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On self-centralizing Sylow subgroups of order four

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On Self-Centralizing Sylow Subgroups of Order Four.

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1. - Introduction.

A well-known result of Gorenstein and Walter [3], which confirms a conjecture of R. Brauer [1], states that if G is a finite group of order $4g$, g odd, with a self-centralizing Sylow 2-subgroup, then it contains a normal subgroup N of odd order such that G/N is isomorphic either to a Sylow 2-subgroup of G or to $PSL(2, q)$ where q is a prime power, $q \equiv 3$, $5 \bmod 8$. Moreover (see [2, p. 348 and p. 356]), N must be metanilpotent.

The objective of this paper is to improve upon this result in the non soluble case by giving more precise information on the structure of N. Indeed, we have:

THEOREM *Let G be a non soluble finite group with a self-centralizing Sylow 2-subgroup of order 4 and let $N = O(G)$ be the maximal normal subgroup of G of odd order. Then one of the following holds:*

a) $G/N \cong PSL(2, q)$ *with* $q = p^f > 5$, $q \equiv 3$, $5 \bmod 8$, *and N is a p-group;*

b) $G/N \cong PSL(2, 5)$ *and N is nilpotent.*

This result is best possible in the sense that, in Case *b)* of the theorem, N need not be a 5-group (see Example 2.4). Moreover the derived length of N in *a)* and *b)* is not bounded. In fact in Example 3.3. groups G are constructed with a self-centralizing Sylow 2-subgroup of order 4

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such that N is a p-group of arbitrary derived length and $G/N \cong \cong PSL(2, p)$ where p is a prime such that $p \equiv 3, 5 \bmod 8$.

In the course of the proof of the theorem, we shall analyze actions of $Q = PSL(2, p^f)$ on a group N of odd order such that a Sylow 2-subgroup of Q acts fixed point freely on N. Note that a series of papers deal with a somewhat similar situation in which the orders of Q and N are coprime, and $C_N(Q) = 1$, see for example [9], [13] and the references given there. By contrast with our result, in that situation N need not be nilpotent.

In [4] it was shown that if $Q = PSL(2, 2^f), f \geq 2$, acts on a 2-group N such that an element of order 3 acts fixed point freely, then N is elementary abelian. If $f = 2$ and if an element of order 5 acts fixed point freely on N, then the nilpotent class of N is ≤ 3 (see [5] and [10]). In these cases, N is of bounded nilpotent class (note that this contrasts with Example 3.3).

In the following, we denote by $G = [N]H$ that G is a split extension of its normal subgroup N by a complement H. Moreover, A_n is the alternating group on n letters, hence $A_4 \cong PSL(2, 3)$ and $A_5 \cong PSL(2, 5)$. In addition, p will denote a prime and q is always a power of p. The order of the element g is denoted by $o(g)$.

All other notation is standard and can be found in [2], [6] and [8], for example. All groups in this paper are finite.

2. - Actions of $PSL(2, q)$.

We start by noting some well-known facts that will be used several times in the sequel.

LEMMA 2.1. *Let G be a group and let $N = O(G)$. Assume that $G/N \cong PSL(2, q)$ where $q \equiv 3, 5 \bmod 8$. If $S \in \mathrm{Syl}_2(G)$ and $C_G(S) = S$, then $|N| = |C_N(\sigma)|^3$ for every involution $\sigma \in S$.*

PROOF. Since all involutions of G are conjugate, the result follows from [2, p. 347]. ■

PROPOSITION 2.2. *Let G be a Frobenius group with kernel N and let F be a finite field of characteristic r not dividing $|N|$. Let M be an FG-module and assume that $C_M(N) = 0$. Let A be a Frobenius complement for G. Then M has a basis which is permuted by A with orbits of size $|A|$. In particular, if $|M| = r^t$, we have $|C_M(A)| = r^{t/|A|}$.*

PROOF. See [8, p. 270]. ■

We now deal with extensions of a group N of odd order by $PSL(2, q)$, which have a self-centralizing Sylow 2-subgroup. The following result can be read off from the (modular) character table of $PSL(2, q)$ and may already be known. We present here an elementary proof.

