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Some Elementary Remarks about \( n \)-Local Fields.

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Local fields (i.e. complete fields relative to a discrete rank one valuation) play a key role in the theory of algebraic functions of one variable and elsewhere. Local fields have been studied from many points of view (see [4], [5], [6], [12], [13], [14], [15]). \( n \)-local fields (see the definition below) are a natural generalization of local fields. These fields appear in the theory of algebraic functions of several variables and in algebraic geometry. In the last years many problems on \( n \)-local fields, as, for example, class-field theory, have developed (see [8], [9]).

The aim of this paper is to make out some elementary results about \( n \)-local fields. In the first section general notations and definitions are given. In the second section we remark that the \( n \)-local field of bounded Laurent power series in many variables over a perfect field (see the definition below), can be defined in the same way as the field of Witt’s vectors, starting from a field of repeated formal power series in several variables over a perfect field. In this case the Witt’s operations are applied only to coefficients in the same way as in the classical case. However, we give all the steps of that construction.

Let \( k \) be a field and \( k(X, Y) \) the field of rational functions of two variables over \( k \).

In the third section we give a description of the maximal completion of the field \( k(X, Y) \) relative to a rank two and discrete valuation, trivial on \( k \).

Finally in the last section is proved that if \( K \) is a \( n \)-local field such that its residue field has the property that for every natural number \( m \) has a finite set of separable extension of degree \( m \), then \( K \) has, for every natural number \( m \), a finite number of tamely ramified extensions of degree \( m \).

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1. Definition.

We refer the reader to [3], [7], [11] and [12] for usual definitions. By valued field we mean a pair \((K, v)\) where \(K\) is a field and \(v\) a valuation on \(K\). As usual, we denote with \(O_v\) the valuation rings of \(v\), \(M_v\) its maximal ideal, \(G_v\) the value group of \(v\) and \(k_v\) the residue field of \(v\). If \(x \in O_v\), \(x^*\) denotes the natural image of \(x\) into \(k_v\). We shall say that \((K, v)\) is a local field if \(v\) is discrete of rank one and \(K\) is complete with respect to \(v\).

Let \((K, v)\) be a valued field where \(v\) is discrete and of rank \(n\). Let \(0\) be the valuation ring of \(v\) and let \(\mathcal{P} = \{M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n\} = (\mathcal{P})\) be all the prime ideals of \(O\). For every \(i < n\), denote \(O_i\) the ring of fractions of \(O\) with respect to the prime ideal \(M_i\). In particular \(O_0 = O \) and \(O_n = K\). Denote \(v_i, i = 0, \ldots, n - 1\), the valuation on \(K\) associated to the valuation ring \(O_i\). One has \(v_0 = v\) and for every \(i\), \(v_i\) is a valuation of rank \(n - i\) on \(K\). Denote \(\bar{v}_i, i = 0, 1, \ldots, n - 1\) the valuation on the field \(K_{i+1} = O_{i+1}/M_{i+1}\), associated with the valuation ring \(O_i/M_{i+1}\). It is easy to see that \(\bar{v}_i\) is of rank one. Particularly one has \(K_n = K\) and \(\bar{v}_{i-1} = v_{n-1}\).

We shall say that \((K, v)\) is \(n\)-local if \((K_{i+1}, \bar{v}_i)\) is local for all \(i = 0, 1, \ldots, n - 1\).

Let \(k\) be a field and \(n\) a natural number. Denote \(k((t_1)) \cdots ((t_n))\) the field of repeated power series in \(n\) indeterminates \(t_1, \ldots, t_n\) over \(k\). (For example one has: \(k((t_1))((t_2)) = (k((t_1))((t_2))), \) etc.). This field is in a natural way endowed with a rank \(n\) and discrete valuation \(v\). Moreover \((k((t_1)) \cdots ((t_n)), v)\) is \(n\)-local.

Another example of \(n\)-local field is obtained as follows. Let \((K, v)\) be a local field. Let us denote \(K\{\{t\}\}\) the set of all formal Laurent series \(\sum_{-\infty}^{+\infty} a_n t^n\) over \(K\) which verify the following two conditions (see [8]):

i) the set \(\{v(a_n)\}\) is lower bounded,

ii) \(v(a_{-n}) \to \infty\) if \(n \to \infty\).

