Centralizers of semisimple subgroups in locally finite simple groups

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Centralizers of Semisimple Subgroups in Locally Finite Simple Groups.

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The classification of finite simple groups has led to considerable progress in the study of the locally finite simple groups or LFS-groups as we will call them. In [7], B. Hartley and the author studied the centralizing properties of elements in LFS-groups. LFS-groups are usually studied in two classes; infinite linear LFS-groups and infinite non-linear LFS-groups. Infinite linear LFS-groups are the Chevalley groups and their twisted analogues over infinite locally finite fields [1], [2], [6] and [12]. Here we are mainly interested in non-linear LFS-groups.

In [9] we have defined semisimple elements for LFS-groups and studied the centralizers of these elements. Here we extend the definition of a semisimple element given in [9] to semisimple subgroups.

**Definition.** Let $G$ be a countably infinite LFS-group and $F$ be a finite subgroup of $G$. The group $F$ is called a $K$-semisimple subgroup of $G$, if $G$ has a Kegel sequence $K = (G_i, M_i)_{i \in \mathbb{N}}$ such that $\langle |M_i|, |F| \rangle = 1$, $M_i$ are soluble for all $i$ and if $G_i/M_i$ is a linear group over a field of characteristic $p_i$, then $\langle p_i, |F| \rangle = 1$.

This definition is a generalization of the $K$-semisimple element in [9]. In particular every element in a $K$-semisimple group is a $K$-semisimple element in the sense of [9]. B. Hartley and the author proved in [7], Theorem B that centralizers of $K$-semisimple elements in non-linear LFS-groups involve infinite non-linear LFS-groups.

In [5], the centralizers of subgroups are studied and the following questions are asked:

Is it the case that in a non-linear LFS-group the centralizer of every finite subgroup is infinite?

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Does the centralizer of every finite subgroup involve an infinite non-linear simple group?

A finite abelian group $F$ in a finite simple group $G$ of classical type or alternating is called a **nice group** if whenever $G$ is of type $B_l$ or $D_l$, then Sylow 2-subgroup of $F$ is cyclic. If $G$ is alternating group or of type $A_l$ or $C_l$, then every abelian subgroup is a nice group. In particular every abelian group of odd order is a nice group.

A finite abelian group in a countably infinite locally finite simple group $G$ is called a K-nice group if $F$ is a nice group in almost all Kegel components of a Kegel sequence $K$ of $G$. We prove here:

**THEOREM 1.** If $F$ is a K-nice abelian subgroup and $K$-semisimple in a non-linear LFS-group $G$, then $C_G(F)$ has a series of finite length in which the factors are either non-abelian simple or locally soluble moreover one of the factors is non-linear simple. In particular $C_G(F)$ is an infinite group.

**THEOREM 2.** Suppose that $G$ is infinite non-linear and every finite set of elements of $G$ lies in a finite simple group. Then

(i). There exist infinitely many finite abelian semisimple subgroups $F$ of $G$ and local systems $L$ of $G$ consisting of simple subgroups such that $F$ is nice in every member of $L$.

(ii) There exists a function $f$ from natural numbers to natural numbers independent of $G$ such that $C = C_G(F)$ has a series of finite length in which at most $f(|F|)$ factors are simple non-abelian groups for any $F$ as in (i). Furthermore $C$ involves a non-linear simple group.

Let us recall the definition of the group theoretical classes $T_n$ and $T_{n,r}$ given in [7].

**Definition.** $T_{n,r}$ consists of all groups (not necessarily locally finite) having a series of finite length in which at most $n$ factors are non-abelian simple and the rest are soluble groups, the sum of whose derived lengths is at most $r$.

**Definition.** $T_n$ consist of all locally finite groups having a series of finite length in which there are at most $n$ non-abelian simple factors and the rest are locally soluble.

The following Lemma is given in [7] Lemma 2.1.
LEMMA 1. (i) The classes $T_n$ and $T_{n,r}$ are closed under taking normal subgroups and quotients.

