Ulrich F. Albrecht
H. Pat Goeters
Charles Megibben

Zero-one matrices with an application to abelian groups

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SUMMARY - An $n \times n$ matrix $E$ is called a 0,1-matrix if each entry of $E$ is either a 0 or a 1. In this case we can view $E$ as either an integer valued matrix, or a matrix over $\mathbb{Z}_2$, the integers mod 2. Matrices of this type, enjoying other properties as well, have recently cropped up in the study of torsion-free abelian group theory. Our aim is to study properties of these matrices in a setting unencumbered by this group theory. As a consequence we are able to answer a question posed in [FM].

1. A 0,1-matrix $E$ is called admissible in [FM] provided $|E_k| = \det E_k \neq 0$ for each $k$, where $E'_k$ is $E$ with its $k^{th}$ columns replaced by the vector 1 containing only 1's. We will say that $F$ is equivalent to $E$ if one can complement (by interchanging 1's and 0's) certain columns of $E$ to get $F$. It is easy to check that admissibility is preserved under this equivalence. This is because if $E'$ is equivalent to $E$ after the $i^{th}$ column only of $E$ was complemented, then $|E'_j| = -|E_j|$ when $j \neq i$, and $|E'_i| = |E_i|$. The admissible matrices play a significant role in abelian group theory, a role which will be summarized in the second section.

We will consider two conditions imposed on a matrix $E$ over $\mathbb{Z}_2$:

(a) Each row sum of $E$, computed in $\mathbb{Z}_2$, is the same, and

(b) $E$ is equivalent to an invertible matrix over $\mathbb{Z}_2$.

Clearly, both conditions are preserved under our equivalence relation. We will compare these conditions to the property of being admiss-
sible. We will call a matrix $E$ over $\mathbb{Z}_2$, admissible mod 2, if for all $k$ the $\mathbb{Z}_2$-determinant of $E_k$, $|E_k|_2$, is not zero where $E_k$ is as defined above. Of course, if $E$ is admissible mod 2 then $E$ is admissible when viewed as a matrix with integer entries.

**Proposition 1.** Let $E$ be an $n \times n$ matrix over $\mathbb{Z}_2$ and $E^*$ the classical adjoint of $E$ (over $\mathbb{Z}_2$). Then $E$ is admissible mod 2 if and only if $E^*1 = 1$.

**Proof.** The $k$th entry of $E^*1$ is $M_{1k} + M_{2k} + \ldots + M_{nk}$ where $M_{ik} = i$, $k$th cofactor (= minor) of $E$. But this sum is just the cofactor expansion of $|E_k|_2$ along its $k$th column. Hence, $|E_k|_2 = 1$, (i.e. $|E_k|_2 \neq 0$) for all $k$ if and only if $E^*1 = 1$.

We will show that $E$ satisfies both (a) and (b) if and only if $E$ is admissible mod 2. In case $E$ satisfies (a) we often refer to $E$ as having row parity. Clearly $E$ has row parity if and only if $1$ is an eigenvector for $E$ over $\mathbb{Z}_2$. In case $E1 = 0$, $E$ has even row parity, and if $E1 = 1$, then $E$ has odd row parity. We will use $\overline{n}$ to denote $\{1, 2, \ldots, n\}$ when no confusion is possible.

**Theorem 2.** $E$ is admissible mod 2 if and only if (a) and (b) hold for $E$.

**Proof.** The $j$th column of $E$ is the characteristic function on some index set $I \subset \overline{n}$. As such we will call the support of the $j$th column of $E$, $I$.

If $E$ is admissible mod 2 and $I$ is the support of the 1st column of $E$, let $E'$ be the matrix resulting from complementing the 1st column of $E$. Then, the support of the 1st column of $E'$ is $I' = \overline{n} \setminus I$. By performing cofactor expansion of $|E_1|_2$, $|E|_2$ and $|E'|_2$ along their first columns, we see that $|E_1|_2 = 1 = |E|_2 + |E'|_2$. If $|E|_2 = 0$ then $|E'|_2 = 1$ so that $E$ is equivalent to an invertible matrix. Also, by Proposition 1, $EE^*1 = E1 = (\det E)1$, so that $E$ has row parity.

