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A Remark on Global Smooth Solutions for Quasilinear Wave Equations.

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ABSTRACT - In this paper, following a result of S. Klainerman, we prove the existence of global smooth solutions in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$ for quasilinear wave equations with small initial data wich are periodic in at most $n-2$ variables. The proof is based on suitable decay estimates for partially periodic solutions of the homogeneous wave equation.

1. Introduction.

The aim of this paper is to present a result concerning the existence of a global solutions in $\boldsymbol{C}^\infty$ to quasilinear wave equations for initial data and nonlinear perturbations which are 2π-*periodic* with respect to some of the space variables.

Let us consider the initial value problem on $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$:

$$u_{tt} - \Delta u = f(x, u_t, u_{x_1}, \ldots, u_{x_n}), \tag{1.1}$$

$$u(0, x) = \varepsilon u_0(x), \qquad u_t(0, x) = \varepsilon v_0(x); \tag{1.2}$$

where $f: \boldsymbol{R}_x^n \times \boldsymbol{R}_y^{n+1} \to \boldsymbol{R}$ is a $\boldsymbol{C}^\infty$ function which satisfies a condition such as:

$$|f(x, z)| \leq c|z|^{r+1} \quad \textit{for } |z| \leq \delta, \tag{1.3}$$

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where r is an integer $\geqslant 1$, and for every *multi-indices* α, β

$$\sup_{x\in \boldsymbol{R}^n_x, |z|\leqslant\rho} |D^\beta_x D^\gamma_z f(x, z)| < \infty\,, \tag{1.4}$$

when ρ is sufficiently small.

It was proved, under suitable condition on n and r (see in particular [K], [K, P], [T, Y]) that for given initial data $u_0(x)$, $v_0(x)\in \boldsymbol{C}^\infty_0(\boldsymbol{R}^n_x)$ Pb. (1.1), (1.2) admits a unique global solution in $\boldsymbol{C}^\infty$ provided ε is sufficiently small. In [K, P] the proof was based on the $L^p - L^q$ for estimates the solution of the unperturbed equation with $\boldsymbol{C}^\infty_0$ initial data, due to W. von Wahl [W]. Later, in [K], [T, Y] and other papers a more general result was obtained using the so called Γ *and* Ω*-norms.*

Now we assume the periodicity in the first j space variables, with $2\leqslant n-j$, and defining a suitable Ω*-type* norm for smooth functions in $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$, we give a decay estimate for solutions of the homogeneous wave equation wich are periodic in $x_1,\dots,x_j$:

$$|Du(t)|_{L^\infty}\leqslant C_{n,j}(1+t)^{-(n-j-1)/2}\|Du(0)\|_{\tilde{\Omega},[n/2]+3} \tag{1.5}$$

where the $\tilde{\Omega}$*-norms* will be define in § 2.

Using this estimate we obtain, following essentially [K, P], the global existence for smooth solution when ε is sufficiently small.

We can state our main result:

THEOREM 1. *Assume that* (1.3), (1.4) *holds and that* $f(x, z)$ *is* 2π*-periodic in* $x_1,\dots,x_j$. *Then, for any smooth initial data* $u_0(x)$, $v_0(x)$, 2π*-periodic in* $x_1,\dots,x_j$ *and compactly supported with respect to* $x_{j+1},\dots,x_n$ *with* j *satisfying*

$$\frac{n-j-1}{2}>\frac{1}{r}, \tag{1.6}$$

there exists $\varepsilon_0>0$ *such that problem* (1.1), (1.2) *has a unique global smooth solution for* $0\leqslant\varepsilon\leqslant\varepsilon_0$.

If $n-j=3$ *and* $r=1$, *we have the following estimate of the life-span* T_ε *of the solution*

$$T_\varepsilon\geqslant B\exp\{A/\varepsilon\}, \tag{1.7}$$

for some positive constants A, B *and for* ε *sufficiently small.*

If $n-j=2$ *and* $r=2$, *we can say more, namely that:*

$$T_\varepsilon\geqslant B\exp\{A/\varepsilon^2\}. \tag{1.8}$$

REMARK. We recall here that Pb. (1.1), (1.2), without any other assumption on the nonlinear term, does not admit, in general, global solutions periodic in all the space variables. Consider for example the equation

$$u_{tt} - \Delta u = u_t^2 \tag{1.9}$$

then, it easy to see that every solution of (1.9) with constant and positive initial data, blow-up in finite time.

As a special case of Th. 1, taking initial data which are *constant* in the variables $x_1, \dots, x_j$ we re-obtain the results of [K], thus the condition (1.6) is in some sense sharp.

We give here a direct proof of Theorem 1 when the nonlinear term $f(x, Du)$ does not depends explicitely on x, observing that when f depends on $x \in \boldsymbol{R}_x^n$ it is sufficient to use a suitable form of Lemma 2.7 to estimate the Ω-*norms* of the nonlinear term.

REMARK. By the same methods, we can prove global existence in $\boldsymbol{C}^\infty$ for small initial data to the more general equations of the form

$$u_{tt} - \Delta u = \sum_{i,j=1}^{n} a_{i,j}(x, Du)\, u_{x_i, x_j} + f(x, Du) \tag{1.10}$$

where $f(x, z)$ satisfies the same assumptions as before; $a_{i,j}(x, Du)$ satisfies a condition of type (1.4) and a condition of the form

$$|a_{i,j}(x, z)| \leqslant C|z|^r, \qquad |z| \leqslant \delta. \tag{1.11}$$

REMARK. A weaker version of Th. 1, which avoids the use of Γ-*norms*, can be found in [Ma]. More precisely, using merely the Sobolev norms and an appropriate Von Wahl's type estimate for the periodic solutions of the homogeneous wave equation, we are able to prove the conclusion of Th. 1 assuming instead of (1.6)

$$\frac{(n-1)}{2} - j > \frac{1}{r}\left(1 + \frac{1}{r}\right). \tag{1.12}$$

This is the plane of the paper: in § 2 we give the notations and the fundamental decay estimates for partially periodic solutions to the linear wave equations, in § 3 we recall a classical local existence result for the nonlinear wave equation, finally in § 4, 5 we complete the proof of Th. 1.