Define in \(K\{\{t\}\}\) the following valuation \(\bar{w}_1:\)

\[\bar{w}_1\left(\sum_{-\infty}^{+\infty} a_n t^n\right) = \inf_n v(a_n).\]

It is easy to see that \((K\{\{t\}\}, \bar{w}_1)\) is a local field whose residue field is just \(k(t)\) where \(k\) is the residue field of \((K, v)\).

We shall say that \((K\{\{t\}\}, \bar{w}_1)\) is the local field of bounded Laurent series over \((K, v)\). Moreover we can define on \(K\{\{t\}\}\) a rank two valua-
tion \( u_1 \) as follows: Let \( x = \sum_{n=-\infty}^{+\infty} a_n t^n \) and let \( n_1 \) be the last integer number such that \( v(a_n) = w_1(x) \). Now we shall define \( u_1(x) = (w_1(x), n_1) \). The residue field of \((K\{t\}, u_1)\) is just \( k \). In fact \( u_1 \) is the composite valuation of \( w_1 \), with the order valuation on \( k((t)) \). (See [11, pag. 43]).

Furthermore we can define the local field \((K\{t_1\}, \{t_2\}, w_2)\) as the local field of bounded Laurent series over \((K\{t_1\}, w_1)\) and by recurrence we can define the local field \((K\{t_1\}, \ldots, \{t_n\}, w_n)\) whose residue field is just \( k\{(t_1)\} \ldots \{(t_n)\} \). The field \( K\{t_1\}, \ldots, \{t_n\} \) is naturally endowed with a rank \( n + 1 \) valuation \( u_n \) and \((K\{t_1\}, \ldots, \{t_n\}, u_n)\) is a \( n + 1 \) local field. Usually we shall say that \( K\{t_1\}, \ldots, \{t_n\} \) is the field of bounded Laurent series in \( n \) variables over \((K, v)\).

### 2. Alternative definition of \( K\{t_1\}, \ldots, \{t_n\} \).

Let \( k \) be a perfect field of characteristic \( p > 0 \) and let \((T(k), v)\) be the Witt's local field associated to \( k \) (see [4], [14]). We remind that \( T(k) \) is the quotient field of the ring of Witt's vectors and \( v \) the natural valuation. The aim of this section is to show that \( T(k)\{t_1\}, \ldots, \{t_n\} \) can be constructed in the same way as the field \( T(k) \) starting with the field \( k_1 = k((t_1)) \) \ldots \((t_n)) \). In other words if \( k \) is a perfect field of characteristic \( p > 0 \), we shall build a rank one and discrete valuation ring of zero characteristic whose residue field is just \( k_n \).

1) Let \( A_0 = \mathbb{Q}[X_i]_{i \in I} \) be the polynomial ring in a set \( X_i \) of indeterminates over \( \mathbb{Q} \), the field of rational numbers. Denote \( S_n = A_0((Y_1)) \ldots ((Y_n)) \). For any natural number \( i \), define inductively the mapping

\[
(p^i): A_n \rightarrow A_n
\]

as follows: The mapping \((p^i): A_0 \rightarrow A_0\) is just the raising at the \( p^i \)-power, i.e. \((p^i)(a) = a^{p^i}\) for all \( a \in A_0 \) and for \( n \geq 1 \), one has: if \( a \in A_n, a = \sum_{k > -\infty} a_k Y_n^k, a_k \in A_{n-1} \) then: \((p^i)(a) = \sum_{k > -\infty} a_k, p, Y_n^k, \) where \( a_k, p_i = (p^i)(a_k) \).

Now let us define \( B_n \) to be the set of all sequences with entries in \( A_n: \alpha = (\alpha_0, \alpha_1, \ldots), \alpha_i = \sum_{j > -\infty} a_{ij} Y_n^j, a_{ij} \in A_{n-1} \). The set \( B_n \) is a ring with operations defined component wise. Define the mapping \( \varphi_n: B_n \rightarrow B_n \) by:

\[
\varphi_n(\alpha) = \varphi_n(\alpha_0, \alpha_1, \alpha_2, \ldots) = (\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \ldots)
\]
where
\[ \alpha^{(0)} = \alpha_0, \quad \alpha^{(1)} = \alpha_{0,p} + p\alpha_1, \ldots, \alpha^{(m)} = \alpha_{0,p^m} + p\alpha_{1,p^{m-1}} + \ldots + p^m\alpha_m. \]

If we denote
\[ P\alpha = (\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p}, \ldots) \]
then as one easily sees:
\[ \alpha^{(n)} = (P\alpha)^{(n-1)} + p^n\alpha_n. \]
Also it is obvious that: \( \alpha_0 = \alpha^{(0)} \) and generally:
\[ \alpha_n = \frac{1}{p^n}(\alpha^{(n)} - \alpha_{0,p^n} - \ldots - p^{n-1}\alpha_{n-1,p}) \]
and so the mapping \( \varphi_n \) is a bijection.