Conversely, it is enough to assume that $E$ is invertible. From this and because of (a), $E1 = 1$. Then $E^*E1 = E^*1 = (\det E)1 = 1$, and $E$ is admissible mod 2 by Proposition 1. ■

**Example 3.** It can be checked that $E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ is admissible, but $E$ does not have row parity so it is not admissible mod 2.

Row parity is easily checked. Any $n \times n$ 0,1-matrix $E$ is equivalent
to a matrix \( E' = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \) where \( F \) is an \((n - 1) \times (n - 1)\) 0,1-matrix and \( I \in Z_2^{n-1} \). Hence to check that (\( \beta \)) holds for \( E \) we need only compute \( |F|_2 \), which is clearly preferable to the computation of \( n \) determinants for admissibility mod 2.

**Lemma 4.** There are \( \prod_{j=0}^{m-1} (2^m - 2^j) \) invertible \( m \times m \) matrices over \( Z_2 \).

**Proof.** To form an invertible \( m \times m \) matrix, we must select \( X_1 \in Z_2^m \setminus \{0\} \) for the first column, \( X_2 \in Z_2^m \setminus \text{span} \{X_1\} \) for the second, \( X_3 \in Z_2^m \setminus \text{span} \{X_1, X_2\} \) for the third, and so on. There are \((2^m - 1) \cdot (2^m - 2) \cdots (2^m - 2^{m-1})\) ways for this selection to occur.

It is desirable to know just how many admissible mod 2 matrices there are. Let \( \mathcal{E} = \{E | \text{E is n \times n, admissible mod 2 and invertible}\} \). Since \( \mathcal{E} \) is finite and is closed under multiplication, \( \mathcal{E} \) is a group. Let \( \mathcal{F} \) be the subgroup of \( \mathcal{E} \) consisting of those \( E \in \mathcal{E} \) with \( E = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \), as above.

**Theorem 5.** (i) \( |\mathcal{F}| = \prod_{j=0}^{n-2} (2^{n-1} - 2^j) \).

(ii) \( |\mathcal{E}| = 2^{n-1} \prod_{j=0}^{n-2} (2^{n-1} - 2^j) \).

(iii) There are \( 2^n \prod_{j=0}^{n-2} (2^{n-1} - 2^j) \) admissible mod 2 matrices.

**Proof.** Any \( E \in \mathcal{F} \) can be expressed as \( E = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \) with \( F \) an \((n - 1) \times (n - 1)\) invertible matrix uniquely determined by \( E \). Since \( E \) has row parity, \( I + F\bar{1} = \bar{1} \), since the first row of \( E \) has parity 1, so that \( I = (F\bar{1})' \) (the complement of \( F\bar{1} \)) is determined by \( F \). Conversely, any \((n - 1) \times (n - 1)\) invertible matrix \( F \) determines the matrix \( \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \in \mathcal{F} \) where \( I = (F\bar{1})' \), so the computation of \( |\mathcal{F}| \) follows from lemma 4.

Any \( E \in \mathcal{E} \) is equivalent to a matrix in \( \mathcal{F} \). Now suppose that \( E \in \mathcal{F} \), and that \( E' \) is a matrix equivalent to \( E \) as the result of complementing the \( j \)th column (only) of \( E \). An in the proof of Theorem 2, \( 1 = |E|_2 + \ldots + |E'|_2 \) so that \( E' \notin \mathcal{E} \). If \( E'' \) is matrix resulting from complementing only one column of \( E' \), then as before \( |E''|_2 + |E'|_2 = 1 \) and \( E'' \in \mathcal{E} \). It fol-
lows that if \( E^{(s)} \) results from \( E \) by complimenting some \( s \) columns of \( E \), then \( E^{(s)} \in \mathcal{E} \) if and only if \( s \) is even.

Let \( a_n \) = number of subsets of \( 2^n \) containing an even number of elements. We have just shown that \( |\mathcal{E}| = a_n |\mathcal{F}| \). Set \( b_n = 2^n - a_n \), and define \( \delta : 2^n \to \mathbb{Z}_2 \) by letting \( \delta(T) \) = remainder of \( \text{card}(T) \) mod 2. Let \( S + T \) denote the symmetric difference of \( S \) and \( T \) so that \( 2^n \) is an abelian group under +. Since \( \text{card}(S + T) = \text{card}(S) + \text{card}(T) - 2 \text{card}(S \cap T) \), \( \delta(S + T) = \delta(S) + \delta(T) \) and \( \delta \) is a homomorphism. Hence \( a_n = b_n = 2^n/2 = 2^{n-1} \).