2. Preliminaries.

Following S. Klainerman [K], we introduce a set of partial differential operators:

$$\partial_0 = -\frac{\partial}{\partial_t}, \qquad \partial_i = \frac{\partial}{\partial_{x_i}} \qquad (x_0 = t), \tag{2.1}$$

$$\begin{aligned} &L_0 = t\partial_t + x_1\partial_1 +, \dots, + x_n\partial_n, \\ &\Omega_{a,b} = x_a\partial_b - x_b\partial_a, \qquad a, b = 0, \dots, n. \end{aligned} \tag{2.2}$$

The first order operators $\Omega_{a,b}$ and L_0 satisfy the following commutation properties with the wave operator $\Box$ in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$:

$$[\Box, \Omega_{a,b}] = 0, \qquad [\Box, L_0] = 2\Box. \tag{2.2a}$$

Moreover we have

$$[L_0, \Omega_{a,b}] = 0, \qquad [L_0, \partial_i] = -\partial_i,$$

$$[\Omega_{a,b}, \Omega_{c,d}] = \eta_{b,c}\Omega_{a,d} + \eta_{a,d}\Omega_{b,c} - \eta_{b,d}\Omega_{a,c} - \eta_{a,c}\Omega_{b,d},$$

$$[\Omega_{a,b}, \partial_c] = \eta_{b,c}\partial_a - \eta_{a,c}\partial_b.$$

Thanks to the commutation properties (2.2*a*), if $\Box u = 0$, then also $L_0 u$ and $\Omega_{a,b}u$ are solutions of the homogeneous wave equation. Thus, the energy identities hold

$$\|D\Omega_{a,b}u(t)\|_{L^2} = \|D\Omega_{a,b}u(0)\|_{L^2}, \qquad \|DL_0u(t)\|_{L^2} = \|DL_0u(0)\|_{L^2}.$$

In the following, these relations will be used to obtain the basic decay estimates for the $\boldsymbol{L}^\infty$*-norm* of $u(t, x)$ (see Lemma 2.6 below).

For semplicity of notations, in the following sections we shall consider our problems in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n \times \boldsymbol{R}_y^m$, where we assume periodicity in $\boldsymbol{R}_y^m$, and compactness in $\boldsymbol{R}_x^n$. Let $u(t, x, y)$: $\boldsymbol{R}_t \times \boldsymbol{R}_x^n \times \boldsymbol{R}_y^m \to \boldsymbol{R}$ be a smooth function, then we define:

$$\begin{aligned} &\nabla_x u = (u_{x_1}, \dots, u_{x_n}), \\ &Du = (u_t, u_{x_1}, \dots, u_{x_n}), \\ &\tilde{D}u = (u_t, u_{x_1}, \dots, u_{x_n}, u_{y_1}, \dots, u_{y_m}), \end{aligned} \tag{2.3}$$

besides we introduce the following sets of first order operators

$$
(2.3a)\quad
\begin{aligned}
&\Omega = (\Omega_{i,j}), \quad 1 \leqslant i < j \leqslant n, \\
&\bar{\Omega} = (\Omega_{a,b}), \quad 0 \leqslant a < b \leqslant n, \\
&\Omega^* = (\partial_i, \Omega_{a,b}), \quad 0 \leqslant i \leqslant n, \ 0 \leqslant a < b \leqslant n, \\
&\tilde{\Omega} = (\Omega^*, \partial_{y_1}, \dots, \partial_{y_m}) = (\partial_t, \partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_m}, \Omega_{a,b}).
\end{aligned}
$$

Taking $u(t, x, y)$ a smooth function compactly supported with respect to the variables $x = (x_1, \dots, x_n)$ and 2π-*periodic* with respect to $y = (y_1, \dots, y_m)$, in corrispondence of one of the sets of operators $\Omega, \bar{\Omega}, \Omega^*, \tilde{\Omega}$, say $A = (A_1, \dots, A_\sigma)$, we define the norm:

$$
(2.4a)\qquad \|u(t)\|_{A,k} = \Big(\sum_{|\alpha| \leqslant k} \|A^\alpha u(t)\|^2_{L^2(\boldsymbol{R}^n_x \times [0, 2\pi]^m)} \Big)^{1/2}
$$

and observe that a different orderings of the operators $A_1, \dots, A_\sigma$, will produce equivalent norms. In the following, we also use the norm

$$
(2.4b)\qquad \|u(t)\|^2_{\Omega^*, k, l} = \sum_{|\alpha| \leqslant k,\ |\beta| \leqslant l} \int_{\boldsymbol{R}^n_x} \int_{[0, 2\pi]^m} |\Omega^{*\alpha} D^\beta_y u(t, x, y)|^2 dx\, dy .
$$

We quote now the fundamental decay estimates, due to S. Klainerman, with respect to the Γ-*norms* (which hold for any smooth function with compact support) referring to [K] for the proof.