If \( \alpha, \beta \in B_n \), then one defines:
\[ \alpha + \beta = \varphi_n^{-1}(\varphi_n(\alpha) \oplus \varphi_n(\beta)), \]
\[ \alpha\beta = \varphi_n^{-1}(\varphi_n(\alpha) \odot \varphi_n(\beta)), \]
where \( \oplus \) and \( \odot \) are the addition and multiplication (defined component wise) in \( B_n \). It is clear that relative to the above defined operations, \( B_n \) is a commutative ring with identity \((1, 0, 0, \ldots)\).

Furthermore, denote:
\[ A_0' = Z[X_i]_{i \in I}, \quad A_n' = A_0'((Y_1)) \ldots ((Y_n)), \quad n \geq 1. \]

It is clear that \( A_n' \) is a subring of \( A_n \).

If
\[ \alpha_i = \sum_{k > -\infty} a_{ik}Y_n^k, \quad \beta_i = \sum_{k > -\infty} b_{ik}Y_n^k \]
are two elements of \( A_n' \), then we write:
\[ \alpha_i \equiv \beta_i (\text{mod } p^e), \quad e \geq 0 \]
if and only if \( a_{ij} \equiv b_{ik} (\text{mod } p^e) \) for all \( k \). We remark that in \( A_0' \) the congruence relation is defined in an obvious way: two polynomials with integral coefficients are congruent modulo \( p^e \) if and only if the coefficients of similar terms are congruent modulo \( p^e \).

The following remarks are easy to prove and are left to the reader (see [4], pag. 157).
REMARK 2.1. Let \( a_i, b_i \in \mathbb{A}_n', \ 0 \leq i \leq m. \) Then the system of congruences:
\[
a_i \equiv b_i (\text{mod } p^e), \quad 0 \leq i \leq m
\]
is equivalent to:
\[
a^{(i)} \equiv b^{(i)} (\text{mod } p^{e+1}), \quad 0 \leq i \leq m
\]
where \( a^{(i)} = a_{0,i} p^i + p a_{1,i} p^{i-1} + \ldots + p^i a_i \) and similar \( b^{(i)}. \)

Let us denote \( B'_n \) the subset of \( B_n \) consisting of all elements \( \alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots) \) such that \( \alpha_m \in \mathbb{A}_n' \) for all \( m \geq 0. \)

REMARK 2.2. Let \( \alpha, \beta \in B'_n. \) If \( \alpha \circ \beta \) means one of the elements \( \alpha + \beta, \alpha - \beta, \alpha \beta \) defined as above, then for all \( m \geq 0, (\alpha \circ \beta)_m \) is a polynomial with integral coefficients and without free terms in \( \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m: \)
\[
(\alpha + \beta)_m = s_m(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m),
\]
\[
(\alpha \beta)_m = p_m(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m).
\]

2) Now let \( k \) be a perfect field of characteristic \( p > 0. \) Let us define
\[
W(k_n) = \{ \alpha/\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \}
\]
with \( \alpha_i \in k((t_1)) \ldots ((t_n)) = k_n. \)

If
\[
\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \ldots),
\]
\[
\beta = (\beta_0, \beta_1, \ldots, \beta_m, \ldots),
\]
are elements of \( W(k_n), \) let us define:
\[
\alpha + \beta = ((\alpha + \beta)_0, (\alpha + \beta)_1, \ldots, (\alpha + \beta)_m, \ldots),
\]
\[
\alpha \beta = ((\alpha \beta)_0, (\alpha \beta)_1, \ldots, (\alpha \beta)_m, \ldots),
\]
where \( (\alpha + \beta)_m, (\alpha \beta)_m \) are defined as above.

REMARK 2.3. \( W(k_n) \) endowed with these operations is an integral domain.

