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A lattice of homomorphs

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A Lattice of Homomorphs.

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Preliminary notes.

In this paper all groups are finite and soluble. The homomorph \( h(\mathcal{B}) \) for a boundary \( \mathcal{B} \) consists of all \("\mathcal{B}\)-perfect groups\), namely all those groups that have no \( \mathcal{B} \)-groups among their epimorphic images. The boundary \( b(\mathcal{K}) \) for a homomorph \( \mathcal{K} \) consists of all groups \( G \) such that \( G \notin \mathcal{K} \) and if \( 1 \neq N \leq G \), then \( G/N \notin \mathcal{K} \). The maps \( h \) and \( b \) are mutually inverse bijections between the set of non-empty homomorphs and the set of boundaries. Let \( \mathcal{K} \) be a homomorph. We recall from [4] that the class \( D\mathcal{K} \) of \( \mathcal{K} \) comprises all groups \( G \) such that \( \operatorname{Cov}_\mathcal{K}(G) \neq \emptyset \) namely all those groups that have \( \mathcal{K} \)-covering subgroups. \( D\mathcal{K} \) is also a homomorph. We study in [6] the set

\[
\mathcal{H}(\mathcal{U}) = \{ \mathcal{K} | D\mathcal{K} = \mathcal{U} \}, \text{ where } \mathcal{U} \text{ is a homomorph.}
\]

Those homomorphs \( \mathcal{K} \) such that \( D\mathcal{K} = \mathcal{U} \) behave with regard to \( \mathcal{U} \) in a somewhat similar way to the Schunck classes with regard to the whole universe of soluble groups. The class \( \mathcal{C}(\mathcal{U}) \) (see (2.1) of [6]) is introduced in order to characterize the homomorphs \( \mathcal{K} \) of \( \mathcal{H}(\mathcal{U}) \), when \( \mathcal{H}(\mathcal{U}) = \emptyset \) or \( |\mathcal{H}(\mathcal{U})| = 1 \) and to study the relation of usual containment in \( \mathcal{H}(\mathcal{U}) \). The class \( \mathcal{C}(\mathcal{U}) \) consists of those primitive groups \( G \) in \( \mathcal{U} \) that satisfy:

If \( M \leq X \) and \( X/\operatorname{core}_X M \equiv G \), we have \( M \in \mathcal{U} \) if and only if \( X \in \mathcal{U} \).

Let \( \mathcal{P} \) denote the class of finite soluble primitive groups.

If \( \mathcal{H}(\mathcal{U}) \neq \emptyset \), the minimum in \( \mathcal{H}(\mathcal{U}) \) with regard to the relation of containment is \( \mathcal{R} = h((\mathcal{P} - \mathcal{H}(\mathcal{U})) \cup \mathcal{C}(\mathcal{U})) \) (see [6], (3.3)).

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In this paper we study the relation of strong containment in \( H(U) \) given by

1 DEFINITION. Let \( U \) be a homomorph. Let \( \mathcal{X}, \mathcal{Y} \in H(U) \). We say that \( \mathcal{X} \) is strongly contained in \( \mathcal{Y} \), and write \( \mathcal{X} \ll \mathcal{Y} \) if, for each \( G \in U \) an \( \mathcal{X} \)-covering subgroup of \( G \) is contained in some \( \mathcal{Y} \)-covering subgroup of \( G \).

For a homomorph \( \mathcal{X} \), we denote \( \mathcal{X} := h(b(\mathcal{X}) \cap \mathcal{P}) \). For every group \( G \in D\mathcal{X} \) we have: \( \text{Cov}_{\mathcal{X}}(G) = \text{Cov}_X(G) \) (see [6], (1.8)).

2 LEMMA. Let \( \mathcal{X} \) be a homomorph. We denote
\[
a(\mathcal{X}) := \{ G \in D\mathcal{X} \mid \text{ if } H \in \text{Cov}_{\mathcal{X}}(G), H \cap \text{Soc} G = 1 \}.
\]
We have:

a) \( a(\mathcal{X}) = a(\mathcal{X}) \cap D\mathcal{X} \).
b) \( \mathcal{X} = h(a(\mathcal{X})) \).

PROOF. a) It is evident by the definition.
b) Since \( b(\mathcal{X}) = b(\mathcal{X}) \cap \mathcal{P} \), we have \( b(\mathcal{X}) \subseteq a(\mathcal{X}) \cap D\mathcal{X} = a(\mathcal{X}) \) and therefore \( h(a(\mathcal{X})) \subseteq h(b(\mathcal{X})) = \mathcal{X} \). Since \( \mathcal{X} = h(a(\mathcal{X})) \) (see [2], VI (1.4)) and \( a(\mathcal{X}) \subseteq a(\mathcal{X}) \), we have \( \mathcal{X} = h(a(\mathcal{X})) \subseteq h(a(\mathcal{X})) \).