LEMMA 2.1. Let $u(t, x)$ be a smooth compactly supported function in the hyperboloid $H^n_+ = \{\bar{\omega} = (t, x) \in \boldsymbol{R}_t \times \boldsymbol{R}^n_x : t^2 - |x| = 1,\ t > 0\}$, then

$$
(2.5)\qquad \sup_{H^n_+} |u(\bar{\omega})| \leqslant C_n \Big(\sum_{|\beta| \leqslant [n/2]+1} \|\bar{\Omega}^\beta u\|^2_{L^2(H^n_+)} \Big)^{1/2}.
$$

LEMMA 2.2. Let $u(x)$ be a smooth function in $\boldsymbol{R}^n$ compactly supported. Then:

$$
(2.6)\quad |u(x)| \leqslant C_n \Big(\frac{1}{|x|} \Big)^{(n-1)/2} \|u\|^{1/2}_{\Omega, [(n-1)/2]+1} \|\partial_r u\|^{1/2}_{\Omega, [(n-1)/2]+1}, \qquad x \neq 0 .
$$

LEMMA 2.3. Let $u(t,x)$ be a smooth function in $\boldsymbol{R}_t \times \boldsymbol{R}^n_x$ compactly supported in $\boldsymbol{R}^n_x$ or vanishing sufficiently fast at infinity, for any fixed

$t \geqslant 0$. Then for any $(t, x) \in \boldsymbol{R}_t \times \boldsymbol{R}_x^n$ such that $t > 0$, $t \geqslant 2|x|$ we have:

$$(2.7) \qquad |u(t, x)|^2 \leqslant C_n (t^2 - |x|^2)^{-(n+1)} \cdot \int_0^{2\rho} \|u(s)\|_{\bar{\Omega}, [n/2]+1} (\|u(s)\|_{\bar{\Omega}, [n/2]+1} + \|L_0 u(s)\|_{\bar{\Omega}, [n/2]+1}) \, ds$$

where $\rho^2 = t^2 - |x|^2$.

The estimates (2.6) and (2.7) are not convenient fot t and x small, thus we must resort to the Ω^**-norms*, (which include also the derivatives $\partial_{x_1}, \dots, \partial_{x_n}$). On the other hand, for our purpose we must avoid the use of the operator $L_0 = t\partial_t + x_1 \partial_{x_1} + \dots + x_n \partial_{x_n}$ because its commutator with the wave operator in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n \times \boldsymbol{R}_y^m$, has the form

$$[\Box, L_0] = 2\Box + 2\partial_{y_1}^2 + \dots + 2\partial_{y_m}^2$$

so that, if we apply L_0 to each term of the equation $\Box u = f(x, y, Du)$ in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n \times \boldsymbol{R}_y^m$ we obtain a *linear* term in the right side:

$$\Box L_0 u = L_0 f(x, y, Du) + [\Box, L_0] u \, .$$

Thus, we need a result of the following type:

LEMMA 2.4. Let $u(t, x)$ be a smooth function in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$ compactly supported in $\boldsymbol{R}_x^n$, then for any $t \geqslant 0$, $x \in \boldsymbol{R}_x^n$:

$$(2.8) \quad |u(t, x)|^2 \leqslant C_n (1+t)^{-(n-1)} (\|u(t)\|^2_{\bar{\Omega}^*, [n/2]+1} + \|\nabla_x u(t)\|^2_{\bar{\Omega}, [(n-1)/2]+1}) + \\ + C_n (1+t)^{-n} \int_0^{2\rho} (\|u(s)\|^2_{\bar{\Omega}, [n/2]+1} + \|Du(s)\|^2_{\bar{\Omega}, [n/2]+1}) \, ds \, ,$$

where $\rho^2 = t^2 - |x|^2$.

PROOF. We shall first prove inequality (2.8) in the case $t > 0$ and $t \geqslant 2|x|$, then we use Lemma 2.2 and the Sobolev embedding theorem to consider the other cases.

Assume that $t > 0$ and $t \geqslant 2|x|$, putting $(t, x) = \rho\bar{\omega}$, where $\bar{\omega} = (\omega_0, \omega) \in \boldsymbol{R} \times \boldsymbol{R}^n$, with $\omega_0^2 - |\omega|^2 = 1$, we have

$$(2.9) \qquad \rho^{n+1} u^2(\rho\bar{\omega}) \leqslant (n+1) \int_0^{\rho} \{u^2(\lambda\bar{\omega}) + \lambda |u(\lambda\bar{\omega})| \, |\partial_\rho u(\lambda\bar{\omega})|\} \lambda^n \, d\lambda \, .$$

Let $dS_{\bar\omega}$ be the area element of the hyperboloid

$$H^n = \{\bar\omega = (\omega_0, \omega): \omega_0^2 - |\omega|^2 = 1\}$$

then $\lambda^n d\lambda\, dS_{\bar\omega}$ is the area element of $\boldsymbol{R}\times\boldsymbol{R}^n$. Thus, integrating (2.9) on

$$\Sigma = \{\bar\omega \in H^n: \omega_0 \geq 2|\omega|\}$$

we easily find

$$(2.10)\qquad \int_\Sigma \rho^{n+1} u^2(\rho\bar\omega)\, dS_{\bar\omega} \leq (n+1) \int \int_{0\leq s^2-|y|^2\leq\rho^2, |y|\leq s/2} \cdot \{u^2(s, y) + (s^2 - |y|^2)^{1/2}|u(s, y)|\,|\partial_\rho u(s, y)|\}\, ds\, dy$$

$$\leq (n+1)\int_0^{2\rho} ds \int_{|y|\leq s/2} \{u^2(s, y) + s|u(s, y)|\,|\partial_\rho u(s, y)|\}\, ds\, dy .$$

Recalling that $\partial_\rho = \cosh(\theta)\partial_t + \sum_{i=1}^n \operatorname{senh}(\theta) x_i/|x|\, \partial_i$, we have

$$|\partial_\rho u(s, y)| \leq \cosh(\theta)|Du(s, y)|$$

then, since $|y| \leq s/2$ in (2.10) we have $\theta \leq 1/2 \log 3$ in the domain of integration and we can write:

$$(2.11)\qquad \rho^{n+1}\int_\Sigma u^2(\rho\bar\omega)\, dS_{\bar\omega} \leq (n+1) \int_0^{2\rho} (\|u(s)\|_{L^2}^2 + Cs\|Du(s)\|_{L^2}\|u(s)\|_{L^2})\, ds$$

from which it follows that:

$$(2.12)\qquad \rho^{n+1} \sum_{|\beta|\leq k} \int_\Sigma \bar\Omega^\beta u^2(\rho\bar\omega)\, dS_{\bar\omega} \leq$$

$$\leq C(1+\rho)\int_0^{2\rho} \left(\|u(s)\|_{\bar\Omega, k}^2 + \sum_{|\beta|\leq k} \|D(\bar\Omega^\beta u)(s)\|_{L^2}^2\right) ds .$$