If \( \mathcal{E}' \) is the set of admissible mod 2 matrices with zero determinant, then the map sending \( E \in \mathcal{E} \) to the matrix \( E' \) formed by complimenting the first column of \( E \), is a bijection. Thus, there are \( 2 |\mathcal{E}| \) admissible mod 2 matrices.

2. In this section we will attempt to convey the role that the matrices \( E \in \mathcal{E} \) play in abelian group theory without involving the group theory.

The use of admissible matrices in classifying a certain class of Butlwe groups (specifically, the \( B(1) \)-groups) was initiated in [FM], and investigated further in [GM]. Other results concerning the same class of groups were obtained earlier in [AV] and [Ri]. For a deeper involvement of the group theory, see the listed references.

The set of isomorphism classes of subgroups of the rationals form a distributive lattice \( \Delta \). Moreover, any finite distributive lattice \( T \) is isomorphic to a sublattice of \( \Delta \) ([IR] or [GU]). Let us fix an isomorphism. Then for any collection \( \tau_1, \ldots, \tau_n \in T \), the \( n \)-tuple \( \tau = (\tau_1, \ldots, \tau_n) \) determines a certain abelian group \( G = G[\tau_1, \ldots, \tau_n] \). The description of \( G \) is not relevant here but the interested reader should consult the cited references (in fact, \( G \) is only determined up to quasi-isomorphism: see below).

Given an \( n \)-tuple \( \tau = (\tau_1, \ldots, \tau_n) \) with \( \tau_i \in T \), and a 0,1-matrix \( E \) we can let \( E \) operate on \( \tau \) as follows: Set \( \tau_i = \bigwedge_{i \in I} \tau_i \) for any \( \phi \neq I \subseteq \mathbb{n} \). If \( I_i \) is the support of the \( i \)-th column of \( E \), define \( \tau E = (\sigma_1, \ldots, \sigma_n) \) where \( \sigma_i = \tau_{I_i} \cup \tau_{I'_i} \) and \( I'_i = \mathbb{n} \setminus I_i \).

We will now summarize some of the results concerning the groups \( G[\tau_1, \ldots, \tau_n] \) in terms of \( \tau \) and our operation \( \tau E \). Two abelian groups \( G \) and \( H \) are called quasi-isomorphic if each is isomorphic to a subgroup of finite index in the other, in which case we write \( G \sim H \).

**Theorem 6.** Let \( \tau = (\tau_1, \ldots, \tau_n) \) and \( \sigma = (\sigma_1, \ldots, \sigma_n) \) with \( \tau_i, \sigma_j \in T \) for all \( i, j \). Furthermore, assume that \( \tau \neq \tau_i \cup \tau_j \), for any proper \( I \subseteq \mathbb{n} \) except \( I = \{i\} \) or \( \{i\}' \), and \( \sigma_j \neq \sigma_j \cup \sigma_j \), for any proper \( J \subseteq \mathbb{n} \) except, \( J = \{j\} \) or \( \{j\}' \). Let \( G = G[\tau_1, \ldots, \tau_n] \) and \( H = G[\sigma_1, \ldots, \sigma_n] \).
(1) [FM] $G \sim H$ if and only if $\tau E \geq \sigma$, and $\sigma F \geq \tau$ for some admissible matrices $E$ and $F$.

(2) [GM] $G \sim H$ if and only if $\tau E \geq \sigma$ and $\sigma F \geq \tau$ for some matrices $E$ and $F$ which are admissible mod 2. In this case, if we choose $E \in \mathcal{E}$, then $F = E^{-1}$ works.

