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Zero-one matrices with an application to abelian groups

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SUMMARY - An \( n \times n \) matrix \( E \) is called a 0,1-matrix if each entry of \( E \) is either a 0 or a 1. In this case we can view \( E \) as either an integer valued matrix, or a matrix over \( Z_2 \), the integers mod 2. Matrices of this type, enjoying other properties as well, have recently cropped up in the study of torsion-free abelian group theory. Our aim is to study properties of these matrices in a setting unencumbered by this group theory. As a consequence we are able to answer a question posed in [FM].

1. A 0,1-matrix \( E \) is called admissible in [FM] provided \( |E_k| = \det E_k \neq 0 \) for each \( k \), where \( E_k \) is \( E \) with its \( k^{th} \) columns replaced by the vector \( 1 \) containing only 1's. We will say that \( F \) is equivalent to \( E \) if one can complement (by interchanging 1's and 0's) certain columns of \( E \) to get \( F \). It is easy to check that admisibility is preserved under this equivalence. This is because if \( E' \) is equivalent to \( E \) after the \( i^{th} \) column only of \( E \) was complemented, then \( |E'_j| = -|E_j| \) when \( j \neq i \), and \( |E'_i| = |E_i| \). The admissible matrices play a significant role in abelian group theory, a role which will be summarized in the second section.

We will consider two conditions imposed on a matrix \( E \) over \( Z_2 \):

(a) Each row sum of \( E \), computed in \( Z_2 \), is the same, and

(b) \( E \) is equivalent to an invertible matrix over \( Z_2 \).

Clearly, both conditions are preserved under our equivalence relation. We will compare these conditions to the property of being admissi-
sible. We will call a matrix $E$ over $\mathbb{Z}_2$, admissible mod 2, if for all $k$ the $\mathbb{Z}_2$-determinant of $E_k$, $|E_k|_2$, is not zero where $E_k$ is as defined above. Of course, if $E$ is admissible mod 2 then $E$ is admissible when viewed as a matrix with integer entries.

**Proposition 1.** Let $E$ be an $n \times n$ matrix over $\mathbb{Z}_2$ and $E^*$ the classical adjoint of $E$ (over $\mathbb{Z}_2$). Then $E$ is admissible mod 2 if and only if $E^* \overline{1} = \overline{1}$.

**Proof.** The $k^{\text{th}}$ entry of $E^* \overline{1}$ is $M_{1k} + M_{2k} + \ldots + M_{nk}$ where $M_{ik} = i$, $k^{\text{th}}$ cofactor (= minor) of $E$. But this sum is just the cofactor expansion of $|E_k|_2$ along its $k^{\text{th}}$ column. Hence, $|E_k|_2 = 1$, (i.e. $|E_k|_2 \neq 0$) for all $k$ if and only if $E^* \overline{1} = \overline{1}$.

We will show that $E$ satisfies both (a) and (b) if and only if $E$ is admissible mod 2. In case $E$ satisfies (a) we often refer to $E$ as having row parity. Clearly $E$ has row parity if and only if $\overline{1}$ is an eigenvector for $E$ over $\mathbb{Z}_2$. In case $E \overline{1} = 0$, $E$ has even row parity, and if $E \overline{1} = 1$, then $E$ has odd row parity. We will use $\overline{n}$ to denote \{1, 2, ..., $n$\} when no confusion is possible.

**Theorem 2.** $E$ is admissible mod 2 if and only if (a) and (b) hold for $E$.

**Proof.** The $j^{\text{th}}$ column of $E$ is the characteristic function on some index set $I \subset \overline{n}$. As such we will call the support of the $j^{\text{th}}$ column of $E$, $I$.

If $E$ is admissible mod 2 and $I$ is the support of the $1^{\text{st}}$ column of $E$, let $E'$ be the matrix resulting from complementing the $1^{\text{st}}$ column of $E$. Then, the support of the $1^{\text{st}}$ column of $E'$ is $I' = \overline{n} \setminus I$. By performing cofactor expansion of $|E_1|_2$, $|E|_2$ and $|E'|_2$ along their first columns, we see that $|E_1|_2 = 1 = |E|_2 + |E'|_2$. If $|E|_2 = 0$ then $|E'|_2 = 1$ so that $E$ is equivalent to an invertible matrix. Also, by Proposition 1, $EE^* \overline{1} = E \overline{1} = (\det E) \overline{1}$, so that $E$ has row parity.