According to the commutation properties of Ω with ∂, we can easily verify that for every integer $k \geq 0$ there exists $M_k > 0$ such that:

$$(2.13)\qquad \frac{1}{M_k}\|Du(t)\|_{\bar\Omega, k} \leq \sum_{|\beta|\leq k} \|D(\bar\Omega^\beta u)(t)\|_{L^2} \leq M_k \|Du(t)\|_{\bar\Omega, k} .$$

Thus using a localized version of Lemma 2.1 we find that for $t > 0$ and $t \geq 2|x|$:

$$(2.14)\qquad |u(t, x)|^2 \leq$$

$$\leq C_n t^{-(n+1)}(1+t)\int_0^{2\rho} (\|u(s)\|_{\bar\Omega,[n/2]+1}^2 + \|Du(s)\|_{\bar\Omega,[n/2]+1}^2)\, ds .$$

Moreover, by Lemma 2.2 for $x \neq 0$ we have

$$(2.15) \quad |u(t, x)|^2 \leqslant C_n \left(\frac{1}{|x|} \right)^{n-1} (\|u(t)\|^2_{\Omega, [(n-1)/2]+1} + \|\nabla_x u(t)\|^2_{\Omega, [(n-1)/2]+1})$$

and, from the definition of Ω^* it follows:

$$(2.16) \qquad |u(t, x)| \leqslant C\|u(t)\|_{\Omega^*, [n/2]+1}.$$

So combining (2.14) for $t \geqslant 1$, $t \geqslant 2|x|$, (2.15) for $t \geqslant 1$, $t \leqslant 2|x|$ and (2.16) for $0 \leqslant t \leqslant 1$, we obtain (2.8). This completes the proof of Lemma 2.4.

We now apply the previous Lemma to a function u which depends on the variables $y_1, \ldots, y_m$. Let $u(t, x, y)$ be a smooth function in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n \times \boldsymbol{R}_y^m$, compactly supported in $\boldsymbol{R}_x^n$ and 2π*-periodic* in $\boldsymbol{R}_y^m$ for any fixed $t \geqslant 0$. From (2.8), we have:

$$(2.17) \quad |D_y^\beta u(t, x, y)|^2 \leqslant$$

$$\leqslant C_n (1+t)^{-(n-1)} (\|D_y^\beta u(t, \cdot, y)\|^2_{\Omega^*, [n/2]+1} + \|\nabla_x D_y^\beta u(t, \cdot, y)\|^2_{\Omega, [(n-1)/2]+1}) +$$

$$+ C_n (1+t)^{-n} \int_0^{2\rho} (\|D_y^\beta u(s, \cdot, y)\|^2_{\Omega, [n/2]+1} + \|D_y^\beta D u(s, \cdot, y)\|^2_{\Omega, [n/2]+1})\, ds.$$

Thus integrating this expression with respect to y on $[0, 2\pi]^m$ for any multi-index $|\beta| \leqslant [m/2]+1$, by Sobolev immersion theorem we find:

$$(2.18) \quad |u(t, x, y)|^2 \leqslant C_{m,n} (1+t)^{-(n-1)} \cdot$$

$$\cdot \sum \int_{[0, 2\pi]^m} \{\|D_y^\beta u(t, \cdot, y)\|^2_{\Omega^*, [n/2]+1} + \|\nabla_x D_y^\beta u(t, \cdot, y)\|^2_{\Omega, [(n-1)/2]+1}\}\, dy +$$

$$+ C_{m,n} (1+t)^{-n} \cdot$$

$$\cdot \sum \int_0^{2\rho} \int_{[0, 2\pi]^m} \{\|D_y^\beta u(s, \cdot, y)\|^2_{\Omega, [n/2]+1} + \|D_y^\beta D u(s, \cdot, y)\|^2_{\Omega, [n/2]+1}\}\, dy\, ds$$

where the sums are extended to all multi-indices $|\beta| \leqslant [m/2] + 1$.

In conclusion, if we use the norm

$$(2.19)\qquad \|u(t)\|^2_{\Omega^*,k,l} = \sum_{|\alpha|\leqslant k, |\beta|\leqslant l} \int_{R^n_x} \int_{[0,2\pi]^m} |\Omega^{*\alpha} D^\beta_y u(t,x,y)|^2 \, dx\, dy\,,$$

we have proved the following result:

LEMMA 2.5. Let $u(t, x, y)$ be a smooth function compactly supported in $\boldsymbol{R}^n_x$ and 2π*-periodic* in $\boldsymbol{R}^m_y$ for any fixed $t \geqslant 0$, then:

$$(2.20)\qquad |u(t,x,y)| \leqslant C_{m,n}(1+t)^{-(n-1)/2} \cdot \sup_{0\leqslant s\leqslant 2\rho} \left(\|u(s)\|_{\Omega^*,[n/2]+1,[m/2]+1} + \|Du(s)\|_{\Omega^*,[n/2]+1,[m/2]+1}\right).$$

Consider now a solution $u(t, x, y)$ to the homogeneous wave equations on $\boldsymbol{R}_t \times \boldsymbol{R}^n_x \times \boldsymbol{R}^m_y$:

$$(2.21)\qquad \Box u \equiv \partial^2_t u - \partial^2_{x_1} u - \ldots - \partial^2_{x_n} u - \partial^2_{y_1} u - \ldots - \partial^2_{y_m} u = 0\,.$$