(ii) Let $N < M < G$. If $G \in T_{n,r}$ and $M/N$ is soluble, then the derived length of $M/N$ is at most $r$.

(iii) If $M < G$, $M \in T_m$ and $G/M \in T_m$ then $G \in T_{m+n}$.

LEMMA 2. Let $G$ be a group and $A$ be a finite automorphism group of $G$. Let $N$ be a normal $A$-invariant subgroup of $G$ and $C/N = C_{G/N}(A)$.

(i) If $N \leq Z(G)$, then $C_G(A) < C$ and $C/C_G(A)$ is isomorphic to a direct product of subgroups of $N$. In particular $C/G(A)$ is an abelian group.

(ii) If $[N, G, G, \ldots, G] = 1$ with a finite number of terms of $G$ and $C \in T_n$ (respectively $T_{n,r}$), then $C_G(A) \in T_n$ (respectively $T_{n,r}$).

The commutator in (ii) is left normed, so that $N$ lies in the hyper-center of $G$.

PROOF (i) Let $A = \langle a_1, \ldots, a_n \rangle$. For each $i = 1, 2, \ldots, n$ define a map

$$\phi_{a_i} : C \to N,$$

$$\phi_{a_i}(g) = [a_i, g],$$

$\phi_{a_i}$ is a homomorphism with $\ker \phi_{a_i} = C_{C}(a_i)$. So $C/C_{C}(a_i)$ is isomorphic to a subgroup of $N$. Then we get

$$C/C_{C}(a_i) \cong C/C_{C}(a_1) \times C/C_{C}(a_2) \times \cdots \times C/C_{C}(a_n).$$

Since each of $C/C_{C}(a_i)$ is isomorphic to a subgroup of $Z(G)$, the group $C/C_G(A)$ is abelian.

(ii) Let $N_0 = N$, $N_1 = [N, G], \ldots, N_{i+1} = [N_i, G]$. Then $N_k = 1$. We get each $N_i < G$ and a series

$$1 = N_k < N_{k-1} < \cdots < N_1 < N_0 = N.$$

Let $C_{G/N_i}(A) = C_i/N_i$ and $C_k = C_G(A)$. Since $N_{k-1} \leq Z(G)$, by (i) we have $C_{k-1}/C_{C_{k-1}}(A)$ is abelian.

$C_{i+1} < C_i$; to see this we define a map $\phi_{a_j}$ for each $a_j \in A$ as in the first case:

$$\phi_{a_j} : C_i \to N_i/N_{i+1},$$

$$g \to g^{-1}a_j^{-1}ga_jN_{i+1},$$
the intersection of the kernels of these maps is $C_{i+1}$; and $C_i/C_{i+1}$ is abelian. Hence $C_G(A) \lhd C$ and by Lemma 1 we get $C_G(A) \in T_n$.

**Lemma 3.** ([7], 2.3) (i) If $G \in T_{n,r}$, then $G$ has a finite series of length at most $2n + 1$, the factors of which comprise at most $n$ non-abelian simple factors, at most $n + 1$ soluble groups of derived length at most $r$ and no others.

(ii) $L \in T_n = T_n$.

Centralizers of elements in symmetric groups are well known.

**Lemma 4** ([7], 2.4). - Let $G$ be the symmetric group $\text{Sym}(l)$ and $x$ be an element of order $n$ in $G$. Suppose that the cycle decomposition of $x$ involves $k_i$ cycles of length $i$ ($1 \leq i \leq \min(n)$). Then

$$C_G(x) \cong Dr_{\mid n \cap L_i}$$

where $D_r$ denotes direct product, and $L_i$ is a permutational wreath product $C_i \wr \text{Sym}(k_i)$ of the cyclic group $C_i$ of order $i$ and the symmetric group $\text{Sym}(k_i)$ acting naturally on $k_i$ points. If $k_i = 0$, then $L_i$ is to be interpreted as 1.