PROPOSITION 2.3. *Let $G = PSL(2, q)$ with $q = p^f$, $q \equiv 3, 5 \bmod 8$ and assume $q > 5$. Let $S \in \mathrm{Syl}_2(G)$ and let M be a nontrivial and irreducible module for G over a finite field F of characteristic r, where $r \neq 2$, $r \neq p$. Then $C_M(S) \neq 0$.*

PROOF. We proceed by way of contradiction. Suppose that $C_M(S) = 0$. First, Lemma 2.1 implies $|M| = r^{3h}$ for some positive integer h, and for each involution $\sigma \in S$, we have $|C_M(\sigma)| = r^h$. Take $u \in G$ with $o(u) = (q-1)/2$. Since $q \neq 3$, we have that $N_G(\langle u \rangle)$ is a dihedral group and we can write $u = \gamma_1 \gamma_2$ where γ_1, γ_2 are suitable involutions in $N_G(\langle u \rangle)$. We have $[M, \gamma_1] \cap [M, \gamma_2] \leqslant C_M(u)$. Set $C^* = [M, \gamma_1] \cap \cap [M, \gamma_2]$. Then $|C^*| \geqslant r^h$, since $|[M, \gamma_i]| = r^{2h}$ for $i = 1, 2$. Now let P be a Sylow p-subgroup of G, normalized by u. Then $M_1 := [M, P]$ and $M_2 := C_M(P)$ are invariant under the action of u. Since $M = M_1 \oplus M_2$, we have $C_M(u) = C_{M_1}(u) \oplus C_{M_2}(u)$. In particular, every element of C^* can be written in a unique way as sum of an element of $C_{M_1}(u)$ and an element of $C_{M_2}(u)$. We have $C^* \cap C_{M_2}(u) = 0$. In fact, if $x \in C^* \cap \cap C_{M_2}(u)$ then $\langle x \rangle$ is invariant by P and $N_G(u)$. So $\langle x \rangle$ is invariant for G. Since M is irreducible for G, we get $x = 0$. Hence we have $|C_{M_1}(u)| \geqslant \geqslant r^h$. On the other hand, since M_1 is a faithful module for $N_G(P)$ which satisfies the conditions of Proposition 2.2, it follows that $|C_{M_1}(u)| = = r^{6h/(q-1)}$, a final contradiction. ■

It may be observed that there are modules for A_4 and A_5 such that the previous proposition does not hold:

EXAMPLE 2.4. *a) Let F be a field of characteristic different from 2 and let*

$$G = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

with entries in F. Then $G \cong A_4$. Let M be the natural vector space on which G acts and let $S \in \mathrm{Syl}_2(G)$. Then M is an irreducible and faithful FG-module and we have $C_M(S) = 0$.

b) Now let $G \cong A_5$, choose $P \in \mathrm{Syl}_5(G)$ and set $H = N_G(P)$. Let F

be a field of characteristic different from 2. *Suppose that* M_1 *is the non-trivial FH-module of dimension* 1. *Let* $M = M_1^G$ *be the induced module. It is easy to prove that if* $S \in \mathrm{Syl}_2(G)$, *then* $C_M(S) = 0$. *So* $C_{\widehat{M}}(S) = 0$ *for every composition factor* $\widehat{M}$ *of* M.

If we take two such modules M_1, M_2 *over fields of different odd characteristic, then the natural split extension of* $M_1 \oplus M_2$ *by* G *shows that in part b) of the theorem, the group* N *need not a p-group for no prime whatsoever.*

The following result provides a criterion when every proper subgroup of $PSL(2, q)$ has trivial intersection with at least one Sylow 2-subgroup. It will be seen in the proof of 2.6 that the strange-looking hypothesis of the following lemma is satisfied in our case.

LEMMA 2.5. *Let* $G = PSL(2, q)$, *where* $q \equiv 3, 5 \bmod 8$ *and assume that* $q > 5$. *Suppose that the proper subgroups of* G *are either soluble or isomorphic to* A_5. *If* $H < G$, *then there exists* $S \in \mathrm{Syl}_2(G)$ *such that* $H \cap S = 1$.

PROOF. Of course it suffices to consider the case when $|H|$ is even. By Dickson's theorem (see [6, p. 213 f.]) and our hypothesis, the subgroups of G are the following:

1) Dihedral groups D_z of order $2z$ with $z \mid (q \pm 1)/2$.
2) Groups isomorphic to A_4.
3) Groups isomorphic to A_5.
4) A subgroup Q of $N_G(P)$ where $P \in \mathrm{Syl}_p(G)$ if $q \equiv 5 \bmod 8$.

If σ is an involution of G and $S, \overline{S}$ are Sylow 2-subgroups of G satisfying $\sigma \in S \cap \overline{S}$, then $\langle S, \overline{S} \rangle \leq C_G(\sigma)$. If we denote by n_σ the number of Sylow 2-subgroups containing σ, we have:

a) $n_\sigma = (q+1)/4$ if $q \equiv 3 \bmod 8$,
b) $n_\sigma = (q-1)/4$ if $q \equiv 5 \bmod 8$.

