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Representable equivalences for closed categories of modules

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0. Introduction.

0.1. All rings considered in this paper have a nonzero identity and all modules are unital. For every ring $R$, $\text{Mod}-R$ ($R$-Mod) denotes the category of all right (left) $R$-modules. The symbol $M_R$ ($R M$) is used to emphasize that $M$ is a right (left) $R$-module.

Categories and functors are understood to be additive. Any subcategory of a given category is full and closed under isomorphic objects. $N$ denotes the set of positive integers.

0.2. Recall that a non empty subcategory $\mathcal{S}_R$ of $\text{Mod}-R$ is closed if $\mathcal{S}_R$ is closed under taking submodels, homomorphic images and arbitrary direct sums. Clearly $\mathcal{S}_R$ is a Grothendieck category.

It is easy to show that a closed subcategory $\mathcal{S}_R$ of $\text{Mod}-R$ has a generator and for every generator $P_R$ of $\mathcal{S}_R$ we have:

$$\mathcal{S}_R = \text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$$

where $\text{Gen}(P_R)$ is the subcategory of $\text{Mod}-R$ generated by $P_R$ and $\overline{\text{Gen}}(P_R)$ is the smallest closed subcategory of $\text{Mod}-R$ containing $\text{Gen}(P_R)$.

0.3. Let $\mathcal{S}_R$ be a closed subcategory of $\text{Mod}-R$, $P_R$ a generator of $\mathcal{S}_R$, $A = \text{End}(P_R)$. In the search for subcategories of $\text{Mod}-A$ which are equivalent to $\mathcal{S}_R$, the functors:

$$H = \text{Hom}_R(P_R, -): \text{Mod}-R \rightarrow \text{Mod}-A,$$

$$T = - \otimes_A P: \text{Mod}-A \rightarrow \text{Mod}-R,$$

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play a crucial role. Indeed we have the following representation theorem:

Let $A$ and $R$ be two rings, $\mathcal{O}_A$ a subcategory of $\text{Mod-}A$ such that $A_A \in \mathcal{O}_A$, $\mathcal{S}_R$ a closed subcategory of $\text{Mod-}R$. Assume that an equivalence $(F, G)$ between $\mathcal{O}_A$ and $\mathcal{S}_R$ is given:

$$\mathcal{O}_A \cong_{F G} \mathcal{S}_R.$$

Then there exists a bimodule $A P_R$ such that

1) $P_R \in \mathcal{S}_R$, $A \cong \text{End}(P_R)$ canonically.

2) The functors $F$ and $G$ are naturally equivalent to the functors $T|_{\mathcal{O}_A}$ and $H|_{\mathcal{S}_R}$ respectively.

3) $\mathcal{S}_R = \text{Gen}(P_R) = \text{Gen}(P_R)$, $\mathcal{O}_A = \text{Im}(H)$.

On the other hand a remarkable result of Zimmermann-Huisgen [ZH] and Fuller [F] states that, if $P_R \in \text{Mod-}R$ and $A = \text{End}(P_R)$, the following conditions are equivalent:

(a) $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$.

(b) The functor $H : \text{Gen}(P_R) \rightarrow \text{Mod-}A$ is full and faithful and $A P$ is flat.

Therefore $H$ induces an equivalence between $\text{Gen}(P_R)$ and $\text{Im}(H)$. We say that a module $P_R$ of $\text{Mod-}R$ is a W-module if $\text{Gen}(P_R)$ is a closed subcategory of $\text{Mod-}R$ or, equivalently, if $\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)$.

0.4. Let $P_R$ be a W-module, $A = \text{End}(P_R)$. The main purpose of this paper is to find a satisfactory description of $\text{Im}(H)$. Instead of using the Popescu-Gabriel Theorem (cf. [St] Theorem 4.1. Chap. X) we prefer to proceed in a more concrete manner using always the role of the functors $H$ and $T$ that lead to an interesting torsion theory on $\text{Mod-}A$.

Set

$$\text{Ker}(T) = \{ L \in \text{Mod-}A : L \otimes A P = 0 \}.$$ 

Since $A P$ is flat, $\text{Ker}(T)$ is a localizing subcategory of $\text{Mod-}A$, i.e. $\text{Ker}(T)$ is the torsion class of a hereditary torsion theory in $\text{Mod-}A$. The corresponding torsion-free class is obtained in the following manner: let $Q_R$ be a fixed, but arbitrary, injective cogenerator of $\text{Mod-}R$, $K_A = \text{Hom}_R(P_R, Q_R)$, $\mathcal{O}(K_A)$ the subcategory of $\text{Mod-}A$ cogenerated by
$K_A$. Then $\mathcal{O}(K_A)$ is the requested torsion-free class and $K_A$ is injective in $\text{Mod-}A$.

The Gabriel filter $\mathcal{I}'$—consisting of right ideals of $A$—associated to the torsion theory $(\ker(T), \mathcal{O}(K_A))$ is given by

$$\mathcal{I}' = \left\{ I \leq A_A : \frac{A}{I} \in \ker(T) \right\}.$$  

Equivalently

$$\mathcal{I}' = \left\{ I \leq A_A : IP = P \right\}.$$  

For every $L \in \text{Mod-}A$ denote by $L_{\mathcal{I}}$ the module of quotients of $L$ with respect to $\mathcal{I}$. Set

$$\text{Mod} - (A, \mathcal{I}') = \left\{ L \in \text{Mod-}A : L = L_{\mathcal{I}} \right\}$$

The main result on the torsion theory $(\ker(T), \mathcal{O}(K_A))$ is the following: for every $L \in \text{Mod-}A$

$$L_{\mathcal{I}} = HT(L).$$

Then it is easy to show that $\text{Im}(H) = \text{Mod} - (A, \mathcal{I}')$.

0.5. Various properties of $W$-modules are investigated, in particular their connection with Fuller’s Theorem on Equivalences.

The work ends with an example concerning the closed subcategory of $\text{Mod-}R$ consisting of semisimple modules.

0.6. REMARK. The class $\ker(T)$ was also investigated by [WW].

1. Representable equivalences.

1.1. Through this paper we use the following standing notations. Let $A, R$ be two rings and $A_P R$ a bimodule (left on $A$ and right on $R$). Consider the adjoint functors:

$$T = - \otimes_A P : \text{Mod-}A \to \text{Mod-}R,$$

$$H = \text{Hom}_R(P_R, -) : \text{Mod-}R \to \text{Mod-}A.$$  

For every $L \in \text{Mod-}A$ and $M \in \text{Mod-}R$ there exist the natural morphisms:

$$\sigma_L : L \to HT(L) = \text{Hom}_R(P_R, L \otimes_A P)$$

$$\sigma_L(l) : p \mapsto l \otimes p \quad (p \in P, l \in L)$$
In the sequel the functors $T$ and $H$ will be suitably restricted and corestricted.