Let us recall now the following

3 DEFINITION ([5] and [3] (8.2)). Let \( B \subset \mathcal{P} \). We define \( B_0 = B \), and if \( B_i \) has already been defined, let
\[
B_i + 1 = \left\{ (X/C_X(V))[V] \mid H \leq X \leq K \leq G = KF(G) \in B_i, \text{ } H \in \text{Cov}_{h(B_i)}(K), \text{ is } X \text{-composition factor of } F(G) \right\}.
\]
We denote by \( B^* \) the union of all class \( B_i \) previously defined.

In a similar way to (8.3) from [3] we have

4 PROPOSITION. Let \( \mathcal{X} \) be a homomorph and \( B \subset \mathcal{P} \) such that \( B \subseteq a(\mathcal{X}) \). We have that \( B^* \subseteq a(\mathcal{X}) \) (in particular \( a(\mathcal{X})^* = a(\mathcal{X}) \)).

PROOF. Let us prove that \( B_i \subseteq a(\mathcal{X}) \) for every \( i \in \mathbb{N} \). We proceed by induction on \( i \). We have that \( B = B_0 \subseteq a(\mathcal{X}) \). Suppose \( B_i \subseteq a(\mathcal{X}) \). Let \( B \in B_{i+1} \). There exists \( G \in B_i \subseteq a(\mathcal{X}) \), \( Y \leq X \leq K \), \( K \) complement of \( F(G) \), \( H \in \text{Cov}_{h(B_i)}(K) \), \( V, W \), \( X \)-subgroups of \( F(G) \), \( V/W \), \( X \)-composition
of $F(G)$ such that $B = X/C_X(V/W)[(V/W)]$. Since $B_i \subseteq a(\mathcal{C})$, by [1] (2.2), we have $\mathcal{C} \subseteq h(B_i)$, hence there exists $H \in \text{Cov}_a(K)$ such that $H \leq Y$. As $G \in a(\mathcal{C}) \subseteq D\mathcal{C}$, we have $H \in \text{Cov}_a(K) \subseteq \text{Cov}_a(G)$. Besides, it can be confirmed that

$$B = X/C_X(V/W)[(V/W)] \cong XV/C_X(V/W)W.$$ 

By the properties of covering subgroups $H \in \text{Cov}_a(XV)$ and

$$HC_X(V/W)W/C_X(V/W)W \in \text{Cov}_a(XV/C_X(V/W)W),$$

therefore $B \in D\mathcal{C}$. We know from [3] (8.3), that $B \in a(\mathcal{C})$, so we can deduce that $B \in a(\mathcal{C}) \cap D\mathcal{C} = a(\mathcal{C})$.

Below we study the relation $\ll$ in $H(\mathcal{U})$.

5 Proposition. Let $\mathcal{X}, \mathcal{Y} \in H(\mathcal{U})$. We have $\mathcal{X} \ll \mathcal{Y}$ if and only if $\mathcal{X} \ll \mathcal{Y}$.

Proof. $\Rightarrow$ It is evident from that comment before Lemma 2.

$\Rightarrow$ We have $b(\mathcal{Y}) = b(\mathcal{Y}) \cap \mathcal{P}$. By definition of $\ll$ and $a(\mathcal{X})$, we have that $b(\mathcal{Y}) \cap \mathcal{P} = b(\mathcal{Y}) \cap D\mathcal{Y} \subseteq a(\mathcal{X})$. Moreover, $a(\mathcal{X}) \subseteq a(\mathcal{X})$, hence $b(\mathcal{Y}) \subseteq b(\mathcal{X})$ and by [1] (2.2), $\mathcal{X} \ll \mathcal{Y}$.

Since the mapping $\mathcal{C} \rightarrow \mathcal{C}$ from $H(\mathcal{U})$ to the set of Schunck classes is injective (see [6], 3.1), $H(\mathcal{U})$ can be considered a subset of the Schunck classes ordered by $\ll$.

In the examples described in [6] (1.9), (3.8), (3.9), $(H(\mathcal{U}), \ll)$ has a lattice structure. In these examples we have $a(\mathcal{U}) = a(\mathcal{K})$. In this respect, we can say:

6 Proposition. Let $\mathcal{U}$ be a homomorph and $\mathcal{K}$ the minimum for $\subset$ in $H(\mathcal{U})$. The following statements are equivalent:

a) $a(\mathcal{U}) = a(\mathcal{K})$;

b) $a(\mathcal{U})^\subset = a(\mathcal{U})$.