The mapping (1) can be defined for every \( \alpha \in W(k_n) \) and one has:
\[
p^i \alpha = (0, \ldots, 0, \alpha_{0,i}, \alpha_{1,i}, \ldots).
\]
(If $\varepsilon \in k_n$ then $\varepsilon_{p^i}$ is defined by raising the coefficient of $\varepsilon$ to the power $p^i$. Since $k$ is perfect $i$ can be also negative) and so for every element $\alpha \in W(k_n)$ there exists an element $\alpha_{p^i} \in W(k_n)$ such that $(\alpha_{p^i}^{-1})^{(p^i)} = \alpha$. So the ideal of $W(k_n)$ generated by $p$ is the same to the set of all vectors $\alpha = (\alpha_0, \alpha_1, \ldots)$ such that $\alpha_0 = 0$. But now it is clear that the residue ring $W(k_n)/(p)$ is canonically isomorphic to $k_n$.

Now we shall define a functions:

$$v_n : W(K_n) \to \mathbb{Z}$$

as follows if $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \ldots)$, then $v_n(\alpha) = \infty$ if $\alpha = 0$ and $v_n(\alpha) = r$, where $r$ is the greatest natural number such that $\alpha_i = 0$ for all $i < r$. Particularly $v_n(p) = v(p \cdot 1) = 1$. Now we can prove:

**Remark 2.4.** $W(k_n)$ is a complete rank one and discrete valuation ring whose residue field is $k_n$.

For every $\alpha \in k_n$ denote

$$\{\alpha\} = (\alpha, 0, 0, \ldots, 0, \ldots) \in W(k_n).$$

If $\alpha \in k_n$ then

$$p^\alpha \{\alpha\} = (0, \ldots, 0, \alpha_{p^\alpha}, 0, \ldots).$$

Therefore, if $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \ldots)$ is an element of $W(k_n)$ then it has the $p$-adic representation:

$$\alpha = \sum_{m=0} p^m \{\alpha_{m, p^m}\}. \quad (2)$$

Let $T(k_n)$ be the quotient field of $W(K_n)$ and denote also by $v_n$ the natural extension of $v_n$ to $T(K_n)$. The (rank one and discrete) valuation $v_n$ together with the natural valuation $u$ of $k_n = k(t_1) \cdots (t_n)$ define on $T(k_n)$ a rank $n + 1$ and discrete valuation $u_n$, defined as follows: If $\alpha \in T(k_n)$, then $u_n(\alpha) = (v_n(\alpha), u(\alpha/p^{v_n(\alpha)}))$ (here $\varepsilon^*$ means the image of $\varepsilon \in W(k_n)$ in the residue field). It is easy to see that $(T(k_n), u_n)$ is a $(n + 1)$-complete field. We shall say that $(T(k_n), v_n)$ is the field of Teichmüller-Witt vectors.

**Theorem 2.5.** Let $k$ be a perfect field of characteristic $p > 0$ and let $(T(k), v)$ be the field of Witt’s vectors associated to $k$. Then for every natural number $n \geq 1$, there exists a natural isomorphism of local fields $\varphi_n$ between $(T(k_n), v_n)$ and $(T(k)\{t_1\} \cdots \{t_n\}, w_n)$.

**Proof.** For the sake of simplicity we shall define only $\varphi_1$; the general definition of $\varphi_n$ will be left to the reader.
Let \( \alpha \in T(k_1) \) be given by (2). Let us denote \( \beta_n = \alpha_{m \cdot p^{-m}} \). Then
\[
\sum_{s > -\infty} a_{p, s}^m t_i^s
\]
Since \( b_s = \sum_{m=0}^{+\infty} \{a_{p, s}^m\} p^m \) is an element of \( T(k) \) for all \( s \), we can assign to \( \alpha \) the element \( \varphi_1(\alpha) = \sum_{s > -\infty} b_s t_i^s \) of \( T(k)\{\{t_i\}\} \). That is the definition of \( \varphi_1 \).

### 3. Some rank two completions.

Let \( (K, v) \) be a valued field where \( v \) is a rank one and discrete valuation. Let \( (K(X), u) \) be an extension of \( v \) to \( K(X) \) (\( X \) indeterminate over \( K \)) such that \( u \) is of rank two. In this section we are dealing with the description of the maximal completion of \( (K(X), u) \) (see [12], Ch. II for the notion of maximal completion).

