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Rendiconti del Seminario Matematico della Università di Padova, tome 91 (1994), p. 251-263

<http://www.numdam.org/item?id=RSMUP_1994__91__251_0>
Infinitely Many Spacelike Periodic Trajectories on a Class of Lorentz Manifolds.

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ABSTRACT - Let us consider $\mathbb{R}^4$ equipped with a Lorentzian tensor $g$ with signature $(+, +, +, -)$. In this paper we prove, under suitable assumptions on $g$, the existence of infinitely many spacelike geodesics $z(s) = (x(s), t(s))$ with the periodicity conditions $x(s + 1) = x(s), t(s + 1) = t(s) + T$ ($T > 0$) on the Lorentz manifold $(\mathbb{R}^4, g)$.

1. Introduction.

Let us consider the manifold $(\mathbb{R}^4, g)$, where $g(z) = g(x, t)$ is a Lorentz tensor on $\mathbb{R}^4$, with signature $(+, +, +, -)$. Let $z(s) = (x(s), t(s))$ be a geodesic on $(\mathbb{R}^4, g)$, and suppose that $t(0) = 0$, and there exist $\sigma, T > 0$ such that $x(s + \sigma) = x(s), t(s + \sigma) = t(s) + T$ for every $s \in \mathbb{R}$. Then we shall say that $z$ is a $\sigma$-periodic $T$-trajectory on $(\mathbb{R}^4, g)$. Moreover, if $z$ is a geodesic, there exists $E_z \in \mathbb{R}$ such that $g(z(s))\{\dot{x}(s), \dot{t}(s)\} = E_z$, and $z$ called spacelike, null or timelike if $E_z > 0$ or, respectively, $E_z = 0$, or $E_z < 0$ (see [14], p. 69).

Suitable Lorentz manifolds are used in Relativity theory in order to describe the physical space-time. Then, timelike (or, respectively, null) periodic trajectories corresponds to periodic orbits of a particle of positive mass (or, respectively, of a light ray). Spacelike geodesics are not trajectories of particles, but they are important in order to study geometrical properties of a semiriemannian manifold.

Some multiplicity results for timelike periodic trajectories on $(\mathbb{R}^4, g)$ are given, for instance, in [5] and [9] under the assumption that

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Work supported by MURST and by GNAFA of CNR.
the gravitational field vanish at infinity, so that \( g \) tends to the Minkowski metric at infinity (see Remark 1.3 below for further informations).

In this paper we consider a completely different behavior at infinity for \( g \), and we are able to prove that, for any \( T > 0 \), there are infinitely many spacelike 1-periodic \( T \)-trajectories on the semiriemannian manifold \((\mathbb{R}^4, g)\).

Let \( \{g_{ij}\}_{i,j=1,\ldots,4} \) be the components of \( g \). We suppose that \( g \) not depend to the time, \( g_{ij} = g_{ji} \in C^1(\mathbb{R}^3, \mathbb{R}) \), and \( g_{i4} = 0 \) for \( i = 1, 2, 3 \). We set, for simplicity, \( \alpha = \{\alpha_{ij}\}_{i,j=1,2,3} = \{g_{ij}\}_{i,j=1,2,3} \), and \( \beta = -g_{44} \), so that we have, for every \( x \in \mathbb{R}^3 \) and every \( \begin{pmatrix} \xi \\ \tau \end{pmatrix} \in \mathbb{R}^4 \):

\[
g(x)\begin{pmatrix} \xi \\ \tau \end{pmatrix} = \alpha(x)[\xi, \xi] - \beta(x) \tau^2.
\]

Moreover we assume that there exist \( \alpha_0, \alpha_1, R > 0, p > 2 \) and \( q \in ]0, p-2[ \) such that for every \( x, \xi \in \mathbb{R}^3 \):

\[
\alpha(x)[\xi, \xi] \geq \alpha_0 |\xi|^2,
\]

\[
(q\alpha(x) - \alpha'(x)(x))[\xi, \xi] \geq \alpha_1 |\xi|^2,
\]

\[
p|\beta(x)| \leq (\beta'(x)|x|) \quad \text{if} \quad |x| \geq R,
\]

\[
0 < \beta_0 \equiv \beta(0) = \min_{\mathbb{R}^3} \beta,
\]

\[
\lim_{|x| \to 0} \frac{\beta(x) - \beta_0}{|x|^2} = 0,
\]

\[
\alpha(x) = \alpha(-x), \quad \beta(x) = \beta(-x).
\]

Then we have the following theorem.