Proof. a) $\Rightarrow$ b) It follows immediately from Proposition 4.

b) $\Rightarrow$ a) By b) we obviously have $a(\mathcal{U})^\subset \cap h(a(\mathcal{U})) = \emptyset$. By [3] (8.4), we have $a(\mathcal{U}) \subseteq a(h(a(\mathcal{U})))$. By [6] (3.3), $h(a(\mathcal{U})) = \mathcal{K}$ and therefore $a(\mathcal{U}) \subseteq a(\mathcal{K})$. 
Besides, \( \mathcal{C}(U) \subseteq U = D\mathcal{M} \) implies \( \mathcal{C}(U) \subseteq \mathcal{C}(\mathcal{M}) \cap D\mathcal{M} = a(\mathcal{M}) \). By [6] (1.7), we have \( a(\mathcal{M}) \subseteq \mathcal{C}(U) \) and therefore the equality.

7 Theorem. Let \( U \) be a homomorph such that \( b(U) \cap \mathcal{P} = \emptyset \). (These homomorphs are known as totally unsaturated).

\((\mathcal{H}(U), \ll)\) is a lattice if and only if \( \mathcal{C}(U)^* = \mathcal{C}(U) \).

Proof. \( \Rightarrow \) By the proposition above and [6] (1.7), it suffices to prove that \( \mathcal{C}(U) \subseteq a(\mathcal{M}) \). Let \( G \in \mathcal{C}(U) \). Let \( \mathcal{K} = h(b(U) \cup \{G\}) \). By [6] (2.3), \( \mathcal{K} \in \mathcal{H}(U) \). Since \( \ll \) implies \( \subseteq \), the infimum of \( \{\mathcal{K}, \mathcal{M}\} \) must be \( \mathcal{M} \).

Thus \( \mathcal{M} \ll \mathcal{K} \), therefore \( \mathcal{M} \ll \mathcal{K} \) and consequently \( b(\mathcal{K}) \subseteq a(\mathcal{M}) \). As \( \{G\} = b(\mathcal{K}) \), we have that

\[ G \in a(\mathcal{M}) \cap U = a(\mathcal{M}) \cap D\mathcal{M} = a(\mathcal{M}). \]

\( \Leftarrow \) Let \( \mathcal{X}, \mathcal{Y} \in \mathcal{H}(U) \). Recall from [5] Theorem A that

\[ \mathcal{X} \land \mathcal{Y} = h((b(\mathcal{X}) \cup b(\mathcal{Y}))^*). \]

By Proposition 6 we have \( \mathcal{C}(U)^* = \mathcal{C}(U) = a(\mathcal{M}) \). Since \( b(\mathcal{X}) \cup b(\mathcal{Y}) \subseteq a(\mathcal{M}) \), by Proposition 4, we have that \( (b(\mathcal{X}) \cup b(\mathcal{Y}))^* \subseteq a(\mathcal{M}) \) and therefore \( b(\mathcal{X} \lor \mathcal{Y}) \subseteq a(\mathcal{M}) \). By [6] (2.3), we have that \( \mathcal{K} = h(b(U) \cup \cup b(\mathcal{X} \lor \mathcal{Y})) \in \mathcal{H}(U) \) and it can easily be confirmed that \( \mathcal{K} = \mathcal{X} \land \mathcal{Y} \).

Now let, \( j = h(a(\mathcal{X}) \cap a(\mathcal{Y})) \). Again by the characterization in [6] (2.3) and (3.1), of the homomorphs in \( \mathcal{H}(U) \) we have that \( Z = j \cap U \in \mathcal{H}(U) \), and \( j = Z \). It can be confirmed that \( Z = \mathcal{X} \lor \mathcal{Y} \).

8 Proposition. Let \( U \) be a totally unsaturated homomorph such that \( (\mathcal{H}(U), \ll) \) is a lattice. For every \( \mathcal{X}, \mathcal{Y} \in \mathcal{H}(U) \) we have:

a) \( \mathcal{X} \land \mathcal{Y} = \mathcal{X} \land \mathcal{Y} \).

b) \( \mathcal{X} \ll Z \neq U \) implies \( \mathcal{X} = Z \) if and only if \( |b(\mathcal{X}) \cap \mathcal{P}| = 1 \).

Proof. a) It is clear from the previous proof that

\[ b(\mathcal{X} \land \mathcal{Y}) \cap \mathcal{P} = b(\mathcal{X} \land \mathcal{Y}). \]

b) \( \Rightarrow \) If \( |b(\mathcal{X}) \cap \mathcal{P}| \neq 1 \), we can have \( \emptyset \neq \mathcal{B} \subset b(\mathcal{X}) \cap \mathcal{P} \subseteq \mathcal{C}(U) \). Now \( Z = h(b(U) \cup \mathcal{B}) \in \mathcal{H}(U) \), \( Z \neq \mathcal{X} \) and \( \mathcal{X} \ll Z \neq U \) in contradiction with the hypothesis.

\( \Leftarrow \) As \( \mathcal{X} = h(b(\mathcal{X}) \cap \mathcal{P}) \), \( \mathcal{X} \) is maximal, hence \( \mathcal{X} \ll \mathcal{Z} \neq \mathcal{S} \) implies \( \mathcal{X} = \mathcal{Z} \) and by Proposition 5 we have the thesis.
REFERENCES