**Lemma 5.** Let $F = \langle a_1, \ldots, a_n \rangle$ be an abelian subgroup of $G = \text{Sym}(l)$ and $|F| = m$. Then $C_G(F) \in T_{g(m)}$ where $g$ is a function of $m$ independent of $G$.

The proof of the Lemma 5 goes along the lines of the proof of the Lemma 4. We replace the argument on cycles of an element of equivalent length with the equivalent representations of $F$ on the orbits of $F$. But the bound in the Lemma 4 is no longer valid; the number of non-abelian simple factors in Lemma 5 is less than or equal to the number of subgroups of $F$.

Similarly this Lemma holds for alternating group $\text{Alt}(l)$.

If $l$ is sufficiently large, then $C_G(F)$ involves alternating groups of arbitrary high orders.

**Lemma 6.** Suppose that $G$ is infinite and every finite set of elements of $G$ lies in a finite alternating subgroup. Let $F$ be an abelian subgroup of order $m$ in $G$. Then $C = C_G(F)$ has a series of finite length in which the factors consist of at most $g(m)$ simple non-abelian groups. Further $C$ involves a non-linear simple group.

**Proof.** $G$ has a local system consisting of alternating subgroups and each subgroup in the local system contains $F$. Now by Lemma 5 we have $C_{G_i}(F) \in T_{g(m)}$, where $G_i$ is isomorphic to an alternating group and $i$
is taken from the index set \( I \). \( C_G(F) \) becomes locally \( T_{g(m)} \). By Lemma 3 we get \( C \in T_{g(m)} \) and we are done.

Now we will mention some of the facts about infinite LFS-groups. Some of the questions about infinite LFS-groups can be reduced to questions about countably infinite LFS-groups by using [8], Theorem 1.1.L.9 and Theorem 4.4. The question of whether the centralizer of a finite subgroup involves an infinite simple group or not is one of these types of questions. If in every countably infinite non-linear LFS-group the centralizer of every finite subgroup involves an infinite simple group, then in any infinite non-linear LFS-group centralizer of a finite subgroup also involves an infinite simple group. Therefore we confine ourselves to countable LFS-groups. For countable LFS-groups [8] Theorem 4.5 says that for every countably infinite LFS-group there exists a Kegel sequence \( K = (G_i, N_i) \) where \( G_i \)'s form a tower of finite subgroups of \( G \) satisfying \( G = \bigcup_{i=1}^{\infty} G_i, N_i \triangleleft G_i \), such that \( G_i/N_i \) is a finite simple group and \( G_i \cap N_{i+1} = 1 \) for each \( i \). By [8], Theorem 4.6 if \( G \) is an infinite linear LFS-group one can always choose an infinite subsequence \( (G_j, N_j) \) such that \( N_j = 1 \) for all \( j \).

By using classification of finite simple groups one can find that every LFS-group is either linear or \( G_i/N_i \) are all alternating or a fixed type of classical linear group over various fields with unbounded rank parameter. See [7] for more details about Kegel sequences.

**Theorem 3.** Let \( G \) be a connected reductive linear algebraic group and \( F \) be a finite subgroup of order \( m \) contained in a maximal torus \( T \) in \( G \). Then \( C_G(F) \in T_{f(m)k} \) where \( k \) is the number of simple components of the semisimple part of \( G \) when it is written as a product of simple linear algebraic groups and \( f \) is a function from natural numbers to natural numbers and is independent of \( G \).

By using the above theorem we prove:

**Theorem 4.** Let \( G \) be a connected simple linear algebraic group of classical type. Let \( F \) be a finite subgroup of order \( m \) contained in a maximal torus of \( G \). If \( F \) is fixed pointwise by a Frobenius automorphism \( \sigma \) of \( G \), then \( (C_G(F))^{\sigma} \in T_{f(m)} \) where \( f \) is a function from natural numbers to natural numbers and is independent of \( G \).