**Theorem 1.1.** If (1.1)-(1.6) are satisfied, then, for every \( T > 0 \), there exist infinitely many spacelike 1-periodic \( T \)-trajectories on \((\mathbb{R}^4, g)\).

**Remark 1.1.** If \( x_0 \in \mathbb{R}^3 \) and \( \beta'(x_0) = 0 \), it is easy to check that \( z(s) = (x_0, Ts) \) is a trivial periodic trajectory. We shall see later that the trajectories given by Theorem 1.1 are not trivial, and are geometrically distinct.
REMARK 1.2. Condition (1.3) is a sort of superquadraticity condition at infinity. It has been introduced by P. H. Rabinowitz in the theory of Hamiltonian systems. (1.4) implies that there exists $c_1 > 0$ such that, for every $x \in \mathbb{R}^3$, with $|x| \geq R$:

\begin{equation}
\beta(x) \geq c_1 |x|^p.
\end{equation}

Condition (1.3) means that 
\[ \sum_{i,j=1}^{3} [q \alpha_{ij}(x) - (x'_{ij}(x)|x|)] \xi_i \xi_j \geq \alpha_1 |\xi|^2; \]
it is satisfied, for instance, if 
\[ \alpha(x) = \{\delta_{ij}\}_{i,j=1,2,3}. \]
Moreover, because of (1.3), there exists $c_2 > 0$ such that

\begin{equation}
\|\alpha(x)\| \leq c_2 |x|^q
\end{equation}

for $|x| \geq 1$. In fact, let $x \in \mathbb{R}^3$ with $|x| \geq 1$. Since

\[ d(t^{-q} \alpha(tx/|x|)[\xi, \xi])/dt \leq 0, \]

we have

\[ |x|^{-q} \alpha(x)[\xi, \xi] \leq \alpha(x/|x|)[\xi, \xi] \leq c_2 |\xi|^2 \quad \text{where} \quad c_2 = \max_{|y| = 1} \|\alpha(y)\|. \]

REMARK 1.3. The problem of geodesics for a Lorentz manifold $(M, g)$ has been recently studied by many authors (see [2]-[5], [7]-[12]). If particular, in the papers [5],[9], are given multiplicity results for timelike periodic trajectories on $(\mathbb{R}^4, g)$ under the assumption $\beta(x)$ bounded.

The main difficult in the variational approach of this kind of problems is that the action functional

\[ \int g(z)[\dot{z}, \dot{z}] = \int \alpha(x)[\dot{x}, \dot{x}] - \int \beta(x) \dot{t}^2 \]

is strongly indefinite, i.e. it is not of the form identity + compact, even «modulo compact perturbations». In order to avoid this difficult, we use the convexity of the functional with respecto to $t$ and search for the critical points of a functional $f$ depending only on $x$.

If $\beta(x)$ is bounded as in [9] (or it is subquadratic), the functional $f$ is bounded from below, and satisfies easily the Palais-Smale compactness condition. In our case $f$ is unbounded, so we need some linking argument; moreover more care is required in order to prove compactness conditions.

In Section 2 we expose the functional framework and we prove the compactness condition using assumptions (1.1)-(1.5). Then we prove Theorem 1.1 with a mountain pass argument by using (1.6).
2. Proof of the results.