1) Now we shall make some considerations about the valuation \( u \) (see [10]).

Let \( O_u \) be the valuation ring of \( u \) and let \( (0) \subset M_1 \subset M_2 \) be all the prime ideals of \( O_u \). Let \( O_w \) be the quotient ring of \( O_u \) relative to the complement of \( M_1 \) and let \( w \) be the valuation of the \( K(X) \) associated to \( O_w \). Denote also by \( v' \) the valuation of the field \( k_w = O_w / M_1 \) associated to the valuation ring \( O_u / M_1 \). Since one has \( G_w = Z = G_v \), we can assume that \( G_u = G_w \times G_v \), ordered lexicographically. Moreover the valuation \( u \) can be defined (up to equivalence) as follows. Let \( t \) be an element of \( K(X) \) such that \( u(t) = (1, 0) \). Then \( w(t) = 1 \). If \( x \in K(X) \) then \( u(x) = (a, b) \) where \( a = w(x) \) and \( b = v'(x t^{-a}) \), \((y^*)\) being the image in \( k_w \) of \( y \in O_w \).

With the above notations, two cases are possible (see [10]):

- \( O_w \cap K = K \) (then \( u \) is called of the first kind) and \( O_w \cap K = O_v \) (we shall that \( u \) is of second kind). We shall describe the maximally completion of \( u \) in both cases, separately.

2) Let us assume that \( u \) is of first kind. Then \( w \) is trivial on \( K \) and so it is defined by an irreductible polynomial of \( K[X] \) or is the valuation at the infinity. Since in the case when \( w \) is the valuation at the infinity the things are quite similarly to the case \( f = X \), we can consider only the case when \( w \) is defined by an irreductible polynomial \( f \). In this case \( k_w = K(\alpha) \) where \( \alpha \) is a suitable root of \( f \). Then \( v' \) is an extension of \( v \) to \( K(\alpha) \). According to [10], \( u \) can be defined as follows: if \( F \in K[X] \), let us write:

\[
F = F_0 + F_1 f + \ldots + F_r f^r
\]

where \( \deg F_i < \deg f, 0 \leq i \leq r \). Then one has

\[
u(F) = \inf \bigl( i, v'(F_i(\alpha)) \bigr) .
\]
Let \((K'_2, \tilde{v}')\) be the completion of \(K(X)\) relative to \(w\). It is well known that \(K'_2\) is canonically isomorphic to the field of formal power series in one variable \(t'\) over the field \(K(\alpha)\). By this isomorphism we identify the polynomial \(F = F_0 + F_1 f + \ldots + F_r f'^r\), \(\deg F < \deg f\) to \(F_0(\alpha) + F_1(\alpha) t' + \ldots + F_r(\alpha) t'^r\).

Let \((K_1, \tilde{v}')\) be the completion of \((K(\alpha), v')\) and let \(K_2 = K_1[[t]]\). Define on \(K_2\) a valuation \(\tilde{u}\) as follows: if \(a = \sum_{n > -\infty} a_n t^n \in K_2\), then \(\tilde{u}(a) = (n, \tilde{v}(a_n))\) where \(n\) is the last integer such that \(a_n \neq 0\). It is clear that \((K'_2, \tilde{w}')\) is contained naturally as a subfield in \((K_2, \tilde{u})\). Moreover \(\tilde{u}\) is a rank two discrete valuation whose residue field is just \(k_w\). If \(\tilde{w}\) is the rank one valuation on \(K_2\), associated to \(\tilde{u}\) then \(O_{\tilde{w}} = K_1[[t]]\) and its residue field is just \((K_1, \tilde{v}_1)\). Finally it is clear that \((K_2, \tilde{u})\) is a maximally completion of \((K(X), u)\).