Given $\tau = (\tau_1, \ldots, \tau_n)$ and $G = G[\tau_1, \ldots, \tau_n]$, we will say that $\tau$ is strongly indecomposable if $\tau_i \neq \tau_i \vee \tau_{i'}$ for all $0 \neq I \subset \mathbb{N}$ except $I = \{i\}$ or $\{i\}'$ for each $i$. Following [FM], $\tau$ will be called regular if $\tau_i = \tau_i \vee \bigvee_{j \neq i} \tau_j$ for each $i$, so that $\tau_i = \tau_i \vee \tau_{i'}$, when $I = \{i\}$ or $\{i\}'$. Assuming that $\tau$ is regular and strongly indecomposable, they say that $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a representation type of $G$ if $\sigma$ is regular, strongly indecomposable, they say that $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a representation type of $G$ if $\sigma$ is regular, strongly indecomposable, and $G[\tau_1, \ldots, \tau_n] \sim G[\sigma_1, \ldots, \sigma_n]$. By Theorem 6(2), and a mild computation, we may replace this last condition with the condition that $\tau E = \sigma$ and $\sigma F = \tau$ for two admissible mod 2 matrices $E$ and $F$.

Two representation types $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ are called equivalent if $\sigma = (\gamma_{f(1)}, \ldots, \gamma_{f(n)})$ for some $f$ in the permutation group $S_n$. Fuchs and Metelli ask for an upper bound on the number of nonequivalent representation types of $G[\tau_1, \ldots, \tau_n]$ given $\tau = (\tau_1, \ldots, \tau_n)$, in terms on $n$ (problem 3 in [FM]).

**Theorem 7.** Let $\tau = (\tau_1, \ldots, \tau_n)$ be strongly indecomposable and regular and let $G = G[\tau_1, \ldots, \tau_n]$. There are at most $\prod_{i=0}^{n-2} (2^{n-1} - 2^i)/n!$ nonequivalent representation types of $G$.

**Proof.** Let $\mathcal{R}_\tau$ denote the collection of representation types of $G$. If $\sigma \in \mathcal{R}_\tau$, then $\sigma = \tau E$ for some admissible mod 2 matrix $E$. If $I$ is the support of the $i^{th}$ column of $E$ and $E'$ is formed by complementing the $i^{th}$ column of $E$, then the support of the $i^{th}$ column of $E'$ is $I'$, and for $\delta = \tau E'$, and for $\delta = \tau E'$, $\delta_i = \tau_i \vee \tau_{i'} \vee \bigvee_{j \neq i} \tau_j$. Therefore we may assume that $E \in \mathcal{F}$, and theorem 5(i) implies that $\mathcal{R}_\tau$ has at most $\prod_{i=0}^{n-2} (2^{n-1} - 2^i)$ members.

Let $\mathcal{P} \subset \mathcal{E}$ be the collection of all $n \times n$ permutation matrices. The assignment of $f \in S_n$ to $P_f \in \mathcal{P}$ whose $i, j^{th}$ entry is 1 if and only $f(j) = i$, is a group isomorphism. We will show that $\mathcal{P}$ acts on $\mathcal{R}_\tau$.

If $\sigma \in \mathcal{R}_\tau$, then $\sigma = \tau E$ for some $E \in \mathcal{E}$. Set $\delta = \tau (EP)$ and $\mu = \sigma P$ for $P = P_f \in \mathcal{P}$. For each $j$, since $\sigma$ is regular, $\mu_j = \sigma_{f(j)} \vee \sigma_{\{f(j)\}'} \vee \bigvee_{k \neq i} \sigma_k = \sigma_i$ where $f(j) = i$. But if the $i^{th}$ column of $E$ is $I_i$, then...
\[ \delta_j = \tau_i \lor \tau_{i'} = \sigma_i, \text{ so } \delta = \sigma P = (\sigma_{f(1)}, \ldots, \sigma_{f(n)}) \]. Note that \( \delta \) is strongly indecomposable and regular, and that \( \delta P^{-1} = \sigma \).

Now suppose that \( \tau = \sigma F \) for some \( F \in \mathcal{S} \). We must show that \( \delta(P^{-1}F) = \tau = (\delta P^{-1})F \). Let \( \rho = \delta(P^{-1}F) \) and suppose that the support of the \( k \)th column of \( F \) is \( J_k \). Now \( P^{-1} \) has a 1 in the \( i, j \)th entry if and only if \( f^{-1}(j) = i \), so the support of the \( k \)th column of \( P^{-1}F \) is \( \{ i | i = f^{-1}(j) \} \) for some \( j \in J_k \). Hence \( \rho_k = \delta_{f^{-1}(J_k)} \lor \delta_{f^{-1}(J_k)} \). Also, \( \tau = (\delta P^{-1})F = (\delta_{f^{-1}(1)}, \ldots, \delta_{f^{-1}(n)})F \) has \( \tau_k = \bigwedge_{i \in J_k} \delta_{f^{-1}(i)} \lor \bigwedge_{i \in J_k} \delta_{f^{-1}(i)} = \delta_{f^{-1}(J_k)} \lor \delta_{f^{-1}(J_k)} = \rho_k \). Thus, if \( \sigma P = \delta \), then \( \delta(P^{-1}F) = \tau \) and \( \tau(EF) = \delta \) so that \( \delta \in \mathcal{R}_\tau \). If \( P = P_f \) and \( Q = P_g \) then mimicking the computation given above, we can show that \( (\sigma P)Q = \sigma(PQ) = (\sigma_{f(1)}, \ldots, \sigma_{f(n)}) \) so that \( \mathcal{P} \) acts on \( \mathcal{R}_\tau \).