Conversely, it is enough to assume that $E$ is invertible. From this and because of (a), $E \overline{1} = 1$. Then $E^* E \overline{1} = E^* \overline{1} = (\det E) \overline{1} = \overline{1}$, and $E$ is admissible mod 2 by Proposition 1.

**Example 3.** It can be checked that $E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ is admissible, but $E$ does not have row parity so it is not admissible mod 2. Row parity is easily checked. Any $n \times n$ 0,1-matrix $E$ is equivalent
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to a matrix \( E' = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \) where \( F \) is an \((n - 1) \times (n - 1)\) 0,1-matrix and \( I \in Z_2^{n-1} \). Hence to check that (β) holds for \( E \) we need only compute \( |F|_2 \), which is clearly preferable to the computation of \( n \) determinants for admissibility mod 2.

**Lemma 4.** There are \( \prod_{j=0}^{m-1} (2^m - 2^j) \) invertible \( m \times m \) matrices over \( Z_2 \).

**Proof.** To form an invertible \( m \times m \) matrix, we must select \( X_1 \in Z_2^m \setminus \{0\} \) for the first column, \( X_2 \in Z_2^m \setminus \text{span} \{ X_1 \} \) for the second, \( X_3 \in Z_2^m \setminus \text{span} \{ X_1, X_2 \} \) for the third, and so on. There are \((2^m - 1) \cdot (2^{m-1} - 2) \cdots (2^m - 2^{n-1})\) ways for this selection to occur.

It is desirable to know just how many admissible mod 2 matrices there are. Let \( \mathcal{E} = \{ E | E \text{ is } n \times n, \text{admissible mod } 2 \text{ and invertible} \} \).

Since \( \mathcal{E} \) is finite and is closed under multiplication, \( \mathcal{E} \) is a group. Let \( \mathcal{F} \) be the subgroup of \( \mathcal{E} \) consisting of those \( E \in \mathcal{E} \) with \( E = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \), as above.

**Theorem 5.** (i) \( |\mathcal{F}| = \prod_{j=0}^{n-2} (2^{n-1} - 2^j) \).

(ii) \( |\mathcal{E}| = 2^{n-1} \prod_{j=0}^{n-2} (2^{n-1} - 2^j) \).

(iii) There are \( 2^n \prod_{j=0}^{n-2} (2^{n-1} - 2^j) \) admissible mod 2 matrices.

**Proof.** Any \( E \in \mathcal{F} \) can be expressed as \( E = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \) with \( F \) an \((n - 1) \times (n - 1)\) invertible matrix uniquely determined by \( E \). Since \( E \) has row parity, \( I + F^\top = \bar{1} \), since the first row of \( E \) has parity 1, so that \( I = (F^\top)^\prime \) (the complement of \( F^\top \)) is determined by \( F \). Conversely, any \((n - 1) \times (n - 1)\) invertible matrix \( F \) determines the matrix \( \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \in \mathcal{F} \) where \( I = (F^\top)^\prime \), so the computation of \( |\mathcal{F}| \) follows from lemma 4.

Any \( E \in \mathcal{E} \) is equivalent to a matrix in \( \mathcal{F} \). Now suppose that \( E \in \mathcal{F} \), and that \( E' \) is a matrix equivalent to \( E \) as the result of complementing the \( j \)th column (only) of \( E \). An in the proof of Theorem 2, \( 1 = |E|_2 + |E'|_2 \) so that \( E' \notin \mathcal{E} \). If \( E'' \) is matrix resulting from complementing only one column of \( E' \), then as before \( |E''|_2 + |E'|_2 = 1 \) and \( E'' \in \mathcal{E} \). It fol-
lows that if $E'(s)$ results from $E$ by complimenting some $s$ columns of $E$, then $E'(s) \in \mathcal{E}$ if and only if $s$ is even.