1.2. Let $A$, $R$ be two rings, $\mathcal{A}$ and $\mathcal{S}_R$ subcategories of $\text{Mod-}A$ and $\text{Mod-}R$ respectively. Assume that a category equivalence $(F', G)$ between $\mathcal{A}$ and $\mathcal{S}_R$ is given:

$$\mathcal{A} \xrightarrow{F} \mathcal{S}_R, \quad G \circ F \approx 1_{\mathcal{A}}, \quad F \circ G \approx 1_{\mathcal{S}_R}.$$ 

In this situation we always assume that $A_A \in \mathcal{A}$.

Set $P_R = F(A)$. Then we have the bimodule $A_P R$, with $A = \text{End}(P_R)$ canonically.

1.3. **Lemma.** in the situation (1.2) the functor $G$ is naturally equivalent to the functors $\text{Hom}_R(P_R, -) \mid_{\mathcal{S}_R}$.

**Proof.** Let $M \in \mathcal{S}_R$ and consider the following natural isomorphisms:

$$G(M) \cong \text{Hom}_A(A, G(M)) \cong \text{Hom}_R(F(A), FG(M)) \cong \text{Hom}_R(P_R, M).$$

Thus $G \approx H \mid_{\mathcal{S}_R}$.

1.4. **Definition.** We say that the equivalence $(F, G)$ is representable by the bimodule $A_P R = F(A)$ if $F \approx T \mid_{\mathcal{A}}$ and $G \approx H \mid_{\mathcal{S}_R}$. In this case we say that the bimodule $A_P R$ represents the equivalence $(F, G)$.

1.5. Let $P_R \in \text{Mod-}R$ and let $\text{Gen}(P_R)$ be the subcategory of $\text{Mod-}R$ generated by $P_R$. Recall that a module $M \in \text{Mod-}R$ is in $\text{Gen}(P_R)$ if there exists an exact sequence $P_R(X) \rightarrow M \rightarrow 0$ where $X$ is a suitable set. $\text{Gen}(P_R)$ is closed under taking epimorphic images and arbitrary direct sums. Denote by $\text{Gen}(P_R)$ the smallest closed subcategory of $\text{Mod-}R$ containing $\text{Gen}(P_R)$. $\text{Gen}(P_R) = \text{Gen}(P_R)$ if and only if $\text{Gen}(P_R)$ is closed under taking submodules. Let $A_P R$ be a bimodule and let $Q_R$ be a fixed, but arbitrary, cogenerator of $\text{Mod-}R$. Set $K_A = \text{Hom}_R(P, Q)$ and denote by $\mathcal{O}(K_A)$ the subcategory of $\text{Mod-}A$ cogenerated by $K_A$. 
1.6. LEMMA. Let $A P_R$ be a bimodule. Then $\text{Im}(T) \subseteq \text{Gen}(P_R)$ and $\text{Im}(H) \subseteq \otimes(K_A)$.

PROOF. See [MO2] Prop. 2.2.

For every $M \in \text{set}$
then $t_p(M) \in \text{Gen}(P_R)$ and $\text{Hom}_R(P_R, M) \equiv \text{Hom}_R(P_R, t_p(M))$ in a natural way.

1.7. LEMMA. Let $A P_R$ be a bimodule. Then

a) $\text{Im}(H) = H(\text{Gen}(P_R))$;

b) $M \in \text{Gen}(P_R)$ if and only if $\varphi_M$ is surjective;

c) $L \in \otimes(K_A)$ if and only if $\sigma_L$ is injective.

PROOF. See [MO2] page 207.

1.8. PROPOSITION. The equivalence $(F, G)$ is representable by the bimodule $A P_R (P_R = F(A))$ if and only if for every $L \in \varnothing_A$ and for every $M \in \mathcal{S}_R$ the canonical morphisms $\sigma_L$ and $\varphi_M$ are both isomorphisms.

2. W-modules.

2.1. Let $\mathcal{S}_R$ be a closed subcategory of Mod-$R$. Then $\mathcal{S}_R$ has a generator $P_R$ and

\[
\mathcal{S}_R = \text{Gen}(P_R) = \overline{\text{Gen}}(P_R).
\]

Indeed let $\varnothing$ the filter of all right ideals $I$ of $R$ such that $R/I \in \mathcal{S}_R$. Then $P_R = \bigoplus_{I \in \varnothing} R/I$ is a generator of $\mathcal{S}_R$ and it is easy to check that (1) holds.

2.2. DEFINITION. Let $P_R \in \text{Mod} - R$, $A = \text{End}(P_R)$. Consider the functors $H = \text{Hom}_R(P_R, -)$ and $T = - \otimes_A P$. We say that $P_R$ is a $W_0$-module if

\[ (*) \text{ the functor } H: \text{Gen}(P_R) \to \text{Mod} - A \text{ subordinates an equivalence between } \text{Gen}(P_R) \text{ and } \text{Im}(H) \]

(whose inverse is given by $T|_{\text{Im}(H)}$).

2.3. REPRESENTATION THEOREM. Let $\varnothing_A$ and $\mathcal{S}_R$ be subcategories of Mod-$A$ and Mod-$R$ respectively. Assume that $A_A \in \varnothing_A$ and that $\mathcal{S}_R$ is closed under taking arbitrary direct sums and homomorphisms.
Suppose that a category equivalence \((F, G)\) between \(\mathcal{O}_A\) and \(\mathcal{S}_R\) is given:

\[
\mathcal{O}_A \xrightarrow{F} \mathcal{S}_R.
\]

Then \((F, G)\) is representable by the bimodule \(\mathcal{A}P_R\) \((P_R = F(A), A = \text{End}(P_R))\) and \(\mathcal{S}_R = \text{Gen}(P_R)\), \(\mathcal{O}_A = \text{Im}(H)\). Therefore \(P_R\) is a \(W_0\)-module.

**Proof.** By Lemma (1.2), since \(\text{Gen}(P_R) \subseteq \mathcal{S}_R\) and by Lemma (1.6), the functor \(T|_{\mathcal{O}_A}\) is a left adjoint of the functor \(G\). Since \((F, G)\) is an equivalence \(F\) is a left adjoint of \(G\). Therefore \(F \simeq T|_{\mathcal{O}_A}\). Thus by Lemma (1.6) \(\mathcal{S}_R = \text{Gen}(P_R)\). Finally, by Lemma (1.7), \(\mathcal{O}_A = \mathcal{S}_R = \text{Gen}(P_R)\).

2.4. Under the assumptions of Theorem (2.3) suppose that \(\mathcal{S}_R\) is a closed subcategory of \(\text{Mod}-R\). Then \(P_R\) is a \(W_0\)-module such that \(\mathcal{S}_R = \text{Gen}(P_R)\).

2.5. Let \(P_R \in \text{Mod}-R\) and assume that \(\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)\). Then the condition \((*)\) of 2.2. holds by the following important

2.6. **Theorem.** Let \(P_R \in \text{Mod}-R\), \(A = \text{End}(P_R)\). The following conditions are equivalent:

(a) For every positive integer \(n\), \(P_R\) generates all submodules of \(P^\mathbb{Z}_R\).