If \( \sigma \in \mathcal{R}_\tau \) with \( \sigma_i \leq \sigma_j \), and \( i \neq j \), then \( \sigma_i \leq \sigma_j \lor \bigwedge_{k \neq j} \sigma_k = \sigma_{\{j\}} \lor \sigma_{\{j\}^c} \), which contradicts the strong indecomposability of \( \sigma \). Therefore, \( \sigma P = \sigma \) for \( P \in \mathcal{P} \) if and only if \( P \) is the identity matrix. Since \( \mathcal{P} \) acts on \( \mathcal{R}_\tau \) and the orbit of \( \sigma \) is the equivalent class of \( \sigma \) which contains \( n! \) representation types, there are \( |\mathcal{R}_\tau|/n! \) inequivalent representation types.

One could show that \( \prod_{i=0}^{n-2} (2^{n-1} - 2^i)/n! \) is an integer by looking at the representation \( \mathcal{P}_0 \) of \( \mathcal{P} \) in \( \mathcal{F} \). Then show that \( \mathcal{P}_0 \) acts on \( \mathcal{F} \). Clearly this bound is achieved if and only if \( \tau E \) is a representation type of \( \tau \) for any \( E \in \mathcal{S} \), which is an intrinsic property of \( T \) and does not depend, in general, solely on \( n \). Of course when \( n = 3 \), \( \prod_{i=0}^{1} (2^2 - 2^i)/6 = 1 \) so the bound is tight in this case, regardless of \( T \).

**Example 8.** Let \( \tau_1 = \{1, 2, 3\} \), \( \tau_2 = \{2, 3, 4\} \), \( \tau_3 = \{1, 5, 6\} \) and \( \tau_4 = \{4, 5, 6\} \) in \( T = 2^\mathcal{S} \), the power set of \( \mathcal{S} \). It is easy to see that \( \tau = (\tau_1, \tau_2, \tau_3, \tau_4) \) is regular and strongly indecomposable. However

\[ \tau_{\{2, 3\}} \lor \tau_{\{1, 4\}} = (\{2, 3, 4\} \cap \{1, 5, 6\}) \cup (\{1, 2, 3\} \cap \{4, 5, 6\}) = \emptyset \]

while \( \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \) is the column of an admissible mod 2 matrix \( E \in \mathcal{F} \). For example, \( E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \). But \( \tau E_1 = \sigma \) cannot be a representation
type of τ since σ_2 ≤ σ_i for all i so σ cannot be strongly indecomposable.
In this case, there are less than \( \prod_{i=0}^{2}(2^3 - 2^i)/24 = 7 \) representation types of σ.

Three are 7 pertinent matrices from \( \mathcal{F}: E_0 = \text{identity}, \)

\[
E_1, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},
\]

\[
E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.
\]

These are the matrices of concern because no complementing and/or interchanging of columns will transform one into the other. Set \( \tau_5 = \{1, 4\} \) and \( \tau_6 = \{2, 3, 5, 6\} \). Of the vectors \( \tau E_i, i = 0, \ldots, 6 \), only \( \tau E_0 = \tau, \sigma = \tau E_2 = (\tau_6, \tau_5, \tau_1, \tau_4) \) and \( \gamma = \tau E_4 = (\tau_5, \tau_6, \tau_2, \tau_3) \) are representation types of τ. One easily checks that σ and γ are strongly indecomposable and regular, and that

\[
\sigma \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tau = \gamma \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

so that there are 3 representation types of τ.

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