Let $a_n = \text{number of subsets of } 2^n \text{ containing an even number of elements. We have just shown that } |\mathcal{E}| = a_n |\mathcal{F}|$. Set $b_n = 2^n - a_n$, and define $\delta: 2^n \rightarrow Z_2$ by letting $\delta(T) = \text{remainder of card}(T) \mod 2$. Let $S + T$ denote the symmetric difference of $S$ and $T$ so that $2^n$ is an abelian group under $\oplus$. Since $\text{card}(S + T) = \text{card}(S) \oplus \text{card}(T) - 2 \text{card}(S \cap T), \delta(S + T) = \delta(S) + \delta(T)$ and $\delta$ is a homomorphism. Hence $a_n = b_n = 2^n/2 = 2^{n-1}$.

If $\mathcal{E}'$ is the set of admissible mod 2 matrices with zero determinant, then the map sending $E \in \mathcal{E}$ to the matrix $E'$ formed by complimenting the first column of $E$, is a bijection. Thus, there are $2 |\mathcal{E}'|$ admissible mod 2 matrices.

2. In this section we will attempt to convey the role that the matrices $E \in \mathcal{E}$ play in abelian group theory without involving the group theory.

The use of admissible matrices in classifying a certain class of Butlwe groups (specifically, the $B(1)$-groups) was initiated in [FM], and investigated further in [GM]. Other results concerning the same class of groups were obtained earlier in [AV] and [R]. For a deeper involvement of the group theory, see the listed references.

The set of isomorphism classes of subgroups of the rationals form a distributive lattice $\Delta$. Moreover, any finite distributive lattice $T$ is isomorphic to a sublattice of $\Delta$ ([R] or [GU]). Let us fix an isomorphism. Then for any collection $\tau_1, \ldots, \tau_n \in T$, the $n$-tuple $\tau = (\tau_1, \ldots, \tau_n)$ determines a certain abelian group $G = G[\tau_1, \ldots, \tau_n]$. The description of $G$ is not relevant here but the interested reader should consult the cited references (in fact, $G$ is only determined up to quasi-isomorphism: see below).

Given an $n$-tuple $\tau = (\tau_1, \ldots, \tau_n)$ with $\tau_i \in T$, and a 0,1-matrix $E$ we can let $E$ operate on $\tau$ as follows: Set $\tau_i = \bigwedge_{i \in I} \tau_i$ for any $\phi \neq I \subseteq \bar{n}$. If $I_i$ is the support of the $i$-th column of $E$, define $\tau E = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_i = \tau_{I_i} \lor \tau_{I'_i}$ and $I'_i = \bar{n} \setminus I_i$.

We will now summarize some of the results concerning the groups $G[\tau_1, \ldots, \tau_n]$ in terms of $\tau$ and our operation $\tau E$. Two abelian groups $G$ and $H$ are called quasi-isomorphic if each is isomorphic to a subgroup of finite index in the other, in which case we write $G \sim H$.

**Theorem 6.** Let $\tau = (\tau_1, \ldots, \tau_n)$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\tau_i, \sigma_j \in T$ for all $i, j$. Furthermore, assume that $\tau \neq \tau_i \lor \tau_j$ for any proper $I \subset \bar{n}$ except $I = \{i\}$ or $\{i\}'$, and $\sigma_j \neq \sigma_j \lor \sigma_j$, for any proper $J \subset \bar{n}$ except, $J = \{j\}$ or $\{j\}'$. Let $G = G[\tau_1, \ldots, \tau_n]$ and $H = G[\sigma_1, \ldots, \sigma_n]$. 
(1) [FM] $G \sim H$ if and only if $\tau E \geq \sigma$, and $\sigma F \geq \tau$ for some admissible matrices $E$ and $F$

(2) [GM] $G \sim H$ if and only if $\tau E \geq \tau$ and $\sigma F \geq \tau$ for some matrices $E$ and $F$ which are admissible mod 2. In this case, if we choose $E \in \mathcal{E}$, then $F = E^{-1}$ works.