(b) \(\text{Gen}(P_R) = \overline{\text{Gen}}(P_R)\).

(c) \(\mathcal{A}P\) is flat and the functor \(H: \text{Gen}(P_R) \rightarrow \text{Mod}-A\) is full and faithful.

Moreover if the above conditions are fulfilled, then

1) \(H\) subordinates an equivalence between \(\text{Gen}(P_R)\) and \(\text{Im}(H)\).

2) The canonical image of \(R\) into \(\text{End}(\mathcal{A}P)\) is dense if \(\text{End}(\mathcal{A}P)\) is endowed with its finite topology.

**Proof.** The equivalences \((a) \iff (b) \iff (c)\) are due to Zimmermann-Huisgen (cf. [ZH], Lemma 2.2). The statement (2) is due to Fuller ([F], Lemma 1.3).

2.7. **Definition.** Let \(P_R \in \text{Mod}-R\). We say that \(P_R\) is a \(W\)-module
if $\operatorname{Gen}(P_R)$ is a closed subcategory of $\text{Mod-}R$ or equivalently $\operatorname{Gen}(P_R) = \overline{\operatorname{Gen}(P_R)}$.


3.1. Proposition. Let $P_R$ be a $W$-module, $A = \text{End}(P_R)$, $B = \text{End}(A_P)$. Then the bimodule $A P_B$ is faithfully balanced and $\operatorname{Gen}(P_B)$ is naturally equivalent to $\operatorname{Gen}(P_R)$.

Proof. By Proposition (4.12) of [AF], $A P_B$ is faithfully balanced. Endow $R$ with the $P$-topology $\tau$. $\tau$ is a right linear topology on $R$ and has as a basis of neighbourhoods of 0 the right ideals of the form $\operatorname{Ann}_R(F)$ where $F$ is a finite subset of $P$. Let $\mathcal{F}_r$ be the filter of all right ideals of $R$ which are open in $(R, \tau)$. Set $\mathcal{F}_r = \{M \in \text{Mod-}R: \forall x \in M, \operatorname{Ann}_R(x) \in \mathcal{F}_r\}$. Then $\mathcal{F}_r = \operatorname{Gen}(P_R)$. Indeed it is obvious that $\operatorname{Gen}(P_R) \subseteq \mathcal{F}_r$. On the other hand let $M \in \mathcal{F}_r$ and $x \in M$. Then $\operatorname{Ann}_R(x) \supseteq \operatorname{Ann}_R(p_1, \ldots, p_n)$ where $\{p_1, \ldots, p_n\}$ is a finite subset of $P$. We have

$$
\bigcap_{i=1}^n \operatorname{Ann}_R(p_i) \subseteq \bigoplus_{i=1}^n \frac{R}{\operatorname{Ann}_R(p_i)} \cong \bigoplus_{i=1}^n p_i R \in \operatorname{Gen}(P_R).
$$

Since $\operatorname{Gen}(P_R) = \overline{\operatorname{Gen}(P_R)}$ it is $R/ \bigcap_{i=1}^n \operatorname{Ann}_R(p_i) \in \operatorname{Gen}(P_R)$. It follows that $x R \in \operatorname{Gen}(P_R)$ since $x R$ is an homomorphic image of $R/ \bigcap_{i=1}^n \operatorname{Ann}_R(p_i)$. By Theorem 2.2, $B$ is the Hausdorff completion of $(R, \tau)$, since $\tau$ is the relative topology on $R/\operatorname{Ann}_R(P)$ of the finite topology of $\text{End}(A_P)$. Let $\bar{\tau}$ the topology of $B$. It is clear that $\bar{\tau}$ is the $P$-topology of $B$. For every $I \in \mathcal{F}_r$, let $\bar{I}$ be the closure of $I/\operatorname{Ann}_R(P)$ in $B$. Then $\mathcal{F}_r = \{\bar{I}: I \in \mathcal{F}_r\}$ is a basis of neighbourhoods of 0 in $(B, \bar{\tau})$ and $R/I \cong B/\bar{I}$ both in $\text{Mod-}R$ and in $\text{Mod-}B$. Therefore $\operatorname{Gen}(P_R) = \overline{\operatorname{Gen}(P_B)}$.

3.2. Remark. Let $P_R$ be a $W$-module, $A = \text{End}(P_R)$, $\Im(H) = \mathcal{O}(K_A)$. Then, in general, $\Im(H) \neq \mathcal{O}(K_A)$ (cf. Lemma 1.6), as the following example shows.

Example. Let $P_R$ a generator of $\text{Mod-}R$, $A = \text{End}(P_R)$. Clearly $P_R$ is a $W$-module. Assume that $\Im(H) = \mathcal{O}(K_A)$. Then by Proposition 3.2 of [Mo1], $\Im(H) = \text{Mod-A}$. (This is a generalization of Fuller’s Theorem on Equivalences [F]). It follows that the functors $T$ and $H$ give an
equivalence between Mod-A and Mod-R. By a well known result of Morita [M], $P_R$ is a progenerator of Mod-R. If $P_R$ is a generator non progenerator in Mod-R then $\text{Im}(H) \neq \mathbb{G}(K_A)$

3.3. The Remark 3.2 shows that the theory of W-modules is not trivial even if $P_R$ is a generator of Mod-R so that $\text{Gen}(P_R) = \text{Mod-R}$. (See [WW]).

3.4. We conclude this section giving another generalization of Fuller's Theorem on Equivalences. Namely, if $P_R$ is a W-module and if $\text{Im}(H)$ is closed under taking homomorphic images, then $\text{Im}(H) = \text{Mod-A}$. For this purpose we need some preliminar results.

3.5. Let $P_R \in \text{Mod-R}$, $A = \text{End}(P_R)$, $M \in \text{Gen}(P_R)$. Consider an epimorphism $h: P_R^{(X)} \to M \to 0$ where $X$ is a suitable set. Clearly $h = \sum_{x \in X} h_x \in \text{Hom}_R(P_R, M)$. Therefore there exists a natural injection

$$i: \sum_{x \in X} h_x A \to \text{Hom}_R(P_R, M).$$

An Azumaya's Lemma (cf. [A], Lemma 1) guarantees that, if $\rho_M$ is injective, then the canonical morphism

$$T(i): \left( \sum_{x \in X} h_x A \right) \otimes_A P \to \text{Hom}_R(P_R, M) \otimes_A P$$

is surjective.