3) Let us assume that \(O_w \cap K = O_v\) and \(M_1 \cap K = M_v\) (i.e. \(u\) is of second kind). Clearly one has \(O_u \cap K = O_v\), \(M_2 \cap K = M_v\) and \(O_v = O_u / M_1\). Since \(O_v' \cap k_v = k_v\) it follows that it exists a valuation on \(k_v\) which is trivial on \(k_v\). This means that \(k_w / k_v\) is a transcendental extension and so \(w\) is a r.t. extension (see [11] of \(v\) to \(K(X)\)). Then according to [1, Theorem 2.1], \(w\) is defined by a minimal pair \((a, s)\) where \(a\) is algebraic and separable over \(K\) and \(v\) has a unique extension, denoted \(v_1\), to \(K(a)\) (see [2, Theorem 3.8]). Let \(f\) be the minimal polynomial of a over \(K\), \(\gamma = w(f)\) and \(e\) the smallest natural number such that \(e \gamma \in G_{v_1}\). Let \(h \in K[X]\) be such that \(\deg h < \deg f\) and that \(w(h(X)) = v_1(h(a)) = e \gamma\), and \(r = f^e / h\). According to [1, Theorem 2.1] one has: \(w(r) = 0\) and \(r^*\) is transcendental over \(k_v\). Moreover \(k_w = k_{v_1}(r^*)\).

Now since \(v'\) is trivial over \(K_{v_1}\) it follows that it is defined by an irreducible polynomial \(G(r^*)\) or is the valuation at the infinity. Let \(g\) be a lifting in \(K[X]\) of \(G(r^*)\) (see[10]). Then by [10] the valuation \(u\) is defined as follows. Let \(F \in K[X]\) and let \(F = F_0 + F_1 g + \ldots + F_s g^s\), \(\deg F_i < \deg g\), \(0 \leq i \leq s\) be the \(g\)-expansion of \(F\). Then one has:

\[
u(F) = \inf_i \left( (\nu(F_i), 0) + i \nu(g) \right).
\]

Before to the maximal completion of \((K(X), u)\) we shall make some comments. Let \(n = \deg G(r^*)\) (relative to the variable \(r^*\)). Since \(((g/h^n)^*) = G(r^*)\) and is transcendental over \(k_v\), then according to [2, Proposition 1.1] there exists a root \(b\) of \(g\) such that \(v(b - a) \leq \delta\). It is easy to see that for any \(F \in K[X]\), \(\deg G < \deg f\), one has: \((F(b)/F(a))^* = 1\), and that \((f(b)^e/h(b))^* = c\) is a root of \(G(r^*)\), i.e. \(G(c) = 0\). Moreover if \(v_2\) denotes a suitable extension of \(v\) to \(K(b)\) (it may be proved that \(v_2\) is in fact the unique extension of \(v\) to \(K(b)\)) then \(k_{v_2} = k_{v_1}(c)\).
Let \((\overline{K(b)}, \overline{v}_2)\) be the completion of \((K(b), v_2)\) and let \((k_v(t), v')\) be the completion of \((k_v(\cdot r^*), v')\) (in this last completion the element \(G(r^*)\) goes onto \(t\)). Let \((\overline{K(b)}\{\{t\}\}, \overline{u})\) be the valued field where as usual \((\overline{K(b)}\{\{t\}\}, w_1)\) is the local field of bounded Laurent series over \((K(b), \overline{v}_2)\) and where \(\overline{u}\) is the rank two valuation defined by

\[
\overline{u}\left(\sum_{n=1}^{\infty} a_n t^n\right) = \left(\inf_n \overline{v}_2(a_n), n_0\right).
\]

Here \(n_0\) is the smallest integer number such that the inf on the first component is reached. The reader is referred to [12, Ch. II], to prove that \((\overline{K(b)}\{\{t\}\}, \overline{u})\) is the maximally completion of \((K(X), u)\), above defined.

4) Let \(k\) be a field. We can apply the above observations to the field \(K = k(X)\), \(X\) an indeterminate. Then \(v\) will be a valuation on \(K\) trivial over \(k\) and \(u\) a rank two extension of \(v\) to \(K(Y) = k(X, Y)\). We leave to the reader the task to describe the maximally completion of \((k(X, Y), u)\) in both cases when \(u\) is of first or second kind.

4. Finiteness of the number of extensions of given degree.

In what follows the expression «a finite number of extensions of degree \(n\) with a given property \(\mathcal{P}\), of a field \(K\)» means: there exists a finite set \(\mathcal{L}\) of extension of degree \(n\), with the property \(\mathcal{P}\), of the field \(K\) such that every extension of \(K\), with the property \(\mathcal{P}\) is \(K\)-isomorphic to an element of \(\mathcal{L}\).

1) LEMMA 4.1. Let \((K, v)\) be a local field. Assume that the residue field \(k_v\) is such that for any natural number \(m\) it has only a finite number of separable extensions of degree \(m\). Then for every natural number \(n\) the field \(K\) has a finite number of extension of degree \(n\) which are *tamely ramified* (see [4], pag. 248).