Given $\tau = (\tau_1, \ldots, \tau_n)$ and $G = G[\tau_1, \ldots, \tau_n]$, we will say that $\tau$ is strongly indecomposable if $\tau_i \not\equiv \tau_j \lor \tau_j'$ for all $0 \neq I \subsetneq \mathbb{N}$ except $I = \{i\}$ or $\{i\}'$ for each $i$. Following [FM], $\tau$ will be called regular if $\tau_i = \tau_i \lor \tau_j$ for each $i$, so that $\tau_i = \tau_i \lor \tau_j$, when $I = \{i\}$ or $\{i\}'$. Assuming that $\tau$ is regular and strongly indecomposable, they say that $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a representation type of $G$ if $\sigma$ is regular, strongly indecomposable, they say that $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a representation type of $G$ if $\sigma$ is regular, strongly indecomposable, and $G[\tau_1, \ldots, \tau_n] \sim \sim G[\sigma_1, \ldots, \sigma_n]$. By Theorem 6(2), and a mild computation, we may replace this last condition with the condition that $\tau E = \sigma$ and $\sigma F = \tau$ for two admissible mod 2 matrices $E$ and $F$.

Two representation types $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ are called equivalent if $\sigma = (\gamma_{f(1)}, \ldots, \gamma_{f(n)})$ for some $f$ in the permutation group $S_n$. Fuchs and Metelli ask for an upper bound on the number of nonequivalent representation types of $G[\tau_1, \ldots, \tau_n]$ given $\tau = (\tau_1, \ldots, \tau_n)$, in terms on $n$ (problem 3 in [FM]).

**Theorem 7.** Let $\tau = (\tau_1, \ldots, \tau_n)$ be strongly indecomposable and regular and let $G = G[\tau_1, \ldots, \tau_n]$. There are at most $\prod_{i=0}^{n-2} (2^{n-1} - 2^i)/n!$ nonequivalent representation types of $G$.

**Proof.** Let $\mathcal{R}_\tau$ denote the collection of representation types of $G$. If $\sigma \in \mathcal{R}_\tau$, then $\sigma = \tau E$ for some admissible mod 2 matrix $E$. If $I$ is the support of the $i^{th}$ column of $E$ and $E'$ is formed by complementing the $i^{th}$ column of $E$, then the support of the $i^{th}$ column of $E'$ is $I'$, and for $\delta = \tau E'$, and for $\delta = \tau E'$, $\delta_i = \tau_i \lor \delta_i' = \tau_i \lor \tau_i = \sigma_i$. Therefore we may assume that $E \in \mathcal{F}$, and theorem 5(i) implies that $\mathcal{R}_\tau$ has at most $\prod_{i=0}^{n-2} (2^{n-1} - 2^i)$ members.

Let $\mathcal{P} \subset \mathcal{E}$ be the collection of all $n \times n$ permutation matrices. The assignment of $f \in S_n$ to $P_f \in \mathcal{P}$ whose $i$, $j^{th}$ entry is 1 if and only $f(j) = i$, is a group isomorphism. We will show that $\mathcal{P}$ acts on $\mathcal{R}_\tau$.

If $\sigma \in \mathcal{R}_\tau$, then $\sigma = \tau E$ for some $E \in \mathcal{E}$. Set $\delta = \tau(EP)$ and $\mu = \sigma P$ for $P = P_f \in \mathcal{P}$. For each $j$, since $\sigma$ is regular, $\mu_j = \sigma_{f(j)} \lor \sigma_{(f(j))'} = \sigma_i \lor \lor \lor \sigma_k = \sigma_i$ where $f(j) = i$. But if the $i^{th}$ column of $E$ is $I_i$, then
Now suppose that $\tau = \sigma F$ for some $F \in \mathcal{S}$. We must show that $\delta(P^{-1}F) = \tau = (\delta P^{-1}) F$. Let $\rho = \delta(P^{-1}) F$ and suppose that the support of the $k^{th}$ column of $F$ is $J_k$. Now $P^{-1}$ has a 1 in the $i$, $j$ entry if and only if $f^{-1}(j) = i$, so the support of the $k^{th}$ column of $P^{-1} F$ is $\{ i | i = f^{-1}(j) \}$ for some $j \in J_k$. Hence $\rho_k = \delta f^{-1}(J_k) \lor \delta f^{-1}(J_k')$. Also, $\tau = (\delta P^{-1}) F = (\delta f^{-1}(i_{1}) \lor \ldots \lor \delta f^{-1}(i_{n})) F$ has $\tau_k = \bigwedge_{i \in J_k} \delta f^{-1}(i) \lor \bigwedge_{i \in J_k'} \delta f^{-1}(i) = \delta f^{-1}(J_k) \lor \delta f^{-1}(J_k') = \rho_k$. Thus, if $\tau \neq \delta$, then $\delta(P^{-1}F) = \tau$ and $\tau(EP) = \delta$ so that $\delta \in \mathcal{R}_\tau$. If $P = P_f$ and $Q = P_g$ then mimicking the computation given above, we can show that $(\sigma P) Q = \sigma(PQ) = (\sigma f(1), \ldots, \sigma f(n))$ so that $\sigma$ acts on $\mathcal{R}_\tau$.