3.6. LEMMA. Let $P_R$ be a W-module, $A = \text{End}(P_R)$ and assume that $\text{Im}(H)$ is closed under taking homomorphic images. Let $M \in \text{Gen}(P_R)$ and let $h = (h_x)_{x \in X}$ an epimorphism of $P_R^{(X)}$ onto $M$. Then

$$\sum_{x \in X} h_x A = \text{Hom}_R(P, M).$$

PROOF. We have in Mod-A the exact sequence

$$0 \to \sum_{x \in X} h_x A \xrightarrow{i} \text{Hom}_R(P, M) \to V \to 0.$$

By assumption $V \in \text{Im}(H)$. Applying the exact functor $- \otimes_A P$ we get the exact sequence:

$$0 \to \left( \sum_{x \in X} h_x A \right) \otimes_A P \xrightarrow{T(i)} \text{Hom}_R(P, M) \otimes_A P \to V \otimes_A P \to 0.$$
Since $P_R$ is a $W$-module, $\varphi_M$ is an isomorphism (cf. Theorem 2.3 and Proposition 1.8). Therefore, by Azumaya's Lemma, $T(i)$ is surjective so that $V \otimes_A P = 0$. It follows $V = 0$ since the bimodule $_A P_R$ represents a category equivalence between $\text{Im}(H)$ and $\text{Gen}(P_R)$.

3.7. **Definition.** Recall that a module $P_R \in \text{Mod-}R$ is $\Sigma$-quasi-projective if for every diagram with exact row

$$
\begin{array}{c}
P_R \\
\downarrow f \\
P_R^{(X)} \rightarrow M \rightarrow 0
\end{array}
$$

there exists $\alpha \in \text{Hom}_R(P_R, P_R^{(X)})$ such that $f = h \circ \alpha$.

3.8. **Definition.** Let $P_R \in \text{Mod-}R$, $A = \text{End}(P_R)$. Recall that $P_R$ is self-small if for every set $X \neq \emptyset$ we have

$$
\text{Hom}_R(P_R, P_R^{(X)}) \cong \text{Hom}_R(P_R, P_R^{(X)}) = A^{(X)}
$$
canonically.

3.9. **Proposition.** Let $P_R \in \text{Mod-}R$, $A = \text{End}(P_R)$. The following conditions are equivalent:

(a) For every $M \in \text{Gen}(P_R)$ and for every epimorphism $h = (h_x)_{x \in X} : P_R^{(X)} \rightarrow M \rightarrow 0$ we have:

$$
\sum_{x \in X} h_x A = \text{Hom}_R(P, M)
$$

(b) $P_R$ is $\Sigma$-quasi-projective and self-small.

**Proof.** (a) $\Rightarrow$ (b). Consider the diagram (1) of 3.7. By assumption we have $f = \sum_{x \in X} h_x a_x$, with $a_x \in A$ and almost all $a_x$'s vanish. Consider the morphism $g : P \rightarrow P_R^{(X)}$ given by $g = (a_x)_{x \in X}$. Then $f = h \circ g$. Therefore $P_R$ is $\Sigma$-quasi-projective. Let us show that $P_R$ is self-small. Let $i_x : P_R \rightarrow P_R^{(X)}$ the $x$-th inclusion and consider the diagram with exact row

$$
\begin{array}{c}
P_R \\
\downarrow f \\
0 \rightarrow P_R^{(X)} \rightarrow P_R^{(X)} \rightarrow 0
\end{array}
$$
We have \( f = \sum_{x \in X} i_x a_x \) with \( a_x \in A \) and almost all \( a_x \)'s vanish. Let \( g = (a_x)_{x \in X} \). Then \( g \in A^{(X)} \) and \( f = i \circ g \), hence \( \text{Hom}_R(P_R, P_R^{(X)}) \cong A^{(X)} \).

(b) \( \Rightarrow \) (a). Let \( f \in \text{Hom}_R(P, M) \) and let \( h: P_R^{(X)} \to M \to 0 \) be an epimorphism. Then there exists a morphism \( g: P_R \to P_R^{(X)} \) such that \( f = h \circ g \). On the other hand \( g = (a_x)_{x \in X} \) with \( a_x \in A \) and almost all \( a_x \)'s vanish, hence \( f \in \prod_{x \in X} h_x A \).

3.10. THEOREM. Let \( P_R \) be a \( W \)-module, \( A = \text{End}(P_R) \) and assume that \( \text{Im}(H) \) is closed under taking homomorphic images. Then \( \text{Im}(H) = \text{Mod-}A \).

PROOF. We have \( T(A^{(X)}) = A^{(X)} \otimes_A P \cong P_R^{(X)} \) in a natural way. By Lemma 3.6 and Proposition 3.9 \( P_R \) is self-small, hence \( H(P_R^{(X)}) = \text{Hom}_R(P_R, P_R^{(X)}) \cong A^{(X)} \). Thus \( A^{(X)} \in \text{Im}(H) \). Let \( L \in \text{Mod-}A \). There exists an exact sequence \( A^{(X)} \to L \to 0 \), so that \( L \in \text{Im}(H) \).

4. The torsion theory \((\text{Im}(T), \omega(K_A))\).

From now on we assume the reader familiar with some elementary facts on torsion theories. See [St] or [N].

4.1. In all this section \( P_R \) is a \( W \)-module with \( A = \text{End}(P_R) \). Set, as usual, \( T = - \otimes_A P \) and \( H = \text{Hom}_R(P_R, -) \). The bimodule \( _A P_R \) represents an equivalence between \( \text{Im}(H) \) and \( \text{Gen}(P_R) = \overline{\text{Gen}(P_R)} \).

4.2. Consider the following subcategory of \( \text{Mod-}A \)

\[ \text{Ker}(T) = \{ L \in \text{Mod-}A : L \otimes_A P = 0 \} . \]

Clearly \( \text{Im}(H) \cap \text{Ker}(T) = 0 \).

4.3. LEMMA. \( \text{Ker}(T) \) is a localizing subcategory of \( \text{Mod-}A \), i.e. \( \text{Ker}(T) \) is the torsion class for a hereditary torsion theory on \( \text{Mod-}A \).

PROOF. It is obvious that \( \text{Ker}(T) \) is closed under taking homomorphic images, direct sums and extensions. On the other hand, since \( _A P \) is flat, \( \text{Ker}(T) \) is closed under taking submodules.

The Gabriel filter \( I' \) canonically associated to the localizing subcate-
gory $\text{Ker}(T)$ is given by setting
\[ \Gamma = \left\{ I \leq A_A : \frac{A}{I} \in \text{Ker}(T) \right\}. \]

Clearly
\[ \Gamma' = \left\{ I \leq A_A : IP = P \right\}. \]

Let $L \in \text{Mod-A}$. The torsion submodule $t_T(L)$ of $L$ is defined by setting
\[ t_T(L) = \left\{ x \in L : \text{Ann}_A(x) \in \Gamma \right\}. \]

Then the category of torsion-free modules is
\[ \mathcal{S}_T = \left\{ L \in \text{Mod-A} : t_T(L) = 0 \right\}. \]

For every $L \in \text{Mod-A}$, $L/t_T(L)$ is torsion free. If no confusion arises, we write $t(L)$ instead of $t_{t_T(L)}$.

Let $Q_R$ be a fixed, but arbitrary, injective cogenerator of $\text{Mod-R}$, $K_A = \text{Hom}_R(P_R, Q_R)$, $\mathcal{O}(K_A)$ the subcategory of $\text{Mod-A}$, cogenerated by $K_A$. Since $A_P$ is flat, $K_A$ is injective in $\text{Mod-A}$.