In the following we assume that (1.1)-(1.5) hold. Let us consider a geodesic \( z(s) = (x(s), t(s)) \) on \((\mathbb{R}^4, g)\); then \( z \) satisfies the geodesic equations:

\[
\frac{d}{ds} \left[ \alpha(x) \dot{x} \right] = \frac{1}{2} \left( \alpha'(x) [\dot{x}, \dot{x}] - \beta'(x) \dot{t}^2 \right),
\]

\[
\frac{d}{ds} [\beta(x) \dot{t}] = 0 .
\]

If \( z \) is a \( \sigma \)-periodic \( T \)-trajectory, we shall call the minimal period of \( x \), the minimal period of \( z \). Notice that, if \( z_1 = (x_1, t_1) \) and \( z_2 = (x_2, t_2) \) are \( \sigma \)-periodic \( T \)-trajectories on \((\mathbb{R}^4, g)\), with \( z_1 \neq z_2 \), then \( z_1 \) and \( z_2 \) are geometrically distinct.

In fact, if \( z_2(s) = z_1(\varphi(s)) \) for some reparametrization \( \varphi(s) \), from geodesic equations we have \( \varphi(s) = as + b \) for some \( a, b \in \mathbb{R} \) (see [14], p. 69), so that \( t_2(s) = t_1(as + b) \). Since \( \dot{t}_1(s) \neq 0 \) for any \( s \in \mathbb{R} \), from \( t_1(0) = 0 = t_2(0) = t_1(b) \), we have \( b = 0 \), and from \( t_1(as + \alpha) = t_2(s + \sigma) = t_2(s) + T = t_1(as + \sigma) \), we have \( \alpha = \sigma = a = 1 \), which is impossible.

In particular, if \( z_1 \) and \( z_2 \) have not the same minimal period, then its are geometrically distinct.

**Remark 2.1.** We observe now that, if \( z(s) = (x(s), t(s)) \) is a \( k^{-1} \)-periodic \( Tk^{-1} \)-trajectory, \( x \) and \( t \) are also 1-periodic and \( t(s + 1) = t(s) + T \). Infact, it is easy to check that \( t(s + 1) = t(s + (k - 1)/h + Th/k \) for every \( h = 1, \ldots, k \); then \( z \) is a 1-periodic \( T \)-trajectory on \((\mathbb{R}^4, g)\), with minimal period less or equal to \( 1/k \). So, in order to prove Theorem 1.1, we can show that there exists \( k_0 \in N \) such that, for every \( k \in N \) with \( k \geq k_0 \), there exists a \( k^{-1} \)-periodic \( Tk^{-1} \)-trajectory \( z(s) = (x(s), t(s)) \), with \( \dot{x} \neq 0 \).

Let \( k \in N \) be free for the moment, and let us consider the functional

\[
I(x, \eta) = \int_0^{1/k} \alpha(x) [\dot{x}, \dot{x}] \, ds - \int_0^{1/k} \beta(x)(T/k + \eta)^2 \, ds ,
\]

defined on \( H^{1,2}(S^{1/k}, \mathbb{R}^3) \times L_0(S^{1/k}, \mathbb{R}) \), where \( H^{1,2}(S^{1/k}, \mathbb{R}^3) \) is the Sobolev space of \( k^{-1} \)-periodic functions \( x: \mathbb{R} \rightarrow \mathbb{R}^3 \) with
$x, \dot{x} \in L^2([0, 1/k])$, and

$$L_0(S^{1/k}, \mathbb{R}) = \left\{ \gamma \in L^2(S^{1/k}, \mathbb{R}) \mid \int_0^{1/k} \gamma \, ds = 0 \right\}.$$  

It is easy to check that, if $(x, \eta)$ is a critical point of $I$, then $z(s) = (x(s), t(s))$, where $t(s) = Ts/k + \int_0^s \eta \, ds$ is a critical point of the action functional

$$\int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] \, ds - \int_0^{1/k} \beta(x) \dot{t}^2 \, ds;$$  

so, it is a 1-periodic $T$-trajectory on $(\mathbb{R}^4, g)$, with minimal period less or equal to $1/k$ (see Remark 2.1).