**Proof of Theorem 3.** Let \( G \) be a connected reductive linear algebraic group. Then by [10] (E 1.4) \( G = Z^0 G' \) where \( Z^0 \) is the connected component of the centre of \( G \) and \( G' \) is the commutator subgroup. \( G' \) is
a connected semisimple group, moreover $G' \cap Z^0$ is a finite normal subgroup of $G$. If

$$C/Z^0 = C_{G/Z^0}(F) \in T_{f(m)k}$$

then by Lemma 1 the group $C \in T_{f(m)k}$. But $G/Z^0$ is a semisimple group. Hence we may assume that $G$ is semisimple. Then $G = G_1 G_2 \ldots G_k$ where $G_i$ are simple linear algebraic groups.

Let $Z = Z_1 \ldots Z_k = Z(G)$ where $Z_i = Z(G_i)$. Then

$$G/Z = G_1 Z/Z \times \ldots \times G_k Z/Z.$$  

But $G_i Z/Z \cong G_i / G_i \cap Z$

Hence $\overline{G} = G/Z = \overline{G}_1 \times \ldots \times \overline{G}_k$. Then

$$C_{G/Z}(F) = C_{G_1}(F_1) \times \ldots \times C_{G_k}(F_k)$$

where $F_i$'s are the images of $F$ under the projection of $G$ onto $G_i$. Now if the number of non-abelian simple factors in $C_{G_i Z/Z}(F_i)$ is at most $f(m)$, then the number of non-abelian simple factors of $C_{G/Z}(F)$ is $f(m)k$. Then by Lemma 1 we have $C_G(F) \in T_{f(m)k}$. For exceptional types the connected components of the Dynkin diagram is already fixed so we may assume that the simple components of the semisimple part of $G$ are of classical type.

Therefore it is enough to prove the following:

If $G$ is a simple linear algebraic group of classical type, $F$ a finite subgroup of $G$ of order $\leq m$ and contained in a maximal torus $T$ of $G$, then $C_G(F) \in T_{f(m)}$.

Let $F = \langle a_1, \ldots, a_n \rangle$ where $|a_i| = m_i$ and $|F| = m = m_1 m_2 \ldots m_n$. Then by [10] Theorem 4.1

$$C_G(F) = \langle T, X, n_w \mid x(a_i) = 1, x \in \Phi, a_i^{n_w} = a_i, \ i = 1, 2, \ldots, n \rangle,$$

$$C_G(F)^0 = \langle T, X, n_w \mid x(a_i) = 1, x \in \Phi, i = 1, 2, \ldots n \rangle$$

where $X$'s are the root subgroups with respect to the torus $T$. The group $C_G(F)^0$ is a reductive group. Since every element in $F$ is semisimple and $C_G(a_i) / C_G(a_i)^0$ is an abelian group by [10], Corollary 4.4, we get that $C_G(F) / C_G(F)^0$ is an abelian group. Now by Lemma 1, it is enough to show that $C_G(F)^0 \in T_{f(m)}$.

Since the maximal torus $T$ and the character group of the root lattice are isomorphic as abelian groups, for every element $a_i \in T$, there exists a character $\chi_{a_i}$ of the root lattice corresponding to $a_i$. 

Let
\[ \Psi = \{ \alpha | \alpha(a_i) = 1, i = 1, 2, \ldots, n \}. \]

\(\Psi\) is a subroot system of \(\Phi\) in the sense that \(\Psi\) is itself a root system and if the sum of any two roots in \(\Psi\) is a root in \(\Phi\), then their sum is again in \(\Psi\). The subroot system may not be connected but it can be written as a union of connected root systems. But by [4], page 25 every root system determines the simple group up to isogeny and the groups corresponding to disjoint connected components centralize each other. Each connected component of \(\Psi\) corresponds to a subgroup \(K\) of \(C_G(F)^0\) such that \(K/Z(K)\) is simple.

Hence in order to find the number of non-abelian simple factors of \(C_G(F)^0\) it is enough to find the number of connected components of \(\Psi\).