According to the commutation properties between the wave operator $\Box$ and $\Omega_{a,b}$, $0 \leqslant a < b \leqslant n$, we have the energy identity for any $t \geqslant 0$,

$$(2.22)\qquad \|\tilde D(\Omega^{*\alpha} D^\beta_y u)(t)\|_{L^2} = \|\tilde D(\Omega^{*\alpha} D^\beta_y u)(0)\|_{L^2},$$

where $\tilde D = (\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_m})$ and, as in (2.13), for any integers k, l there exists $M_{k,l} > 0$ such that

$$(2.23)\qquad \frac{1}{M_{k,l}} \|\tilde D u(t)\|_{\Omega^*,k,l} \leqslant \sum_{|\alpha|\leqslant k, |\beta|\leqslant l} \|\tilde D(\Omega^{*\alpha} D^\beta_y u)(t)\|_{L^2(\boldsymbol{R}^n_x \times [0,2\pi]^m)} \leqslant M_{k,l} \|\tilde D u(t)\|_{\Omega^*,k,l}\,.$$

Hence, combining (2.20), (2.22) and (2.23), we obtain

LEMMA 2.6. Let $u(t, x, y)$ be a smooth solution of the homogeneous wave equation, compactly supported in $\boldsymbol{R}^n_x$ and 2π*-periodic* in $\boldsymbol{R}^m_y$ then, for any $t \geqslant 0$ we have the estimate

$$(2.24)\qquad |\tilde D u(t,x,y)| \leqslant C_{m,n}(1+t)^{-(n-1)/2} \|\tilde D u(0)\|_{\Omega^*,[n/2]+2,[m/2]+1}\,.$$

Hence, setting

$$(2.25)\qquad |u(t)|_{\Omega^*,k,l} = \sum_{|\alpha|\leqslant k, |\beta|\leqslant l} \|\Omega^{*\alpha} D^\beta_y u(t)\|_{L^\infty(\boldsymbol{R}^n_x \times \boldsymbol{R}^m_y)},$$

we have also

$$(2.26)\quad |\tilde{D}u(t)|_{\Omega^* k,\, l} \leq C_{m,\,n,\,k,\,l}(1+t)^{-(n-1)/2} \|\tilde{D}u(0)\|_{\Omega^*,[n/2]+2+k,[m/2]+1+l}.$$

REMARK. In view of the estimate of the nonlinear terms (see Lemma 2.7 below), it is more convenient to use, instead of the norms $\|\cdot\|_{\Omega^*,\,k,\,l}$ the $\|\cdot\|_{\tilde{\Omega},\,k}$ norms. Setting,

$$(2.27)\qquad |u(t)|_{\tilde{\Omega},\,k} = \sum_{|\alpha| \leq k} \|\tilde{\Omega}^\alpha u(t)\|_{L^\infty(\boldsymbol{R}_x^n \times \boldsymbol{R}_y^m)},$$

from (2.24) and (2.27) it is easy to deduce

$$(2.28)\quad |\tilde{D}u(t)|_{\tilde{\Omega},\,k} \leq C_{m,\,n,\,k}(1+t)^{-(n-1)/2} \|\tilde{D}u(0)\|_{\tilde{\Omega},[(m+n)/2]+3+k}.$$

We now estimate the composite functions:

LEMMA 2.7. Let $f\colon \boldsymbol{R}^N \to \boldsymbol{R}$ be a smooth function of $U = (u_1, \ldots, u_N)$ satisfying

$$f(U) = O(|U|^{1+r}) \quad at\ U = 0,$$

where r is an integer ≥ 1; and put

$$(2.29)\qquad \Lambda_k(\rho) = \sum_{|u_i| \leq \rho,\, 1 \leq |\gamma| \leq k+r+1} |D^\gamma f(u_1, \ldots, u_N)|.$$

Then if all the norms appearing at the right side below are bounded, we have

$$(2.30)\ \|f(U(t,\cdot))\|_{\tilde{\Omega},\,k} \leq C_k \Lambda_k(|U(t,\cdot)|_{L^\infty}) |U(t,\cdot)|^r_{\tilde{\Omega},[k/2]} \|U(t,\cdot)\|_{\tilde{\Omega},\,k}.$$

where C_k is a positive constant which depends only on k.

PROOF. When $k = 0$, (2.30) is obvious; if $k \geq 1$ and $1 \leq |\alpha| \leq |k|$, we have

$$(2.31)\quad \tilde{\Omega}^\alpha f(U) =$$

$$= \sum_{|\gamma|=1}^{|\alpha|} D^\gamma f(U) \sum C(\gamma, \omega_{i,j}) \tilde{\Omega}^{\omega_{1,1}} u_1 \ldots \tilde{\Omega}^{\omega_{1,\gamma_1}} u_1 \ldots \tilde{\Omega}^{\omega_{N,1}} u_N \ldots \tilde{\Omega}^{\omega_{N,\gamma_N}} u_N.$$

Where the second sum is extended to all multi-indices $\omega_{1,1}, \ldots, \omega_{1,\gamma_1}; \ldots; \omega_{N,1}, \ldots, \omega_{N,\gamma_N}$ such that $\sum \omega_{i,j} = \alpha$, $|\omega_{i,j}| > 0$ and

$C(\gamma, \omega_{i,j})$ are positive constants. Putting

(2.32) $$\lambda_\gamma(\rho) = \sup_{|u_i| \leq \rho} |D^\gamma f(u_1, \cdot, u_N)|,$$

we then easily compute:

(2.33) $$\|\tilde{\Omega}^\alpha f(U(t, \cdot))\|_{L^2(\boldsymbol{R}^n_x \times [0, 2\pi]^m)} \leq C(\alpha) \sum_{|\gamma|=1}^{|\alpha|} \{\lambda_\gamma(\|U(t, \cdot)\|_{L^\infty}) \times$$

$$\times \sum \|\tilde{\Omega}^{\omega_{1,1}} u_1 \dots \tilde{\Omega}^{\omega_{1,\gamma 1}} u_1 \dots \tilde{\Omega}^{\omega_{N,1}} u_N \dots \tilde{\Omega}^{\omega_{N,\gamma N}} u_N \|_{L^2(\boldsymbol{R}^n_x \times [0, 2\pi]^m)} \},$$

where the second sum is as above. It is easy to see that one at most of the multi-indices $|\omega_{i,j}|$ is greater than or equal to $[|\alpha|/2] + 1$, thus (2.30) follows from (2.33) taking into account that

(2.34) $$|\lambda_\gamma(\rho)| \leq |\rho|^{r+1-|\gamma|} \sum_{|\beta|=r+1} |\lambda_\beta(\rho)|$$

for $|\gamma| \leq r+1$.

3. Local existence for periodic solutions.

Before proving Theorem 1, we recall here a result of *local existence* for solutions which are periodic with respect to m space variables. We refer to [G1], [G2] and [M] for a detailed discussion and proof.

DEFINITION. We denote by for $\boldsymbol{H}^s_{\pi_m}(\boldsymbol{R}^n_x \times \boldsymbol{R}^m_y)$ the space of the functions $g(x, y)$, *2π-periodic* in $y_1, \dots, y_m$, such that

$$g(x, y) \in \boldsymbol{H}^s(\boldsymbol{R}^n_x \times (0, 2\pi)^m).$$

In the same way we define $\boldsymbol{L}^p_{\pi_m}(\boldsymbol{R}^n_x \times \boldsymbol{R}^m_y)$ and $\boldsymbol{W}^{N,p}_{\pi_m}(\boldsymbol{R}^n_x \times \boldsymbol{R}^m_y)$.

Let $f(x, y, z)$ be a $\boldsymbol{C}^\infty$ function on $\boldsymbol{R}^n_x \times \boldsymbol{R}^m_y \times \boldsymbol{R}^{m+n+2}_z$, such that

(3.1) $$f(x, y, 0) \in \boldsymbol{H}^\infty_{\pi_m}(\boldsymbol{R}^n_x \times \boldsymbol{R}^m_y),$$

(3.2) $$\sup_{(x,y) \in \boldsymbol{R}^n_x \times \boldsymbol{R}^m_y, |z| \leq \rho} |D^\alpha f(x, y, z)| < \infty \quad \forall \alpha; \ \forall \rho \geq 0.$$

Then, the following result of local solvability for the nonlinear wave equation holds:

PROPOSITION 3.1. *Assume that the function $f(x, y, u, \tilde{D}u)$, satisfies (3.1), (3.2); then for any initial data $u_0(x, y)$, $v_0(x, y) \in \boldsymbol{H}^\infty_{\pi_m}(\boldsymbol{R}^n_x \times \boldsymbol{R}^m_y)$ there exists $T > 0$ (depending on u_0 and v_0) such that the quasilinear* *wave equation*

$$(3.3) \qquad u_{tt} - \Delta u = f(x, y, u, \tilde{D}u)$$

has a unique (local) solution $u(t, x, y) \in C^\infty([0, T); H^\infty_{\pi_m}(R^n_x \times R^m_y))$.

Such solution has the finite speed of propagation property with respect to $x \in R^n_x$. *Moreover, if f does not depend explicitely on u and satisfies a condition such as*

$$(3.4) \qquad |f(x, y, \tilde{D}u)| \leqslant c|\tilde{D}u|^{r+1}$$

for $|\tilde{D}u|$ *sufficiently small and r integer* $\geqslant 1$, *the life span of the solution can be estimate on below as*

$$(3.5) \qquad T \geqslant C(\|\tilde{D}u(0, \cdot, \cdot)\|_{H^s_{\pi_m}})^{-r}$$

for $s > [(m+n)/2] + 1$.

Remark. (3.5) easily follows from the energy estimates for the quasilinear wave equations, using the inequalities of Gagliardo and Nirenberg in order to estimate the nonlinear terms (see [N]).

4. Estimates for the solutions of the nonlinear wave equations

We now use the result of § 2 in order to estimate the solution $u(t, x)$ of the quasilinear Cauchy problem in $R_t \times R^n_x \times R^m_y$:

$$(4.1) \qquad u_{tt} - \Delta u = f(\tilde{D}u)$$

$$(4.2 \qquad u(0, x, y) = u_0(x, y), \qquad u_t(0, x, y) = v_0(x, y)$$

where $f\colon R^{m+n+1} \to R$ is a smooth function such that:

$$(4.3) \qquad |f(\tilde{D}u)| \leqslant C|\tilde{D}u|^{r+1} \quad \text{for } |\tilde{D}u| \leqslant \delta$$

r is an integer $\geqslant 1$, and $u_0(x, y)$, $v_0(x, y)$ are smooth functions compactly supported in R^n_x and 2π*-periodic* in R^m_y. Let $u(t, x, y)$ be a local solution of (4.1), (4.2), belonging to $C^\infty([0, T]; H^\infty_{\pi_m}(R^n_x \times R^m_y))$, then by the Duhamel's principle we have

$$(4.4) \qquad u(t, x, y) = w(t, x, y) + \int_0^t G(s, t, x, y)\, ds\,,$$

where $w(t, x, y)$ is the solution of the homogeneous problem:

$$w_{tt} - \Delta w = 0\,, \tag{4.5}$$

$$w(0, x, y) = u_0(x, y), \qquad w_t(0, x, y) = v_0(x, y), \tag{4.6}$$

while for any fixed $s \in [0, T)$, $G(s, t, x, y)$ is the solution to the linear problem:

$$g_{tt} - \Delta g = 0 \tag{4.7}$$

$$g(s, x, y) = 0\,, \qquad g_t(s, x, y) = f(\tilde{D}u(s))\,. \tag{4.8}$$

By (2.28) we can easily see that, for every $k \geq 0$, we have

$$|\tilde{D}u(t)|_{\tilde{\Omega}, k} \leq C_k(1 + t)^{-(n-1)/2} \|\tilde{D}w(0)\|_{\tilde{\Omega}, \eta+k} + \\ + C_k \int_0^t (1 + (t - s))^{-(n-1)/2} \|f(\tilde{D}u(s))\|_{\tilde{\Omega}, \eta+k}\, ds\,, \tag{4.9}$$

where $\eta = [(m + n)/2] + 3$.