Notice that, because of (1.4), for every $x \in H \equiv H^{1,2}(S^{1/k}, \mathbb{R}^3)$, the functional $\gamma \mapsto \int_0^{1/k} \beta(x)(T/k + \gamma)^2 \, ds$ is strictly convex, so it possess a unique minimum point $\eta_x \in L_0(S^{1/k}, \mathbb{R})$. Let $f : H \to \mathbb{R}$ be the functional

$$f(x) = \int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] \, ds - \int_0^{1/k} \beta(x)(T/k + \eta_x)^2 \, ds + \frac{\beta_0 T^2}{k^3}.$$  

**Lemma 2.2.** The function $x \mapsto \eta_x$ is continuous from $H$ to $L_0(S^{1/k}, \mathbb{R})$; moreover $f \in C^1(H, \mathbb{R})$ and

$$\langle f'(x), y \rangle = \left\langle \frac{\partial I}{\partial x}(x, \eta_x), y \right\rangle,$$

so that, $x \in H$ is a critical point of $f$ if and only if $(x, \eta_x)$ is a critical point of $I$.

**Proof.** The proof is contained in [9]. We recall it for the reader convenience. First of all we observe that $\int_0^{1/k} \beta(x)(T/k + \eta_x) \eta_x \, ds = 0$, because of $\eta_x$ is a critical point of the functional $\eta \mapsto \int_0^{1/k} \beta(x)(T/k + \gamma)^2 \, ds.$
So \( \int_0^{1/k} \beta(x) \eta_x^2 ds = -(T/k) \int_0^{1/k} \beta(x) \eta_x ds \), and then

\[
\| \eta_x \| \leq \frac{T \| \beta(x) \|_{\infty}}{k \beta_0}.
\]

Now, let \( x, y \in H \). Clearly

\[
I(x, \eta_y) - I(y, \eta_y) \leq f(x) - f(y) \leq I(x, \eta_x) - I(y, \eta_x),
\]

and \( I(x, \eta_x) - I(y, \eta_x) \to 0 \) as \( y \to x \). Moreover, since

\[
I(x, \eta_y) - I(y, \eta_y) = \int_0^{1/k} \alpha(x)[\dot{x}] - \alpha(y)[\dot{y}] ds - \int_0^{1/k} (\beta(x) - \beta(y))(T/k + \tau) ds,
\]

using (2.1) we get \( I(x, \eta_y) - I(y, \eta_y) \to 0 \) as \( y \to x \), so \( f \) is continuous.

We prove now that \( x \mapsto \eta_x \) is continuous. Infact, arguing by contradiction, we suppose that there exist \( x \in H \), \( (x_n) \subset H \) and \( \epsilon > 0 \) such that \( x_n \to x \) and \( \| \eta_x - \eta_{x_n} \| \geq \epsilon \). Since \( \int_0^{1/k} \beta(x)(T/k + \tau)^2 ds \) is strictly convex, we have

\[
\sup \{ I(x, \eta) \mid \eta \in L_0(S^{1/k}, R), \| \eta - \eta_x \| = \epsilon/2 \} \leq I(x, \eta_x) - \delta
\]

for some \( \delta > 0 \). Let \( \mu_n \in \partial B(\eta_x, \epsilon/2) \cap \{ \eta_x + \lambda(\eta_{x_n} - \eta_x) \mid \lambda \in [0, 1] \} \); since \( I(x_n, \cdot) \) is concave, we have \( I(x_n, \mu_n) \geq I(x_n, \eta_x) \), so that

\[
I(x, \eta_x) - \delta \geq I(x, \mu_n) = I(x, \mu_n) - I(x, \mu_n) + I(x_n, \mu_n) \geq I(x, \mu_n) - I(x, \mu_n) + I(x_n, \eta_x).
\]

Since \( (\mu_n) \) is bounded and \( x_n \to x \), we get \( I(x, \mu_n) - I(x_n, \mu_n) \to 0 \), and \( I(x_n, \eta_x) \to I(x, \eta_x) \), and then we have a contradiction.