Let \(L_E\) be the corresponding Lie algebra of the linear algebraic group \(G\) over an algebraically closed field \(E\). Then \(\chi_{a_i}\) acts on the Lie algebra as \(\chi_{a_i}(h) = h\) for all \(h\) in the Cartan subalgebra of \(L_E\) and \(\chi_{a_i}(e_r) = \chi_{a_i}(r)(e_r)\) for all \(e_r \in L_r\).

Given a connected root system and non-trivial characters \(\chi_{a_i}\) of order \(m_i\), \(i = 1, 2, \ldots, n\), we need to show that the number of irreducible components of
\[ \Psi = \{ \alpha \in \Phi | \chi_{a_i}(\alpha) = 1 \text{ for all } i = 1, 2, \ldots, n \} \]
is less than \(f(m)\).

So the problem reduces actually to a root system problem.

In [9] we found that for each \(\chi_{a_i}\) the number of connected components of \(\Psi\) is at most \(m_i + 2\). Here by using similar methods as in [9] we show that the number of connected components of \(\Psi\) is at most \(f(m) = m^n, n \leq m\).

We give the proof only for the type \(A_l\), because the other classical types can be handled easily by adapting the same technique.

Let \(s\) be the least common multiple of \((m_1, m_2, \ldots, m_n)\). Each \(\chi_{a_i}\) is of order \(m_i\), for each \(i\), \(\chi_{a_i}^s\) is identity on the root lattice. So for each \(r \in \Psi\), \(\chi_{a_i}(r)\) is \(s^{th}\) root of unity.

Now let \(\Phi\) be the root system of type \(A_l\). By [3] page 45
\[ \Phi = \{ e_i - e_j | i \neq j, i, j \in \{1, 2, \ldots, l + 1\} \} \]
where \(e_1, e_2, \ldots, e_{l+1}\) is an orthonormal basis of an Euclidean space of dimension \(l + 1\). The following vectors form a fundamental system for \(A_l\):
\[ e_1 - e_2, \ e_2 - e_3, \ldots, e_l - e_{l+1}. \]
\(\chi_{a_i}(e_i - e_k)\) is an \(s^{th}\) root of unity. In order to make calculations
easier we would like to extend $\chi_{a_i}$ for all $i$ from root lattice to $\sum_{i=1}^{l+1} Z_{e_i}$. As root lattice and $\sum_{i=1}^{l} Z_{e_i}$ are abelian groups and $\chi_{a_i}$ is a homomorphism from the root lattice to the divisible abelian group $K^*$ of the multiplicative group of the field $K$, $\chi_{a_i}$ can be extended from root lattice to $\sum_{i=1}^{l} Z_{e_i}$. We can define $\chi_{a_i}(e_{i+1})$ for case $A_l$ as we please.

Let $\chi_{a_i}(e_{i+1}) = 1$. So $\chi_{a_i}(e_i - e_{i+1}) = \chi_{a_i}(e_i)\chi_{a_i}(e_{i+1})^{-1} = \lambda_i$. Hence $\chi_{a_i}(e_i) = \lambda_i$. Therefore $\chi_{a_i}(e_i)$ is an $s^{th}$ root of unity for all $i = 1, 2, \ldots l + 1$. For each $n$-tuple $(\lambda_1, \ldots , \lambda_n)$ of the $s^{th}$ roots of 1 the sets

$$S(\lambda_1, \ldots , \lambda_n) = \{ j : \chi_{a_i}(e_j) = \lambda_i \text{ for all } i = 1, 2, \ldots , n \}$$

form a partition of $\{ 1, 2, \ldots , l + 1 \}$ into not more than $s^n$ disjoint sets. Since the roots of $A_l$ are of the form $e_i - e_j$, $i \neq j$, we have

$$\chi_{a_i}(e_i - e_j) = 1 \text{ iff } \chi_{a_i}(e_i)\chi_{a_i}(e_j)^{-1} = 1 \text{ iff } \chi_{a_i}(e_i) = \chi_{a_i}(e_j)$$

if and only if $i$ and $j$ belong to the same $S(\lambda_1, \ldots , \lambda_n)$. Then the set

$$\{ e_i - e_j : i \neq j \text{ and } j \text{ belong to the same } S(\lambda_1, \ldots , \lambda_n) \}$$

forms a subroot system of $\Phi$.