4.4. LEMMA.
\[ \text{Ker}(T) = \left\{ L \in \text{Mod-A} : \text{Hom}_A(L, K_A) = 0 \right\}. \]

PROOF. For every $L \in \text{Mod-A}$ we have the canonical isomorphisms:
\[ \text{Hom}_A(L, K_A) = \text{Hom}_A(L, \text{Hom}_R(P, Q)) \cong \text{Hom}_R(L \otimes_A P, Q_R). \]

Since $Q_R$ is a cogenerator in $\text{Mod-R}$ we have
\[ \text{Hom}_A(L, K_A) = 0 \iff L \otimes_A P = 0 \iff L \in \text{Ker}(T). \]

4.5. PROPOSITION.
\[ \mathcal{S}_T = \mathcal{O}(K_A). \]

PROOF. Let $L \in \mathcal{S}_T$. Then $t_T(L) = 0$. Let $l \in L$, $l \neq 0$. Then $lA \notin \text{Ker}(T)$ hence, by Lemma 4.4, $\text{Hom}_A(lA, K_A) \neq 0$. Let $f : lA \to K_A$ a non zero morphism. Since $K_A$ is injective in $\text{Mod-A}$, $f$ extends to a morphism $\tilde{f} : L \to K_A$ and $\tilde{f}(l) \neq 0$. It follows $L \in \mathcal{O}(K_A)$.

Conversely let $L \in \mathcal{O}(K_A)$ and let $L' \leq L$ such that $L' \in \text{Ker}(T)$. By
Lemma 4.4 we have $\text{Hom}_A(L', K_A) = 0$. On the other hand there exists an exact sequence $0 \to L \to K_A^X$ where $X$ is a suitable set. Then $L' = 0$, so that $L \in \mathcal{T}_\Gamma$.

4.6. **Corollary.** (a) The torsion theory $(\text{Ker}(T), \mathcal{O}(K_A))$ is cogenerated by the injective module $K_A$.

(b) Since $\text{Im}(H) \subseteq \mathcal{O}(K_A)$, the modules in $\text{Im}(H)$ are torsion-free.

4.7. **Proposition.** For every $L \in \text{Mod}-A$ consider the canonical morphism $\sigma_L: L \to \text{Hom}_R(P_R, L \otimes_A P)$. Then:

$$t_\Gamma(L) = \text{Ker}(\sigma_L).$$

**Proof.** We have:

\[
\text{Ker}(\sigma_L) = \{l \in L: l \otimes p = 0, \forall p \in P\} = \{l \in L: lA \otimes_A P = 0\} = \\
= \{l \in L: lA \in \text{Ker}(T)\} = t_\Gamma(L). 
\]

4.8. Let $L \in \text{Mod}-A$, $I, J \in \Gamma$, $I \geq J$. Consider the natural morphism

$$\text{Hom}_A(I, L) \to \text{Hom}_A(J, L)$$

given by restrictions. For every $L \in \text{Mod}-A$ set:

$$L_\Gamma = \lim_{I \in \Gamma} \text{Hom}_A(I, \frac{L}{t_\Gamma(L)})$$

and, since $A$ is torsion-free

$$A_\Gamma = \lim_{I \in \Gamma} \text{Hom}_A(I, A).$$

It is well known that $L_\Gamma$ is a right $A$-module, $A_\Gamma$ is a ring and moreover $L_\Gamma$ is a right $A_\Gamma$-module.

$A_\Gamma$ is called the ring of quotients of $A$ and $L_\Gamma$ the module of quotients of $L$ with respect to the Gabriel filter $\Gamma$. For every $L \in \text{Mod}-A$, $L_\Gamma$ is also called the localization of $L$ at $\Gamma$.

For every $L \in \text{Mod}-A$ there exists a canonical morphism $\varphi_L: L \to L_\Gamma$
such that
\[
\text{Ker}(\varphi_L) = t_r(L), \quad \frac{L^r}{\varphi_L(L)} \in \text{Ker}(T), \quad \varphi_L(L) \in \mathcal{O}(K_A)
\]

\(\varphi_A: A \to A^r\) is a ring morphism.

**4.9. Lemma.** Let \(I \in \Gamma\).

Then \(\text{Hom}_A(A/I, A) = 0\) and \(\text{Ext}_A^1(A/I, A) = 0\).

**Proof.** See [WW] Proposition 1.2.

**4.10. Corollary.** The canonical morphism \(\varphi_A: A \to A^r\) is a ring isomorphism.

**Proof.** By Corollary 4.6 \(A\) is torsion-free. Let \(I \in \Gamma\) and let \(\alpha_I: I \to A\) be the canonical inclusion. By Lemma 4.9 the exact sequence

\[0 \to I \xrightarrow{\alpha_I} A \to A/I \to 0\]

gives rise to the exact sequence:

\[0 = \text{Hom}_A(A/I, A) \to \text{Hom}_A(A, A) \xrightarrow{\alpha_I^*} \text{Hom}_A(I, A) \to \text{Ext}_A^1(A/I, A) = 0.
\]

Therefore \(\alpha_I^*: \text{Hom}_A(A, A) \to \text{Hom}_A(I, A)\) is an isomorphism i.e. any morphism \(I \to A\) extends uniquely to an element of \(A\). Then, if \(I, J \in \Gamma\) and \(I \geq J\), the restriction map \(\text{Hom}_A(I, A) \to \text{Hom}_A(J, A)\) is an isomorphism.

**4.11. Definitions.** Recall that a module \(L \in \text{Mod}-A\) is \(\Gamma\)-injective if for every \(I \in \Gamma\) the restriction morphism

\[(1) \quad \text{Hom}_A(A, L) \to \text{Hom}_A(I, L)\]

is surjective.

\(L\) is \(\Gamma\)-injective if and only if \(\text{Ext}_A^1(N, L) = 0\) for every \(N \in \text{Ker}(T)\).

A module \(L \in \text{Mod}-A\) is called \(\Gamma\)-closed if for every \(I \in \Gamma\) the above morphism (1) is an isomorphism.

The following results are classical in torsion theories.

**4.12. Theorem.** Let \(L \in \text{Mod}-A\). The following conditions are equivalent:

(a) \(L\) is \(\Gamma\)-closed;
(b) $L \in \mathcal{O}(K_A)$ and $L$ is $\Gamma$-injective;

(c) for every morphism $\alpha: U \to V$ in $\text{Mod-}A$ such that $\text{Ker}(\alpha) \in \text{Ker}(T)$ and $\text{Coker}(\alpha) \in \text{Ker}(T)$, the transposed morphism $\text{Hom}_A(V, L) \to \text{Hom}_A(U, L)$ is an isomorphism;

(d) the canonical morphism $\varphi_L: L \to L_{\Gamma}$ is an isomorphism.