Finally, fix \( x, y \in H \), and let \( \tau > 0 \). From (2.2) we have

\[
\frac{I(x + \tau y, \eta_x) - I(x, \eta_x)}{\tau} \leq \frac{f(x + \tau y) - f(x)}{\tau} \leq \frac{I(x + \tau y, \eta_{x + \tau y}) - I(x, \eta_{x + \tau y})}{\tau}.
\]

For \( \tau \to 0 \) we get \( \langle f'(x), y \rangle = \langle \partial I(x, \eta_x)/\partial x, y \rangle \), so the lemma is proved.

**REMARK 2.3.** Notice that \( \int_0^{1/k} \beta(x)(T/k + \tau) \eta_x ds = 0 \) for every
In other words, there exists \( c_x \in \mathbb{R} \) such that
\[
f(x(s))(\frac{T}{k} + \gamma_x(s)) = c_x \quad \text{for every } s \in \mathbb{R}.
\]
Since \( c_x \leq 0 \) implies \( \frac{T}{k} + \gamma_x(s) \leq 0 \), so \( \frac{T}{k} + \gamma_x(s) \leq 0 \), we have \( c_x > 0 \), and then
\[
\frac{T}{k} + \gamma_x(s) > 0 \quad \text{for every } s.
\]
Moreover \( \beta(x)(\frac{T}{k} + \gamma_x)^2 = c_x (\frac{T}{k} + \gamma_x) \), so \( c_x = (k^2/T) \int_0^{1/k} \beta(x)(\frac{T}{k} + \gamma_x)^2 ds \).

**Lemma 2.4.** Fix \( \rho > 0 \) and \( x \in H \), and set \( I = \{ s \in [0, 1/k] | |x(s)| \leq \rho \} \). Then, if \( |I| > 0 \),
\[
\int_0^{1/k} \beta(x)(\frac{T}{k} + \gamma_x)^2 ds \leq \frac{T^2 M}{k^4 |I|},
\]
where \( M = \max \{ \beta(x) | |x| \leq \rho \} \), and \( |I| \) is the Lebesgue measure of \( I \).

**Proof.** Let \( c_x \) be as in Remark 2.3, so that \( \frac{T}{k} + \gamma_x(s) = c_x/\beta(x(s)) \) for every \( s \in \mathbb{R} \). If \( s \in I \), we have \( c_x/M \leq \frac{T}{k} + \gamma_x(s) \), and then
\[
c_x^2/M \leq \beta(x(s))(\frac{T}{k} + \gamma_x(s))^2.
\]
Integrating on \( I \), we have:
\[
\frac{c_x^2}{M} |I| \leq \int_0^{1/k} \beta(x)(\frac{T}{k} + \gamma_x)^2 ds = \frac{Tc_x}{k^2}.
\]
Then \( c_x \leq TM/k^2 |I| \), so that the lemma is proved. \( \blacksquare \)

**Lemma 2.5.** Let \( 0 < r < \rho \) and \( (x_n) \subset H \) be such that \( \text{dist}(\text{Im}(x_n), 0) \leq r \) and \( \|x_n\|_\infty \geq \rho \). Then
\[
\int_0^{1/k} \beta(x_n)(\frac{T}{k} + \gamma_x)^2 ds \leq \frac{T^2 M}{k^4 (\rho - r^2) \|x_n\|_2^2},
\]
where \( M = \max \{ \beta(x) | |x| \leq \rho \} \).

**Proof.** Let \( I_n = \{ s \in [0, 1/k] | |x_n(s)| \leq \rho \} \); since \( \|x_n\|_\infty \geq \rho \), \( |I_n| > 0 \), so that
\[
\int_0^{1/k} \beta(x_n)(\frac{T}{k} + \gamma_x)^2 ds \leq \frac{T^2 M}{k^4 |I_n|}.
\]
because of Lemma 2.4. Moreover, since \( \text{dist}(\text{Im} (x_n), 0) \leq r \), we have
\[
\varphi - r \leq \int_{I_n} |\dot{x}_n| \, ds \leq \|\dot{x}_n\|_2 \|I_n\|^{1/2},
\]
and the lemma follows. \( \blacksquare \)

We say that a functional \( f: H \to \mathbb{R} \) verifies the Palais-Smale-Cerami (PSC) condition (see [6]) if every sequence \( (x_n) \subset H \) such that \( f(x_n) \to c \in \mathbb{R} \) and \( \langle f'(x_n), x_n \rangle \to 0 \) as \( n \to \infty \), possesses a convergent subsequence.

