Then A is certainly locally cyclic since $Z \times Z$ has non-abelian automorphism group. If A is p-divisible for some prime p, the map $a \rightarrow a^p$ is an automorphism of infinite order; so the type of A is finite at all primes. Thus A is isomorphic to the subgroup A^* of the additive group Q of rationals generated by reciprocals $p_n^{-k_n}$, where p_n is the n-th prime and k_n is a non-negative integer for each n (see Fuchs [1]). Let α be an automorphism of A^* , and write $1\alpha = rs^{-1}$, where r and s are relatively prime integers. We claim that $s = 1$. If not, then s is divisible by p_m for some m, so that $k_m > 0$; but, since A^* is torsion-free, this means that $p_m^{-k_m}\alpha = rs^{-1}p_m^{-k_m}$, whereas this number has denominator divisible by $p_m^{k_m+1}$, and so is not in A^* . Thus $1\alpha = r$, and a similar argument on α^{-1} shows that $r = \pm 1$, so that $\text{Aut } Z = Z_2$, as required. The same holds for all subgroups H of A, of course.

The converse is clear, and the proof of the lemma is complete.

We turn now to the proof of Theorem 2.1. Let G be a group having a subgroup A and an element x with the properties mentioned in the statement, and let H be a non-trivial subgroup of G. If $H \leq A$, then $H \triangleleft G$ and $C_G(H) = A$, so that $N_G(A)/C_G(A) \cong Z_2$, and so $N_G(H)/C_G(H) \cong \text{Aut } H$, by Lemma 2.3. If $H \not\leq A$, then $H = \langle K, bx \rangle$ for some $K \leq A$ and some element b of A. If $K = 1$, all is well. If not, $C_G(K) = A$ and $C_G(bx) = \langle bx \rangle$, so $C_G(H) = 1$. A very quick calculation shows that $N_G(H) = \langle \sqrt{K}, bx \rangle$, where $\sqrt{K} = \{a \in A \text{ and } a^2 \in K\}$. However, we have $\sqrt{K} \cong K$ since K/K^2 is of order precisely 2 (this is true for any nontrivial subgroup of A), and the isomorphism is the squaring map. This means that $N_G(H) \cong H$ and we simply need to show that $\text{Aut } H \cong H$ in this case. Without loss of generality, we may suppose that $b = 1$, so we are considering a group of the form $H = \langle K, x \rangle$, where K is a non-trivial subgroup of A.

Now K is characteristic in H, since every element of $H-K$ is of order 2. Thus, for every automorphism α of H, $\alpha|_K$ is the identity or the inverting map, and $\alpha: x \rightarrow bx$ for some $b \in K$. Conversely, every map $\phi_{\epsilon, l}$ of the form

$$k \rightarrow k^\epsilon \quad \text{for } k \in K, \quad x \rightarrow lx,$$

where $\epsilon = \pm 1$ and $l \in L$, yields an automorphism of K.

Those with $\varepsilon = 1$ form a subgroup isomorphic with K ; the automorphism $\xi = \phi_{-1,1}$ is of order 2, and

$$\xi^{-1} \phi_{1,l} \xi = \phi_{1,l^{-1}},$$

so that $\text{Aut } H \cong H$, as required.

Conversely, let G be an infinite metabelian MD-group, and A a maximal abelian subgroup containing G' . For every subgroup H of A , $C_G(H) \geq A \geq G'$, so $N_G(H)/C_G(H)$ is of exponent at most 2, by 1.3. Thus the periodic part of A is of order at most 6 and so $A = S \times P$ for some finite group P and torsion-free group S . Since $C_G(A) = A$ by maximality we have $S \neq 1$ since G is infinite. If $P \neq 1$, then A contains a subgroup isomorphic to $Z \times Z_n$, $n \neq 1$, which has non-abelian automorphism group. So A is torsion-free and thus it is locally cyclic since $Z \times Z$ has non-abelian automorphism group. By 1.3 and 2.3, A must be a locally cyclic group, and its type is finite for each prime, so $\text{Aut } A$ is of order 2. Thus $Z_2 \cong N_G(A)/C_G(A) = G/A$, so $G = \langle x, A \rangle$ for some x such that $x^2 \in A$. But x^2 is central and this means that $x^2 = 1$ else $N_G(\langle x^2 \rangle)/C_G(\langle x^2 \rangle)$ is of order 1, not 2 as it is for infinite cyclic groups. This completes the proof of the theorem.

Finally, the proof of our last corollary is obvious:

COROLLARY 2.4. *This only infinite finitely generated metabelian MD-group is D .*

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Manoscritto pervenuto in redazione il 17 dicembre 1991.