**PROOF.** Let \(n\) be a fixed natural number and let \(L/K\) be a tamely ramified extension of degree \(\leq n\). Let \(p\) and \(\pi\) be respectively fixed uniformisants in \(K\) and \(L\). Denote also \(v\) the unique extension of \(v\) to \(L\) and let \(l_v\) the residue field of \((L, v)\). Let \(K_1\) be the maximal unramified extension of \(K\) included in \(L\). Then \(L = K_1(\sqrt[n]{p\epsilon})\), where \(\epsilon\) is an unity of \(K_1\) and \(e = e(K/L)\) (see [15], pag. 89) is the ramification index of \(L/K\). By hypothese one has:

\[
(e, q) = 1, \quad \text{where} \quad q = \text{char } k_v.
\]

Let us consider all separable extensions \(l_v\) of \(k_v\) when \(L\) runs over the set \(\mathcal{L}\) of all tamely ramified extensions of \(K\) of degree \(\leq n\). By
hypothesis and by obvious condition \([l_v : k_v] \leq n\) it follows that there exists a finite separable extension \(l\) of \(k_v\), such that \(k_v \subseteq l_v \subseteq l = k_v(b)\) for all \(l_v\). Let \(N\) be an unramified extension of \(k\), \(N = K(a), v(a) = 0, a^* = b\) and whose residue field is just \(l\). Let \(l \in \mathcal{L}\). Since \(l_v \subseteq l\) then \(L_{ur}\), the unramified part of \(L\) is contained in \(N\). One has: \(L = L_{ur}(\sqrt[n]{\rho\varepsilon}), \varepsilon\) an unity of \(L_{ur}\), \(e \leq n\) and \((e, q) = 1\).

We want to find a finite extension of \(N\) which contains \(\sqrt[n]{\varepsilon}\) for all units \(\varepsilon\) of \(N\), and \(e \leq n\), \((e, q) = 1\). For unit \(\varepsilon\) on \(N\) one has \(\varepsilon^* \in l\) and by hypothesis there exists a finite extension \(\Sigma\) of \(l\) which contains all the radicals \(\sqrt[n]{\varepsilon^*}\), when \(\varepsilon\) runs all unites of \(N\) and \(e\) all natural numbers smaller than \(n\) and relatively prime to \(q\). Let \(S = N(c')\) be the unique unramified extension of \(N\) whose residue field is just \(\Sigma\).

It is clear that for every unit \(\varepsilon \in N\) and every \(e, e \leq n\) and \((e, q) = 1\), in \(S\) there exists the element \(\sqrt[n]{\varepsilon}\). Let \(f\) be the smallest common multiple of all numbers \(e \leq n\) and \((e, q) = 1\). It is clear that \(S(\sqrt[n]{p}) = T\) contains all extensions \(L\) in the set \(\mathcal{L}\) above considered and since \(T/K\) is a separable extension, then it has only a finite number of subfields which are extensions of \(K\). Particularly the set \(\mathcal{L}\) has only a finite number of elements, as claimed.

Let \((k, v)\) be an \(n\)-local field. We shall say that a finite extension \((L, w)\) of \((K, v)\) is tamely ramified if \([G_w : G_v]\) the order of the quotient group \(G_w / G_v\) is relatively prime to the characteristic of \(k_v\) and the residue field \(l_w\) is a separable extension of \(l_v\). Look at the notation in the first section. Then for every \(t, 0 \leq t \leq n - 1\), \(L_{t+1}/L_{t+1}\) is a tamely ramified extension of local fields. One has the following result.

**Theorem 4.2.** Let \((K, v)\) be a \(n\)-local field. Assume that for every natural number \(m\) the residue field \(k_v\) has a finite number of separable extensions of degree \(m\). Then for every natural number \(m'\), \(K\) has a finitely many extensions of degree \(m'\) which are tamely ramified.

**Proof.** The proof follows by induction after \(n\). The case \(n = 1\) was treated in Lemma 4.1. Let us assume that \(n > 1\) and the result is true for every \(n' < n\). Look at the notation in the first section. Denote \(K'\) the residue field of \((K, v_{n-1})\) and \(v'\) the valuation on \(K'\) defined by the valuation ring \(O/M_{n-1}\). It is clear that \((K', v')\) is a \(n - 1\) local field whose residue field is just \(k_v\). Hence by inductive hypothesis \((K', v')\) has for every \(m'\) a finite number of tamely ramified extensions of degree at most \(m'\).