We have the following lemma.

**Lemma 2.6.** There exists \( k_0 \in \mathbb{N} \) such that, for every \( k \geq k_0 \), the functional \( f \) satisfies the PSC-condition.

**Proof.** Let \( M = \max \{ \beta(x) \mid x \leq R + 1 \} \) (\( R \) is defined in (1.3)), and let \( k_0 \in \mathbb{N} \) be such that \( \alpha_0 - T^2 M/k_0^2 > 0 \). Fix \( k \in \mathbb{N} \) with \( k \geq k_0 \), and let us consider a sequence \( (x_n) \subset H \) such that \( f(x_n) \to c \in \mathbb{R} \) and \( \langle f'(x_n), x_n \rangle \to 0 \) as \( n \to \infty \). First of all, we prove that \( \|\dot{x}_n\|_2 \) is bounded modulo subsequences. In fact, we distinguish two cases:

1) case: for every \( n \in \mathbb{N} \), \( \text{dist} (\text{Im} (x_n), 0) > R \) (modulo subsequences). Then \( p\beta(x_n(s)) \leq (\beta'(x_n(s)) \langle x_n(s) \rangle) \) for every \( s \) (see (1.3)), so, from \( f(x_n) \to c \) we get (setting \( r_n = r_{x_n} \)):

\[
p \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] \, ds \leq pc + \int_0^{1/k} (\beta'(x_n)) \langle x_n(T/k + r_n)^2 \, ds + o(1). \]

Since \( \langle f'(x_n), x_n \rangle \to 0 \), we have

\[
\int_0^{1/k} \alpha'(x_n)(x_n)[\dot{x}_n, \dot{x}_n] \, ds + 2 \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] \, ds - \int_0^{1/k} (\beta'(x_n)) \langle x_n \rangle \left( \frac{T}{k} + r_n \right)^2 \, ds = o(1),
\]

so that

\[
\int_0^{1/k} (q\alpha(x_n) - \alpha'(x_n)(x_n))[\dot{x}_n, \dot{x}_n] \, ds \leq pc + o(1),
\]

then \( \|\dot{x}_n\|_2 \) is bounded because of (1.2).
2) case: for every $n \in \mathbb{N}$, $\text{dist}(\text{Im}(x_n), 0) \leq R$ (modulo subsequences). Then, if $(\|x_n\|_\infty)$ is bounded, we have $\beta(x_n(s)) \leq M_1$ for $n \in \mathbb{N}$, $s \in \mathbb{R}$, so $\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 \, ds \leq M_1 T^2 / k^3$, and the claim follows from the fact that $f(x_n) \to c$ as $n \to \infty$. So, we can assume $\|x_n\|_\infty \to \infty$. Let $I_n = \{s \in [0, 1/k] | |x_n(s)| \leq R + 1\}$; from Lemma 2.5 (with $r = R$ and $\rho = R + 1$), we have

$$\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 \, ds \leq \frac{T^2 M}{k^4} \|\dot{x}_n\|_2^2.$$

Then, since $f(x_n) \to c$,

$$\int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] \, ds \leq \frac{T^2 M}{k^4} \|\dot{x}_n\|_2^2 + c + o(1),$$

so that (see (1.1)): $(x_0 - T^2 M/k^4)\|\dot{x}_n\|_2^2 \leq c + o(1)$. Since $k \geq k_0$, the claim follows.