The elements $e_j - e_k$ of $\Psi$ having index in $S(\lambda_1, \ldots , \lambda_n)$, with the property that $k, j \in S(\lambda_1, \ldots , \lambda_n)$ and there exists no $m$ between $k$ and $j$ in $S(\mu_1, \ldots , \lambda_n)$, form a basis $\Delta$ for $\Psi$.

Observe that any two roots in $\Delta$ having their indices from different $S(\lambda_1, \ldots , \lambda_n)$ are orthogonal i.e. If $i, j \in S(\lambda_1, \ldots , \lambda_n)$ and $n, m \in S(\mu_1, \ldots , \mu_n)$ where $S(\lambda_1, \ldots , \lambda_n) \neq S(\mu_1, \ldots , \mu_n)$, then $e_i - e_j$ and $e_n - e_m$ are orthogonal. We also observe that the roots in $\Delta$ having their indices from a fixed $S(\lambda_1, \ldots , \lambda_n)$ form a connected root system. If $S(\lambda_1, \ldots , \lambda_n) = \{ j_1, j_2, \ldots , j_{l(i)} \}$ with $j_1 < j_2 \ldots < j_{l(i)}$, then $\{ e_s - e_t : s, t \in S(\lambda_1, \ldots , \lambda_n) \}$ is a root system of type $A_{l(i) - 1}$ as the root system arising from $S(\lambda_1, \ldots , \lambda_n)$ contains only one type of root. Therefore each nonempty $S(\lambda_1, \ldots , \lambda_n)$ with $|S(\lambda_1, \ldots , \lambda_n)| \geq 2$ gives a connected component and the connected components arising from different $S(\lambda_1, \ldots , \lambda_n)$ are orthogonal to each other. Hence the number of irreducible components of $\Psi$ is less than or equal to the number of nonempty $S(\lambda_1, \ldots , \lambda_n)$ with $|S(\lambda_1, \ldots , \lambda_n)| \geq 2$ which is less than or equal to $s^n \leq m^n \leq m^m$. 


Hence $\Phi$ has less than or equal to $m^n$ connected components and each of them is of type $A_k$ for some $k$.

It is clear that if the rank parameter $l$ is greater than $m^n$, then $\Phi$ is non-empty. If moreover $l > t(m^n)$, then $C_G(F)$ involves simple groups of rank greater than $t$. Hence $C_G(F)$ involves subgroups isomorphic to alternating groups $\text{Alt}(t)$.

For the other cases $C_G(F)^0$ is also reductive and we have $C_G(F)^0 = Z^0C$ where $C$ is a semisimple connected linear algebraic group and $C = C_1C_2...C_k$, $k \leq f(m)$ where each $C_i$ is a simple linear algebraic group corresponding to the roots in the corresponding irreducible root system. Hence $C$ has a series of finite length consisting of at most $f(m)$ non-abelian simple factors. Since the type of the root system determines the type of the linear algebraic group up to isogeny, we know the possible types as well.

**Proof of Theorem 4.** Let $F$ be the subgroup satisfying the assumptions of the theorem. Then by [10] Lemma 5.9 there exists a maximal $\sigma$-invariant torus $T$ of $G$ containing $F$. Then $F$ becomes a semisimple subgroup in the linear algebraic group $G$. So we can use all the theory for the semisimple subgroups of linear algebraic groups. By Theorem 3, $C_G(F) \in T_{f(m)}$ and by previous observation $C_G(F)/C_G(F)^0$ is an abelian group. It is clear that $C_G(F)$ is $\sigma$-invariant. Hence $C_G(F)^0$ is $\sigma$-invariant moreover $C_G(F)/C_G(F)^0$ is $\sigma$-invariant.