Let \(\mathcal{L}\) be the set of all tamely ramified extensions of \((K, v)\) of degree at most \(m'\). Let \(L''\) be the composite over \(K'\) of all extension \(L'\), when \((L, w)\) runs \(\mathcal{L}\) (as above \(L'\) is the residue field of \((L, w)\) relative to \(v_{n-1}\)).
According to the inductive hypothesis $L''/K'$ is a finite (separable) extension. Let $F$ be the composite of all unramified extensions $L$ of $K$ such that $(L, w) \in \mathcal{L}$ (we are saying that $(L, w)$ is unramified if $(L, w_{n-1})$ is an unramified extension of $(K, v_{n-1})$).

Furthermore, let $L^{(1)}, w^{(1)}_{n-1} \in \mathcal{L}$ be the unramified extension of $(K, v_{n-1})$ whose residue field is just $L''$. It is clear that $F \subseteq L^{(1)}$. Let $(L, w) \in \mathcal{L}$, then $\bar{L} = L^{(1)}L$ is a totally ramified extension of $(L^{(1)}, w^{(1)}_{n-1})$. Hence on has: $\bar{L} = L^{(1)}(\sqrt[n]{\pi e})$, where $\pi$ is an uniformising element of $(K, v_{n-1})$, $e$ a unity of $L^{(1)}$ and $(e, p) = 1$ ($p = \text{char } k_v$).

If $\varepsilon$ is unit of $L''$ and $e$ is a natural number relative prime to $p$, then by a slight computation one see that $L''(\sqrt[n]{\varepsilon})$, is a tamely extension of $(L'', w'')$, where $w''$ is the unique extension of $v'$ to $L''$. Hence, according to the inductive hypothesis, the composite of all extensions of $L''$ of the form $L''(\sqrt[n]{\varepsilon})$, where $(e, p) = 1$, $e \leq m'$, $\varepsilon$ a unit of $L$, is a (finite) tamely ramified extension of $(L'', w'')$. Then one has $L'' = L''(\gamma)$ and let $L^{(1)}(y)$ be an extension of $(L^{(1)}, w^{(1)}_{n-1})$ such that $\bar{y}$ (the residue of $y$ relative to the unique extension of $w^{(1)}_{n-1}$ to $L^{(1)}(y)$) is just $\gamma$ and that $[L^{(1)}(y): L^{(1)}] = [L''(\gamma): L'']$. According to Hensel’s Lemma, it follows that for every $\varepsilon \in L^{(1)}$ and every natural $e$, $e \leq m'$ and relatively prime to $p$, one has: $\sqrt[n]{\varepsilon} \in L^{(1)}(y)$. But then we can deduce that for every $(L, w) \in \mathcal{L}$ one has: $K \subseteq L \subseteq L^{(1)}(y, \sqrt[n]{\pi})$ Since $L^{(1)}(y)/K$ is a finite separable extension, then $\mathcal{L}$ is a finite set, as claimed.

**Corollary 4.3.** Let $(K, v)$ be a $n$-local field. Assume that its residue field $k_v$ is of zero characteristic and for every natural number $m$ has only a finite number of extensions of degree $m$. Then for every natural number $n'$, $K$ has a finite number of extensions of degree $n'$.

Let $(K, v)$ be a $n$-local field. We utilise the notations of § 1. Assume that the residue field $k_v$ is finite and the local field $(K_1, \bar{v}_0)$ is of characteristic 0. Then $(K, v_1)$ is $n - 1$ local field whose residue field is just $K_1$. Since $K_1$ is a finite extension of a $p$-adic field it has for every number $n$ only a finite number of extensions of degree $m$. Therefore by Corollary 4.3, for every natural number $n'$, $K$ has only a finite number of extensions of degree $n'$, hence one has the following result:

**Corollary 4.4.** Let $(K, v)$ be $n$ local field. Assume that its residue field $k_v$ is finite and the local field $(K_1, v_0)$ (see § 1) is of characteristic zero. Then for every natural number $n'$, $K$ has a finite number of extensions of degree $n'$. 
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