4.13. COROLLARY. For every $L \in \text{Mod-}A$, $L_{\Gamma}$ is $\Gamma$-closed.

5. A characterization of $\text{Im}(H)$.

5.1. In all this section we work in situation 4.1.

Denote by $\text{Mod-}(A, \Gamma)$ the subcategory of $\text{Mod-}A$ whose objects are all the $\Gamma$-closed modules in $\text{Mod-}A$. By Theorem 4.12 we can write

$$\text{Mod-}(A, \Gamma) = \{ L \in \text{Mod-}A : L = L_{\Gamma} \} .$$

Our main result is the following theorem which, together with Theorem 2.6, gives easily the Popescu-Gabriel Theorem in our setting.

5.2. THEOREM. Let $P_R \in \text{Mod-}R$ be a $W$-module, $A = \text{End}(P_R)$, $H = \text{Hom}_R(P_R, -)$, $T = - \otimes_A P$. Then for every $L \in \text{Mod-}A$ we have

$$L_{\Gamma} = HT(L) .$$

PROOF. For every $L \in \text{Mod-}A$, we have

$$(1) \quad T(L_{\Gamma}) = T(L) .$$

Indeed, consider the exact sequence

$$0 \to t_{\Gamma}(L) \to L \xrightarrow{\varphi_L} L_{\Gamma} \to L_{\Gamma}/\varphi_L(L) \to 0 .$$

Tensoring by $A P$ and since $t_{\Gamma}(L)$ and $L_{\Gamma}/\varphi_L(L)$ are in $\text{Ker}(T)$, we get (1).

We now prove that, for every $L \in \text{Mod-}A$, $L_{\Gamma} \in \text{Im}(H)$ from which it will follow

$$L_{\Gamma} \cong HT(L)$$

by Theorem 2.3.

Indeed assume $L = L_{\Gamma}$. Since $L \in \mathcal{O}(K_A)$ (cf. Theorem 4.12), $\sigma_L$ is injective. Consider the exact sequence

$$(2) \quad 0 \to L \xrightarrow{\varphi_L} HT(L) \to \text{Coker}(\sigma_L) \to 0 .$$
Since $T$ and $H$ are adjoint functors there exists the commutative diagram

\[
\begin{array}{c}
T(L) \xrightarrow{T(a_L)} THT(L) \\
\downarrow T \circ p_L \\
T(L)
\end{array}
\]

Since $T(L) \in \text{Gen}(P_R)$, $\varphi_{T(L)}$ is an isomorphism hence $T(\sigma_L)$ is an isomorphism too. Applying $T$ in (2) we get the exact sequence

\[0 \to T(L) \xrightarrow{T(\sigma_L)} THT(L) \to T(\text{Coker}(\sigma_L)) = 0.\]

It follows $\text{Coker}(\sigma_L) \in \text{Ker}(T)$. Since $L$ is $I'$-injective, the exact sequence (2) splits hence:

\[HT(L) \cong L \oplus \text{Coker}(\sigma_L);\]

therefore $\text{Coker}(\sigma_L) = 0$ because $HT(L)$ is torsion-free. Thus $\sigma_L$ is an isomorphism and $L \in \text{Im}(H)$.

5.3. COROLLARY. Under the assumptions of Theorem 5.2

\[\text{Im}(H) = \text{Mod} - (A, I').\]

PROOF. Let $L \in \text{Im}(H)$. Then $L \cong HT(L)$, hence $L = L_{I'}$. If $L = L_{I'}$ then $L = HT(L)$, hence $L \in \text{Im}(H)$.

6. The trace ideal of $A^P$ in $A$.

6.1. Let $P_R$ be a $W$-module, $A = \text{End}(P_R)$. Define the trace ideal $\tau$ of $A^P$ in $A$ by setting

\[\tau = \sum \{\text{Im}(f): f \in \text{Hom}_A(P, A)\};\]

$\tau$ is a two-sided ideal of $A$.

6.2 LEMMA ([WW], Proposition 1.5 and Theorem 1.6). Let $P_R$ be a $W$-module, $A = \text{End}(P_R)$. Then $\tau \subseteq \bigcap_{I \in \Gamma} I$.

If moreover $P_R$ is a generator of $\text{Mod}-R$, then:

a) $\tau P = P$ so that $I \in \Gamma$ if and only if $I \supseteq \tau$;

b) $\tau^2 = \tau$;
c) the left annihilator of \( \tau \) is 0;

d) \( \tau \) is finitely generated as a two-sided ideal;

e) \( \tau \) is essential as a right ideal.

6.3 COROLLARY. Let \( P_R \) be a generator of \( \text{Mod-}R \), \( A = \text{End} (P_R) \). Then for every \( L \in \text{Mod-A} \)

\[
L_r = \text{Hom} \left( \tau, \frac{L}{t_r(L)} \right).
\]

7. An example: closed spectral subcategories of \( \text{Mod-}R \).

7.1. Let \( \mathcal{S}_R \) be a closed subcategory of \( \text{Mod-}R \), \( P_R \) a generator of \( \mathcal{S}_R \), \( A = \text{End} (P_R) \). Set, as usual, \( T = - \otimes_A P \), \( H = \text{Hom}_R (P_R, -) \). Let \( \Gamma \) be the Gabriel filter associated to the hereditary torsion theory \( (\text{Ker} (T), \mathcal{O}(KA)) \). Then \( \mathcal{S}_R \) is naturally equivalent to the subcategory \( \text{Im} (H) = \text{Mod-} (A, \Gamma) \) of \( \text{Mod-A} \).

Recall that the subcategory \( \text{Mod-} (A, \Gamma) \) is closed under taking injective envelopes and direct products in \( \text{Mod-A} \).

7.2. We are interested in finding conditions in order that every module \( L \in \text{Mod-} (A, \Gamma) \) is injective in \( \text{Mod-} (A, \Gamma) \) or, equivalently, in \( \text{Mod-A} \).

7.3 LEMMA. The sequence in \( \text{Mod-} (A, \Gamma) \)

\[
0 \to L \to^f M \to^g N \to 0
\]

is exact in \( \text{Mod-} (A, \Gamma) \) if and only if

1) \( f \) is injective;

2) \( \text{Im} (f) = \text{Ker} (g) \);

3) \( N/\text{Im} (g) \in \text{Ker} (T) \).

PROOF. Assume that (1) is exact in \( \text{Mod-} (A, \Gamma) \). Then we have the exact sequence

\[
0 \to T(L) \to^{T(f)} T(M) \to^{T(g)} T(N) \to 0 \quad \text{in } \mathcal{S}_R.
\]
Since $\mathcal{S}_R$ is closed, (2) is exact in Mod-$R$. Therefore the sequence
\[ 0 \to HT(L) \xrightarrow{HT(f)} HT(M) \xrightarrow{HT(g)} HT(N) \]
is exact in Mod-$A$; thus $f$ is injective and $\text{Im}(f) = \text{Ker}(g)$.

Assume that $N/\text{Im}(g) \not\approx \text{Ker}(T)$. Then we have the exact sequence in Mod-$A$:
\[ 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to N/\text{Im}(g) \to 0. \]

Applying $T$ we get $T(N/\text{Im}(g)) \not= 0$, in contrast with (2).