$C_G(F)$ is closed and $C_G(F)^0$ is closed and connected, then by [4] page 33

$$((C_G(F))/(C_G(F)^0))^\sigma \equiv (C_G(F))^\sigma/(C_G(F)^0)^\sigma$$

which is abelian. Since we are interested in the number of non-abelian simple factors it is enough to find the number of non-abelian simple factors of $(C_G(F)^0)^\sigma$. The group $C_G(F)^0$ is a reductive group. So $C_G(F)^0 = CZ^0$ where $C$ is a connected semisimple subgroup of $G$. Since $Z$ and $C$ are $\sigma$ invariant and $Z^0$ is abelian, by Lemma 1 it is enough to find the number of non-abelian composition factors of $C^\sigma$. Let $C = C_1C_2...C_k$ where each $C_i$ is a simple linear algebraic group $k \leq f(m)$. Let $Z = Z(C) = Z(C_1)...Z(C_k)$. Then $C/Z = \overline{C} = \overline{C}_1\overline{C}_2...\overline{C}_k$ and $\overline{C}_i = C_iZ/Z$. By Krull Schmidt Theorem $(C_iZ)^\sigma = C_jZ$, then by taking the derived group we see that $(C_i)^\sigma = ((C_iZ)^\sigma)^\sigma = (C_jZ)^\sigma = C_j$. Therefore $\sigma$ permutes the $C_i$'s. Let $O_i$, $i = 1, 2, ... , \tau$ be the orbits of $\sigma$ on \{ $C_1, C_2, ..., C_k$ \} and let $K_i = \prod_{D \in O_i} D$. Hence $C$ is the central product of $K_1...K_\tau$. Let $\overline{K}$ be any one of the orbits of $\sigma$ on $\overline{C}$ say for simplicity the
one containing \(\overline{C}_1\)

\[
\overline{K} = \overline{C}_1 \overline{C}_1^\sigma \overline{C}_1^{\sigma^2} \ldots \overline{C}_1^{\sigma^{(1)\cdot 1}}
\]

and \((\overline{C}_1)^{\sigma^{(1)}} = \overline{C}_1\). Then \(\overline{K}\) is the direct product of groups \(\overline{C}_1^\sigma\) and

\[
\overline{K}^\sigma = \{ c_0 c_1^\sigma \ldots c_{t-1}^{\sigma^{t-1}} \mid (c_0 c_1^\sigma \ldots c_{t-1}^{\sigma^{t-1}})^\sigma = c_0 c_1^\sigma c_{t-1}^{\sigma^{t-1}} \}
\]

where \(c_i \in \overline{C}_1\). This implies that \(c_i = c_0\) for all \(i = 1, \ldots, t - 1\). Hence

\[
\overline{K}^\sigma = \{ c^\sigma \ldots c^{\sigma^{t-1}} \mid c \in \overline{C}_1, c^{\sigma^t} = c \} \cong \overline{C}_1^{\sigma^t}
\]

Since \(\sigma\) is a Frobenius automorphism, \(\sigma^t\) is also a Frobenius automorphism. \(\overline{C}_1\) is a simple group and the fixed points of a Frobenius automorphism of a simple linear algebraic group form a group of the same type possibly the twisted version of it. So \(\overline{C}_1^{\sigma^{(1)}} \in T_1\). Since \(\overline{C} = \overline{K}_1 \overline{K}_2 \ldots \overline{K}_r\). We have \(\overline{C}^\sigma \in T_r\) where \(r = f(m)\). Hence \(C^\sigma \in T_{f(m)}\) as required.

**Corollary.** Let \(X\) be a finite simple group of classical type and \(F\) be a nice abelian subgroup of order \(m\) consisting of semisimple elements. Then \(C_X(F) \in T_{f(m)}\).