If $\sigma \in \mathcal{R}_\tau$ with $\sigma_i \leq \sigma_j$, and $i \neq j$, then $\sigma_i \leq \sigma_j \lor \bigwedge_{k \neq j} \sigma_k = \sigma_{\{j\}} \lor \sigma_{\{j\}'}$, which contradicts the strong indecomposability of $\sigma$. Therefore, $\sigma P = \sigma$ for $P \in \mathcal{P}$ if and only if $P$ is the identity matrix. Since $\mathcal{P}$ acts on $\mathcal{R}_\tau$ and the orbit of $\sigma$ is the equivalent class of $\sigma$ which contains $n!$ representation types, there are $|\mathcal{R}_\tau|/n!$ inequivalent representation types.

One could show that $\prod_{i=0}^{n-2} (2^{n-1} - 2^i)/n!$ is an integer by looking at the representation $\mathcal{R}_0$ of $\mathcal{P}$ in $\mathcal{F}$. Then show that $\mathcal{R}_0$ acts on $\mathcal{F}$. Clearly this bound is achieved if and only if $\tau E$ is a representation type of $\tau$ for any $E \in \mathcal{S}$, which is an intrinsic property of $T$ and does not depend, in general, solely on $n$. Of course when $n = 3$, $\prod_{i=0}^{1} (2^2 - 2^i)/6 = 1$ so the bound is tight in this case, regardless of $T$.

**Example 8.** Let $\tau_1 = \{1, 2, 3\}$, $\tau_2 = \{2, 3, 4\}$, $\tau_3 = \{1, 5, 6\}$ and $\tau_4 = \{4, 5, 6\}$ in $T = 2^6$ the power set of 6. It is easy to see that $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$ is regular and strongly indecomposable. However

\[ \tau_{\{2,3\}} \lor \tau_{\{1,4\}} = (\{2, 3, 4\} \cap \{1, 5, 6\}) \cup (\{1, 2, 3\} \cap \{4, 5, 6\}) = \emptyset \]

while

\[
\begin{bmatrix}
0 \\
1 \\
1 \\
0
\end{bmatrix}
\]

is the column of an admissible mod 2 matrix $E \in \mathcal{F}$. For example, $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. But $\tau E_1 = \sigma$ cannot be a representation
type of $\tau$ since $\sigma_2 \leq \sigma_i$ for all $i$ so $\sigma$ cannot be strongly indecomposable.

In this case, there are less than $\prod_{i=0}^{2}(2^3 - 2^i)/24 = 7$ representation types of $\sigma$.

Three are 7 pertinent matrices from $\tilde{\sigma}$: $E_0 =$ identity,

\begin{align*}
E_1, E_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},
E_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
E_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},
E_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix},
E_6 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.
\end{align*}

These are the matrices of concern because no complementing and/or interchanging of columns will transform one into the other. Set $\tau_5 = \{1, 4\}$ and $\tau_6 = \{2, 3, 5, 6\}$. Of the vectors $\tau E_i, i = 0, \ldots, 6$, only $\tau E_0 = \tau$, $\sigma = \tau E_2 = (\tau_5, \tau_5, \tau_1, \tau_4)$ and $\gamma = \tau E_4 = (\tau_5, \tau_6, \tau_2, \tau_3)$ are representation types of $\tau$. One easily checks that $\sigma$ and $\gamma$ are strongly indecomposable and regular, and that

\[
\sigma \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tau = \gamma \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

so that there are 3 representation types of $\tau$.

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