Now applying Lemma 2.7 and observing that $\|v\|_{L^\infty} \leq |v|_{\tilde{\Omega},[l/2]}$ we find, for every integer $l \geq 0$,

$$\|f(\tilde{D}u(s))\|_{\tilde{\Omega}, l} \leq C_l \Lambda_l(|\tilde{D}u(s)|_{\tilde{\Omega},[l/2]})\, |\tilde{D}u(s)|^r_{\tilde{\Omega},[l/2]} \|\tilde{D}u(s)\|_{\tilde{\Omega}, l} \tag{4.10}$$

where $\Lambda_l(\rho)$ is given by (2.29), and using energy estimates we can estimate $\|\tilde{D}u(s)\|_{\tilde{\Omega}, l}$. Putting:

$$E_l^2(t) = \sum_{|\alpha| \leq l} \|\tilde{D}(\tilde{\Omega}^\alpha u)(t)\|^2_{L^2(\boldsymbol{R}^n_x \times [0, 2\pi]^m)}\,, \tag{4.11}$$

by the commutation properties of $\tilde{\Omega}$ and $\Box$ we find:

$$\frac{d}{dt} E_l^2(t) = \sum_{|\alpha| \leq l} \int_{\boldsymbol{R}^n_x \times [0, 2\pi]^m} \tilde{\Omega}^\alpha f(\tilde{D}u(t)) \frac{\partial}{\partial t} \tilde{\Omega}^\alpha u(t)\, dx\, dy \tag{4.12}$$

from (4.10) and (4.12) it follows:

$$\frac{d}{dt} E_l(t) \leq C_l \Lambda_l(|\tilde{D}u(t)|_{\tilde{\Omega},[l/2]})\, |\tilde{D}u(t)|^r_{\tilde{\Omega},[l/2]} \|\tilde{D}u(t)\|_{\tilde{\Omega}, l}\,. \tag{4.13}$$

Since as in (2.13) there exists $M_l > 0$ such that

$$\frac{1}{M_l} \|\tilde{D}u(t)\|_{\tilde{\Omega}, l} \leq E_l(t) \leq M_l \|\tilde{D}u(t)\|_{\tilde{\Omega}, l}\,, \tag{4.14}$$

applying Gronwall's lemma we have

$$(4.15)\quad \|\tilde{D}u(t)\|_{\tilde{\Omega},l} \leqslant \|\tilde{D}u(0)\|_{\tilde{\Omega},l} \exp\left\{\int_0^t C_l \Lambda_l(|\tilde{D}u(s)|_{\tilde{\Omega},[l/2]})|\tilde{D}u(s)|^r_{\tilde{\Omega},[l/2]}\,ds\right\}.$$

From, (4.9), (4.10) and (4.15) it follows:

$$(4.16)\quad |\tilde{D}u(t)|_{\tilde{\Omega},k} \leqslant C_k(1+t)^{-(n-1)/2}\|\tilde{D}w(0)\|_{\tilde{\Omega},\eta+k} +$$

$$+ C_k\int_0^t\Bigg((1+(t-s))^{-(n-1)/2}\Lambda_{\eta+k}(|\tilde{D}u(s)|_{\tilde{\Omega},[(\eta+k)/2]})|\tilde{D}u(s)|^r_{\tilde{\Omega},[(\eta+k)/2]}\times$$

$$\times\|\tilde{D}u(0)\|_{\tilde{\Omega},\eta+k}\exp\left\{\int_0^s \Lambda_{\eta+k}(|\tilde{D}u(\tau)|_{\tilde{\Omega},[(\eta+k)/2]})|\tilde{D}u(\tau)|^r_{\tilde{\Omega},[(\eta+k)/2]}\,d\tau\right\}\Bigg)\,ds.$$

Setting now:

$$(4.17)\quad Y_k(t) = \sup_{0\leqslant s\leqslant t}(1+s)^{(n-1)/2}|\tilde{D}u(s)|_{\tilde{\Omega},k}$$

and observing that $[(\eta+k)/2] \leqslant k$ as soon as

$$(4.17)\quad k \geqslant 2\eta = 2\left[\frac{m+n}{2}\right]+6,$$

we find:

$$(4.18)\quad |\tilde{D}u(t)|_{\tilde{\Omega},[(\eta+k)/2]} \leqslant |\tilde{D}u(t)|_{\tilde{\Omega},k} \leqslant (1+t)^{-(n-1)/2}Y_k(t)$$

if k satisfies (4.17). Finally from (4.16), (4.18) it follows that:

$$(4.19)\quad Y_k(t) \leqslant C_k\|\tilde{D}w(0)\|_{\tilde{\Omega},\eta+k} + C_k\|\tilde{D}u(0)\|_{\tilde{\Omega},\eta+k}\sup_{0\leqslant\xi\leqslant t}(1+\xi)^{(n-1)/2}\cdot$$

$$\cdot\int_0^\xi\Bigg\{(1+(\xi-s))^{-(n-1)/2}(1+s)^{-((n-1)/2)r}\Lambda_{\eta+k}(Y_k(s))Y_k(s)^r\times$$

$$\times\exp\left\{\int_0^s(1+\tau)^{-((n-1)/2)r}\Lambda_{\eta+k}(Y_k(\tau))Y_k(\tau)^r\,d\tau\right\}ds\Bigg\}.$$