**Proof.** Let \(X\) be a simple group of classical type and \(F\) be an abelian subgroup as above. We may assume that \(X = O_p^\sigma(G^\sigma)\) where \(G\) is a linear algebraic group of adjoint type and \(\sigma\) is a Frobenius automorphism of \(G\). Then \(F\) becomes an abelian subgroup of commuting semisimple elements of \(G\). Now by [10], Theorem 5.8(c) and 5.11 there exist a maximal torus of \(G\) containing \(F\). By [10], Corollary 5.9 there exists a \(\sigma\)-invariant maximal torus of \(G\) containing \(F\). Then \(C_X(F) = C_G(F) \cap O_p^\sigma(G^\sigma) = C_G^\sigma(F) \cap O_p^\sigma(G^\sigma)\). Now by Theorem 3, Theorem 4 and Lemma 1 we have the result.

**Proof of Theorem 2.** If all finite subgroups are contained in an alternating group, then every finite subgroup is semisimple and by Lemma 6 we are done. Hence we may assume that, all finite subgroups are contained in linear groups of fixed classical type. Now if there exists a prime \(p\) such that every finite subset of \(G\) is contained in a finite simple group of fixed classical type over a field of characteristic \(p\), then \(F\) can be chosen as any abelian subgroup of \(G\) such that \(p\) does not divide \(|F|\); in case simple groups of type \(B_t\) or \(D_t\) we choose \(F\) abelian with cyclic Sylow 2-subgroup. If there exists no such a prime \(p\), then each prime appears only finitely many times as a characteristic of the
field. Now choose any abelian subgroup $F$ of $G$, let $|F| = m$ and consider $\{p_i | p_i | m\}$.

Since each $p_i$ appears only finitely many times we may discard finitely many of the $G_i$'s and obtain a local system in such a way that $F \leq G_i$ for all $i$ where $i$ is in some index set $I$ and $F$ is semisimple in every member $G_i$ of the new local system. If necessary we take a subgroup of $F$ with cyclic Sylow 2-subgroup. Now by Corollary we have $C_i = C_{G_i}(F) \in T_{f(m)}$ for all $i$. Now using Lemma 3 $C \in T_{f(m)}$. Hence we are done.

PROOF OF THEOREM 1. Let $F$ be a $K$-nice abelian subgroup and $K = (G_i, M_i)$ be the given Kegel sequence of $G$. If necessary by passing to a subsequence, we may assume that $G_i/M_i$ are all alternating or all belong to a fixed classical family. Then either Theorem 4 applies and $C_{G_i/M_i}(F) \in T_{g(m)}$ or Lemma 4 applies and $C_{G_i/M_i}(F) \in T_{g(m)}$. Since $(|M_i|, |F|) = 1$ we get $C_{G_i/M_i}(F) = C_{G_i}(F)M_i/M_i \in T_{g(m)}$ in the first case and in $T_{g(m)}$ in the second case. By assumption $M_i$'s are soluble hence by Lemma 1 either $C_{G_i}(F) \in T_{f(m)}$ for all $i$ or $C_{G_i}(F) \in T_{g(m)}$ for all $i$. By Lemma 3 we get $C_G(F) \in T_{f(m)}$ or $T_{g(m)}$.

Since $C_{G_i}(F)$ involves alternating groups of unbounded orders and $C_G(F)$ has a series of finite length in which each factor is either non-abelian simple or locally soluble, one of the factors of the series of $C_G(F)$ must be non-linear.

We may extend Theorem 1 to somewhat more general LFS-groups. This will be the extension of [7] Theorem B' from a single semisimple element to an abelian subgroup consisting of semisimple elements.

THEOREM 5. Let $G$ be a non-linear LFS-group and $F$ be an abelian subgroup of order $m$ in $G$. Let $\pi$ be the set of prime divisors of the order of $F$. Suppose there exists a Kegel sequence $K = (G_i, N_i)$ of $G$ such that $F$ is nice for all Kegel components $G_i/N_i$ and

(i) $O_{\pi'}(N_i)$ is soluble.

(ii) $N_i/O_{\pi'}(N_i)$ is hypercentral in $G_i$.

(iii) $G_i/N_i$ is either an alternating group or a classical group defined over a field of characteristic not in $\pi$.

Then $C_G(F)$ belongs to $T_{f(m)}$ and involves a non-linear simple group.

We omit the proof as the technique is similar to the proof of Theorem 1.
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