We set now $x_n = \xi_n + y_n$, where $\xi_n \in \mathbb{R}^3$, and $\int_0^{1/k} y_n(s) \, ds = 0$; we shall prove that $(\xi_n)$ is bounded. In fact, we can assume that $y_n \to y$ weakly in $H^{1,2}$ and strongly in $L^\infty$; then

$$|\xi_n| - (\|y\|_\infty + 1) \leq |x_n(s)| \leq |\xi_n| + (\|y\|_\infty + 1)$$

for $n$ large enough, so that, since

$$\alpha(x_n(s)) [\dot{x}_n(s), \dot{x}_n(s)] \leq c_2 |x_n(s)|^q |\dot{x}_n(s)|^2$$

(see (1.8)), we have $\int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] \, ds \leq c_3 |\xi_n| + c_4$ for some $c_3, c_4 > 0$.

On the other hand, $\beta(x_n(s)) \geq c_1 |x_n(s)|^p$, then $\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 \, ds \geq c_5 |\xi_n|^p + c_6$. Since $f(x_n) \to c$, we have

$$c_5 |\xi_n|^p + c_6 \leq \int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 \, ds = \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] \, ds - c + o(1) \leq c_3 |\xi_n| + c_4 + c + o(1),$$
so \( (\xi_n) \) is bounded. Let us suppose \( x_n \to x \) weakly in \( H^{1,2} \) and strongly \( L^\infty \). Then

\[
\langle f'(x_n), x - x_n \rangle = \int_0^{1/k} \alpha'(x_n)(x - x_n)[\dot{x}_n, \dot{x}_n] \, ds +
\]

\[
+ 2 \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] \, ds - \int_0^{1/k} (\beta'(x_n)|x - x_n|(T/k + \eta_n)^2) \, ds;
\]

because of (2.1) we have that \( (\eta_n) \) is bounded, so, the fact that

\[
\langle f'(x_n), x - x_n \rangle = o(1) \implies \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] \, ds = o(1).
\]

Then

\[
\int_0^{1/k} |\dot{x} - \dot{x}_n|^2 \, ds \leq \alpha_0^{-1} \int_0^{1/k} \alpha(x_n)[\dot{x} - \dot{x}_n, \dot{x} - \dot{x}_n] \, ds = o(1),
\]

so that \( x_n \to x \) strongly in \( H \), and the lemma is proved. ■

Let \( H = H^{1,2}(S^{1/k}, R^3) = R^3 \times Y \), where

\[
Y = \left\{ x \in H \mid \int_0^{1/k} x(s) \, ds = 0 \right\}.
\]

As well-known (see e.g. [13], p. 9), for every \( y \in Y \) we have \( \|\dot{y}\|_2 \geq a\|y\| \), and \( \|y\|_\infty \leq b\|\dot{y}\|_2 \), where \( a = 2k\pi(1 + 4k^2\pi^2)^{-1/2} \), and \( b = (1/12k)^{1/2} \).

We have now the following lemma.

**Lemma 2.7.** There exist \( \delta, \rho > 0 \) such that \( f(y) \geq \delta \) for every \( y \in Y \) with \( \|y\| = \rho \). Moreover \( \delta \) is independent of \( k \).

**Proof.** Fix \( \varepsilon > 0 \) such that \( \alpha_0 - \varepsilon T^2/\sqrt{12} > 0 \). (1.5) implies that there exists \( \rho_1 > 0 \) such that \( \beta(x) \leq \beta_0 + \varepsilon \|x\|^2 \) for \( \|x\| \leq \rho_1 \). Set \( \rho = \rho_1 / b \) and

\[
\delta = \frac{4\pi^2}{1 + 4\pi^2} \left( \alpha_0 - \frac{\varepsilon T^2}{12} \right) \rho_1^2 12.
\]