Conversely, if conditions 1), 2) and 3) hold for the sequence (1), then the sequence (2) is exact in $\mathcal{S}_R$ and (1) is exact in Mod-$\langle A, I^\prime \rangle$.

7.4. Assume that every module in Mod-$\langle A, I^\prime \rangle$ is injective. Let
\[ 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \]
be an exact sequence in Mod-$\langle A, I^\prime \rangle$. Since $L$ is injective, we have
\[ M = L \oplus L' \quad \text{in Mod-}A, \]
where $L' \equiv \text{Im}(g) \leq N$. Let us show that
\[ L' \equiv N \text{ canonically.} \]

Observe that $L' \in \text{Mod-} (A, I^\prime)$. In fact $L'$ is torsion free and, being injective, it is $I'$-injective. We have $N \equiv L' \oplus L''$, with $L'' \equiv N/\text{Im}(g)$. Since $L' \in \mathcal{O}(K_A)$ and $L'' \in \text{Ker}(T)$ we get $N \equiv L'$.

7.5 PROPOSITION. Assume that every module in Mod-$\langle A, I^\prime \rangle$ is injective. The sequence
\[ 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \]
with $L, M, N \in \text{Mod-} (A, I^\prime)$ is exact in Mod-$A$ if and only if it is exact in Mod-$A$.

In this case (1) splits.

7.6 LEMMA. Let $M \in \mathcal{S}_R = \text{Gen}(P_R)$, $N \in \text{Mod-}R$ and let $f: M \to N$ be a morphism. Then $\text{Im}(f) \leq t_P(N)$.

PROOF. Assume that $M = P^{(X)}_R$, where $X \neq \emptyset$ is a set. Let $h: P^{(X)} \to N$ be a morphism. Then $h = (h_x)_{x \in X}$, with $h_x \in \text{Hom}_R(P, N)$. Let $p \in P^{(X)}$. Then $p = (p_x)_{x \in X}$, with $p_x \in P$ and $p_x = 0$ for almost all $x \in X$. 

We have
\[ h(\mu) = \sum_{x \in X} h_x(p_x) \in t_p(N). \]

Let \( M \in \text{Gen}(P_R), f \in \text{Hom}_R(M, N) \). There exists a diagram
\[ P^{(X)} \xrightarrow{h} M \xrightarrow{f} N \]
with \( h \) a surjective morphism. It is \( f \circ h \in \text{Hom}_R(P^{(X)}, N) \), hence
\[ \text{Im}(f \circ h) \leq t_p(N) \text{ and } \text{Im}(f \circ h) = \text{Im}(f). \]

7.7 Proposition. Let \( \mathcal{S}_R \) be a closed subcategory of \( \text{Mod}-R \), \( P_R \) a generator of \( \mathcal{S}_R \), \( A = \text{End}(P_R) \). The following conditions are equivalent:

(a) every module in \( \text{Mod}-(A, I') \) is injective;

(b) \( \mathcal{S}_R \) is a spectral category.

In this case every module in \( \mathcal{S}_R \) is semisimple.

Proof. (a) \( \Rightarrow \) (b) By Proposition 7.5 every short exact sequence in \( \mathcal{S}_R \) splits. Therefore such a sequence splits in \( \text{Mod}-R \). Then every module in \( \mathcal{S}_R \) is semisimple so that \( \mathcal{S}_R \) is spectral.

(b) \( \Rightarrow \) (a) Let \( L \in \text{Mod}-(A, I') = \text{Im}(H) \). Then \( L = H(M) \), with \( M \in \mathcal{S}_R \). Since \( Q_R \) is a cogenerator in \( \text{Mod}-R \), there exists an exact sequence in \( \text{Mod}-R \)
\[ 0 \rightarrow M \rightarrow Q^X \]
where \( X \) is a suitable set. By Lemma 7.6, \( \text{Im}(f) \leq t_p(Q^X_R) \in \text{Gen}(P_R) = \mathcal{S}_R \). Since \( \mathcal{S}_R \) is spectral, \( M \) is a direct summand of \( t_p(Q^X_R) \). Therefore \( L = H(M) \) is a direct summand of \( H(t_p(Q^X_R)) \). On the other hand, \( H(t_p(Q^X_R)) \equiv H(Q^X_A) = K^X_A \) which is injective.

7.8 Proposition. Let \( \mathcal{S}_R \) be a closed spectral subcategory of \( \text{Mod}-R \), \( P_R \) a generator of \( \mathcal{S}_R \) and \( A = \text{End}(P_R) \). Then:

a) for every \( L \in \text{Mod}-A \) the following conditions are equivalent:

(i) \( L \in \text{Mod}-(A, I') \);

(ii) \( L \) is a direct summand of a module of the form \( A^X \), where \( X \) is a non empty set;

b) the ring \( A \) is von Neumann regular and right self-injective.
PROOF. a) (i) ⇒ (ii) Let \( X \) be a non empty set. We show that 
\( H(P_R^{(X)}) \) is a direct summand of \( A^X \). In fact:
\[
H(P_R^{(X)}) \leq H(P_R^X) \cong H(t_P(P_R^X)) \cong A^X \in \text{Mod} - (A, \Gamma').
\]
Since \( H(P_R^{(X)}) \) is injective, \( H(P_R^{(X)}) \) is a direct summand of \( A^X \). Let 
\( L \in \text{Mod} - (A, \Gamma') \) be an injective module. Then \( L = H(M) \), for some 
\( M \in \mathcal{S}_R \). Then \( H(M) \) is a direct summand of a module of the form 
\( H(P_R^{(X)}) \), hence \( L \) is a direct summand of \( A^X \).

(ii) ⇒ (i) If \( L \) is a direct summand of \( A^X \), \( L \) is torsion free and it is 
\( \Gamma' \)-injective, being injective. Therefore \( L \in \text{Mod} - (A, \Gamma') \).

b) Since \( P_R \) is semisimple, \( A \) is von Neumann regular (cf.[St], 

7.9. Let \( \mathcal{S}_R \) be a closed spectral subcategory of \( \text{Mod}-R \), \( P_R \) a generator 
of \( \mathcal{S}_R \) and \( A = \text{End}(P_R) \). In this case the filter \( \Gamma' \) has a nice descrip-
tion using the trace ideal of \( _A^P \) in \( _A A \).

7.10. Fix a simple module \( S \in \text{Mod}-R \) and denote by \( \Sigma(S) \) the spectral 
subcategory of \( \text{Mod}-R \) consisting of all semisimple modules which 
are a direct sum of copies of \( S \).

Fix a positive cardinal number \( \alpha \). Then
\[
P_R = S^{(\alpha)}
\]
is a projective generator and an injective cogenerator of \( \Sigma(S) \). Let 
\( D = \text{End}(S_R) \), \( A = \text{End}(P_R) \). Then \( D \) is a division ring and \( A \) is the 
ring of all \( \alpha \times \alpha \) matrices, with entries in \( D \), whose columns have only a 
finite number of non zero elements. It follows that \( A \cong \text{End}(D^{(\alpha)}) \), 
where \( D^{(\alpha)} \) is considered as a right vector space over the division ring 
\( D \).