In conclusion, we have proved the following:

LEMMA 4.1. Let $u(t, x, y)$ be a local solution of Problem (4.1), (4.2), belonging to $C^\infty([0, T]; H^\infty_{\pi_m}(R^n_x \times R^m_y))$ (which exists by Proposition 3.1), then for $0 \leqslant t \leqslant T$ and $k \geqslant 2[(m+n)/2] + 6$, the following estimate holds

$$(4.20) \qquad Y_k(t) \leqslant C_k \|\tilde{D}w(0)\|_{\tilde{\Omega}, \eta+k} + C_k \|\tilde{D}u(0)\|_{\tilde{\Omega}, \eta+k} \Lambda_{\eta+k}(Y_k(t)) \cdot Y_k(t)^r \times$$

$$\times \sup_{0 \leqslant \xi \leqslant t} (1+\xi)^{(n-1)/2} \int_0^\xi \Bigg\{ (1 + (\xi - s))^{-(n-1)/2} (1+s)^{-((n-1)/2)r} \times$$

$$\times \exp \left[\Lambda_{\eta+k}(Y_k(t)) \cdot Y_k(t)^r \int_0^s (1+\tau)^{-((n-1)/2)r} d\tau \right] ds \Bigg\},$$

where $w(t, x, y)$ is the solution of the homogeneous problem (4.5), (4.6); $\Lambda_{\eta+k}$ is a nondecreasing continuous function which depends only on $f(x, y, z)$ (see (2.29)), and k while

$$Y_k(t) = \sup_{0 \leqslant s \leqslant t} (1+s)^{(n-1)/2} |\tilde{D}u(s)|_{\tilde{\Omega}, k}.$$

5. Proof of Theorem 1.

Applying (4.20), we now prove that the (local) solution of Problem (1.1), (1.2) is uniformly bounded with respect to t, as $t \to \infty$, provide ε is sufficiently small. We use the following

REMARK 5.1. Given $h(y) \geqslant 0$ a continuous non decreasing function, let us consider the function

$$(5.1) \qquad H(y) = C\varepsilon(1 + h(y) y^r \gamma \exp\{h(y) y^r \gamma\}) - y, \quad (y \geqslant 0;\ C, \varepsilon > 0)$$

where r is an integer $\geqslant 0$. Then for every $\alpha > C$ and $\varepsilon_0 > 0$, there exists $A = A(\alpha, \varepsilon_0) > 0$, such that if $\gamma \leqslant A/\varepsilon^r$ and $0 < \varepsilon \leqslant \varepsilon_0$, then $H(y)$ vanish at some point of $(0, \alpha\varepsilon)$.

PROOF. Observing that $H(0) = C\varepsilon > 0$, we have, for $\gamma \leqslant A/\varepsilon^r$

$$(5.2) \qquad H(\alpha\varepsilon) \leqslant C\varepsilon(1 + h(\alpha\varepsilon)\alpha^r A \exp\{h(\alpha\varepsilon)\alpha^r A\}) - \alpha\varepsilon,$$

and hence putting

$$\bar{h} = \sup_{0 \leqslant \varepsilon \leqslant \varepsilon_0} h(\alpha\varepsilon),$$

it follows that

$$H(\alpha\varepsilon) \leqslant C\varepsilon(1 + \bar{h}\alpha^r A \exp\{\bar{h}\alpha^r A\}) - \alpha\varepsilon. \tag{5.3}$$

Thus if A is sufficiently small (with respect to α and ε_0), we have $H(\alpha\varepsilon) < 0$ for some ε, $0 < \varepsilon \leqslant \varepsilon_0$.

We can now prove Theorem 1.

If $(n-1)/2 > 1/r$, there exists $\gamma \geqslant 0$ such that

$$\int_0^\xi (1+\tau)^{-((n-1)/2)r} d\tau \leqslant \gamma, \tag{5.4}$$

$$(1+\xi)^{(n-1)/2} \int_0^\xi (1+(\xi - s))^{-(n-1)/2} (1+s)^{-((n-1)/2)r} ds \leqslant \gamma, \tag{5.5}$$

for every $\xi \geqslant 0$.

Now, since for the solution of (1.1), (1.2) one has $\|\tilde{D}w(0)\|_{\bar{\Omega}, \eta+k} + \|\tilde{D}u(0)\|_{\bar{\Omega}, \eta+k} \leqslant C\varepsilon$, provided $\varepsilon > 0$ is sufficiently small, by Remark 5.1 and estimate (4.20) we see that $Y_k(t)$ is uniformly bounded (independently of t) provided ε is sufficiently small. From the local existence theorem 3.1 it follows that problem (1.1), (1.2) has a global solution.

If $n = 3$ and $r = 1$, the integrals in (5.4), (5.5) can be estimate by the function

$$\gamma = C \log(1+\xi), \tag{5.6}$$

so, by Lemma 4.1 and Remark 5.1 the solution is uniformly bounded provided that

$$C \log(1+t) \leqslant A/\varepsilon. \tag{5.7}$$

Thus the life-span T_ε of the solution satisfies

$$T_\varepsilon \geqslant B \exp\{A/\varepsilon\}, \tag{5.8}$$

for $\varepsilon > 0$ sufficiently small and A and B positive constants.

Finally if $n = 2$ and $r = 2$ we obtain for the integrals in (5.4) and (55) the same bound as in the case $n = 3$, $r = 1$, and the life-span now satisfies

(5.9) $$T_\varepsilon \geqslant B \exp\{A/\varepsilon^2\}.$$

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