For \( y \in Y \) with \( \|y\| = \rho \), we have \( \|y\|_\infty \leq b\|\dot{y}\|_2 \leq b\|y\| = b\rho = \rho_1 \), so that \( \beta(y(s)) \leq \beta_0 + \varepsilon |y(s)|^2 \leq \beta_0 + \varepsilon b^2 \|\dot{y}\|^2_2 \). Then

\[
\int_0^{1/k} \beta(y)(T/k + \eta_y)^2 \, ds \leq (\beta_0 + \varepsilon b^2 \|\dot{y}\|^2_2) T^2 / k^3,
\]
since assumption (1.3) is not superquadratic at infinity on finite-dimensional subspaces of $H$. This fact makes not possible to apply the standard linking theorem of $f$. In order to avoid this difficult, we consider the subspace $E = \{ x \in H \mid x(s + 1/2k) = x(s) \}$. Clearly $E \subset Y$; moreover we have the following lemma.

**LEMMA 2.9.** Let us suppose that (1.6) holds. Then, every critical point $x \in E$ of the functional is a critical point of $f$.

**PROOF.** Let $x \in E$ be a critical point and $z \in H$; we shall prove that \( \langle f'(x), z \rangle = 0 \). In fact, set $z_1(s) = z(s) - z(s + 1/2k)$, and $z_2(s) = z(s) - z_1(s)$, so that $z_1 \in E$, and $z = z_1 + z_2$. Since $\langle f'(x), z_1 \rangle = 0$, we have $\langle f'(x), z \rangle = \langle f'(x), z_2 \rangle$. From Remark 2.3, there exists $c_x > 0$ such that $\beta(x(s))(T/k + \gamma_x(s)) = c_x$. Since $\beta$ is even and $x \in E$, we have that $\gamma_x(s + 1/2k) = \gamma_x(s)$, and then it is easy to check, by using (1.6), that $\langle f'(x), z_2 \rangle = -\langle f'(x), z \rangle$, and the lemma is proved. \( \blacksquare \)

**PROOF OF THEOREM 1.1.** Let us suppose that (1.1)-(1.6) hold, let $k_0 \in \mathbb{N}$ be as in Lemma 2.6, $\delta > 0$ as in Lemma 2.7, and such that $ka > k_0$. From Lemma 2.6, the functional $f_{|E}$ satisfies the PSC condition on $E$. Let $w(s) = r\cos(2k\pi s), \sin(2k\pi s), 0)$; clearly $w \in E$, and since $\beta(w(s)) \geq ar^p + b$, we have

$$\int_0^{1/k} \beta(w)(T/k + \gamma_v)^2 ds \geq (ar^p + b) T^2 / k^3,$$

so that (see Remark 1.2) $f(w) \leq 4k\pi^2 c_2 r^{q+2} - (ar^p + b) T^2 / k^3 + \beta_0 T^2 / k^3$, and $f(w) < 0$ for $r$ large enough (we recall that $q + 2 < p$). Set
Let $p$ be as in Lemma 2.7; since $f(0) = 0$ and we can assume $\|w\| > P$, we have $\delta \leq c < + \infty$. From the mountain pass lemma (see [1]), we have that $c$ is a critical value for the functional $f_{\Sigma E}$. From Lemma 2.9 we get a critical point $x \in H$ of $f$ with $f(x) = c$. Since $c > 0$, we have $\dot{x} \neq 0$. Because of Remark 2.1, $z(s) = (x(s), t(s))$, where $t(s) = Ts/k + \int_0^s \gamma_x(\tau) d\tau$, is a 1-periodic $T$-trajectory on $(\mathbb{R}^4, g)$.

Finally, in order to prove that $z$ is spacelike, we observe that

$$E_z = \int_0^1 \alpha(x) [\dot{x}, \dot{x}] ds - \int_0^1 \beta(x) t^2 ds =$$

$$= k \left( \int_0^{1/k} \alpha(x) [\dot{x}, \dot{x}] ds - \int_0^{1/k} \beta(x)(T/k + \gamma_x)^2 ds \right) =$$

$$= k \left( f(x) - \frac{\beta_0 T^2}{k^3} \right) \geq k\delta - \frac{\beta_0 T^2}{k^2} > 0,$$

so that $E_z > 0$, and Theorem 1.1 is proved. ■

REFERENCES