Let \( \Gamma \) be the usual Gabriel filter on \( A \). Let \( \tau \) be the trace ideal of \( _A^P \) in \( _A A \):
\[
\tau = \sum \{ \text{Im}(g) : g \in \text{Hom}_{_A}(A^P, A) \}.
\]

a) \( _A^P \) is a semisimple module in \( A-\text{Mod} \).

PROOF. Since \( P_R \) is an injective cogenerator of \( \Sigma(S) \), then \( P_R \) is 
strongly quasi-injective in the sense of[MO1]. Applying Proposition 
6.10 of[MO1] we have 
\[
\text{Soc}(A^P) = \text{Soc}(P_R) = P
\]
and thus \( _A^P \) is semisimple.
Let $L_\omega$ be the minimal two-sided non zero ideal of $A$. As it is well known, $L_\omega$ consists of all the endomorphism of $D^{(a)}$ whose image is finite dimensional. $L_\omega$ has the following properties:

i) $L_\omega = \text{Soc}(A) = \text{Soc}(A_A)$;

ii) the right ideals of $A$ containing $L_\omega$ are exactly the essential right ideals of $A$.

Therefore we have, by i),

$$\tau = \sum \{\text{Im}(g) : g \in \text{Hom}_A(A_P, L_\omega)\}.$$ 

Thus $\tau \leq L_\omega$.

b) The trace ideal $\tau = L_\omega$.

**PROOF.** Let us show that $\tau \neq 0$; it will follow that $\tau = L_\omega$, since $\tau$ is two-sided and $L_\omega$ is the minimal two-sided non zero ideal of $A$.

Let $J$ be a maximal right ideal of $R$ such that $R/J = S_R$. The exact sequence

$$0 \rightarrow J \rightarrow R \rightarrow R/J \rightarrow 0$$

gives rise, by applying $\text{Hom}_R(-, P_R)$, to the exact sequence

$$0 \rightarrow \text{Hom}_R(R/J, P_R) = \text{Ann}_P(J) \rightarrow A_P.$$ 

Thus

$$\text{Ann}_P(J) \cong \text{Hom}_R(S, P_R) \cong D^{(a)} \hookrightarrow A_P,$$

since $S_R$ is finitely generated. Therefore $A_P$ contains a direct summand of the form $\text{Ann}_P(J) \cong \text{Hom}_R(S, P_R) \cong D^{(a)}$ and it is well known that $\text{Hom}_A(D^{(a)}, A) \neq 0$.

\[c) \text{Let } f \text{ be an endomorphism of } P_R \text{ such that } \text{Im}(f) \text{ is finitely generated. Then } f \in L_\omega.\]

**PROOF.** In fact $f$ may be represented by an $\alpha \times \alpha$ matrix having only a finite number of non zero rows. Then this matrix represents an endomorphism of $D^{(a)}$ whose image is finite dimensional. Therefore $f \in L_\omega$.

\[d) \text{ } L_\omega P = P; \text{ hence } L_\omega \in \Gamma \text{ and thus}\]

$$\Gamma = \{I \subseteq A_A : I \ni \tau\}.$$ 

**PROOF.** Let $x \in P_R$, $x \neq 0$, and let $f$ be the projection of $P_R$ onto the
submodule $F$ generated by $x$, such that $f(x) = x$. Since $F$ is finitely generated, $f \in L_\omega$. Thus $L_\omega P = P$.

The last statement follows from Lemma 6.2.

We now consider closed spectral subcategories of Mod-$R$ in the general case.

7.11 PROPOSITION. Let $S_R$ be a closed spectral subcategory of Mod-$R$, $P_R$ a generator of $S_R$ and $A = \text{End}(P_R)$. Let $\Gamma$ be the usual Gabriel filter on $A$ and $\tau$ be the trace of $AP$ in $A A$. Then:

a) $\text{Soc}(A_A) = \text{Soc}(A_A) = \tau \neq 0$;

b) $\tau \in \Gamma$ and $\Gamma = \{I \subseteq A_A : I \geq \tau\}$.

Consequently $\Gamma$ consists of all essential right ideals of $A$.

PROOF. Let $(S_\delta)_{\delta \in \Delta}$ be a system of representatives of all non isomorphic simple modules in $S_R$. Set $D_\delta = \text{End}_R(S_\delta)$. We have

$$P_R = \bigoplus_{\delta \in \Delta} S_\delta^{(\alpha_\delta)},$$

where the $\alpha_\delta$'s are non zero cardinal numbers. $P_R$ is a projective generator and an injective cogenerator of $S_R$. Next we have:

$$A = \text{Hom}_R(P_R, P_R) \cong \text{Hom}_R \left( \bigoplus_{\delta \in \Delta} S_\delta^{(\alpha_\delta)}, \bigoplus_{\delta \in \Delta} S_\delta^{(\alpha_\delta)} \right) \cong$$

$$\cong \prod_{\delta \in \Delta} \text{Hom}_R(S_\delta^{(\alpha_\delta)}, S_\delta^{(\alpha_\delta)}) \cong \prod_{\delta \in \Delta} A_\delta,$$

where $A_\delta = \text{End}_R(S_\delta^{(\alpha_\delta)})$.

Let $\tau$ be the trace ideal of $AP$ in $A$; note that $\bigoplus_{\delta \in \Delta} A_\delta$ is essential in $A = \prod_{\delta \in \Delta} A_\delta$. Therefore:

$$\text{Soc}(A_A) = \bigoplus_{\delta \in \Delta} \text{Soc}(A_{\delta}) = \bigoplus_{\delta \in \Delta} L_{\omega}(\delta),$$

where $L_{\omega}(\delta)$ is the smallest two-sided ideal of the ring $A_\delta$. Hence

$$\text{Soc}(A_A) = \text{Soc}(A_A).$$

Then

$$\tau = \sum \left\{ \text{Im}(g) : g \in \text{Hom}_A \left( AP, \bigoplus_{\delta \in \Delta} L_{\omega}(\delta) \right) \right\}.$$

Hence

$$\tau = \bigoplus_{\delta \in \Delta} L_{\omega}(\delta) = \text{Soc}(A_A).$$
As we know, \( \tau \subseteq \bigcap \{ I : I \in \Gamma \} \). Let us show that \( \tau \in \Gamma \). In fact
\[
\left( \bigoplus_{\delta' \in \mathcal{D}} L_{\omega}(\delta') \right) \left( \bigoplus_{\delta \in \mathcal{D}} S_{\delta}(\alpha_{\delta}) \right) = \bigoplus_{\delta \in \mathcal{D}} L_{\omega}(\delta) S_{\delta}(\alpha_{\delta}) = \bigoplus_{\delta \in \mathcal{D}} S_{\delta}(\alpha_{\delta}).
\]

7.12 REMARK. We think that a number of more interesting examples may be constructed from the recent paper of Albu and Wisbauer [AW].

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