Now let μ_H be the number of involutions which are contained in the subgroup H of G. Then we have:

$$\mu_{D_z} \leq \frac{q+1}{2} \quad \text{if } z \mid \frac{q \pm 1}{2},$$
$$\mu_{A_4} = 3,$$
$$\mu_{A_5} = 15,$$
$$\mu_Q \leq q \quad \text{when } q \equiv 5 \bmod 8.$$

So if μ^* is the maximum number of involutions which are contained in a proper subgroup of G, we have:

$$\mu^* \leqslant \max\{q, 15\} \quad \text{if } q \equiv 5 \bmod 8 \text{ and}$$

$$\mu^* \leqslant \max\left\{\frac{q+1}{2}, 15\right\} \quad \text{if } q \equiv 3 \bmod 8.$$

If m denotes the number of Sylow 2-subgroups which intersect non trivially with a proper subgroup H of G, we have:

$$m \leqslant \mu^* \cdot n_\sigma \leqslant \begin{cases} \mu^* \cdot \dfrac{q+1}{4} & \text{if } q \equiv 3 \bmod 8, \\ \mu^* \cdot \dfrac{q-1}{4} & \text{if } q \equiv 5 \bmod 8. \end{cases}$$

On the other hand, the number γ of Sylow 2-subgroups of G is equal to $q(q+1)(q-1)/24$. By an easy calculation we get $m < \gamma$ for $q \geqslant 11$. Thus if $q > 5$ and H is a proper subgroup of G, there is a Sylow 2-subgroup S such that $H \cap S = 1$. ∎

The next two results deal with modules for the groups occurring in the theorem. They will be used to exclude nonnilpotent normal subgroups $O(G)$.

PROPOSITION 2.6. *Let $\mathcal{A}$ be the class of all groups G for which there exists a normal subgroup N of G such that:*

a) N is an abelian p-group (possibly the identity) where $p \neq 2$;

b) $G/N \cong PSL(2, q)$ where $q = p^f$, $q \neq 3, 5$ and $q \equiv 3, 5 \bmod 8$.

If $G \in \mathcal{A}$ and M is an FG-module where F is a finite field of characteristic $\neq 2$, $\neq p$, then for all $S \in \mathrm{Syl}_2(G)$ we have $C_M(S) \neq 0$.

PROOF. By way of contradiction, assume that there exists a counterexample (G, M) where $G \in \mathcal{A}$ and M is an FG-module satisfying the hypothesis of the proposition. Choose this pair such that $|G| + |M|$ is minimal. Then (G, M) has the following properties:

1) If $p \neq 3, 5$, then $G/N \cong PSL(2, p)$.

In fact, by Dickson's theorem there exists a subgroup $H \leqslant G$ such that $H/N \cong PSL(2, p)$. If $p \neq 3, 5$, we have $H \in \mathcal{A}$ and H contains some Sylow 2-subgroup S of G. If $H < G$ then $C_M(S) \neq 0$ by minimality of (G, M). But this is a contradiction and so we have $G = H$.

2) If $p = 3$ or $p = 5$, then $G/N \cong PSL(2, 3^f)$ or $G/N \cong PSL(2, 5^m)$ where f and m are primes.

In fact, if $G \in \mathfrak{A}$ we have $f \geq 1$, and so there is a prime t with $t \mid f$ (in a similar way there is a prime $\bar{t}$ such that $\bar{t} \mid m$). By Dickson's theorem there exists $H \leqslant G$ such that $H/N \cong PSL(2, 3^t)$ or $H/N \cong PSL(2, 5^{\bar{t}})$. As in 1), it follows that $G = H$.

We observe that by 1) and 2), the only subgroups of G/N are either soluble or isomorphic to A_5, so that the hypothesis of Lemma 2.5 holds.

3) M is an irreducible and faithful FG-module.

In fact, let M_1 be an irreducible FG-module with $M_1 < M$. By minimality of (G, M), we have $C_{M_1}(S) \neq 0$ and so $C_M(S) \neq 0$, a contradiction. Thus $M_1 = M$ and M is irreducible. Let K be the kernel of the action of G on M. Of course, we have $K \leqslant N$ and $(G/N, M)$ satisfies the hypotheses of the proposition. If $K \neq 1$, then $0 \neq C_M(SK/K) = C_M(S)$ by minimality of (G, M), a contradiction. So we have $K = 1$.

4) $N \neq 1$.

This follows from Proposition 2.3.

5) N is not contained in $Z(G)$.

In fact, suppose $N \leqslant Z(G)$. Then, by properties of the Schur multiplicator of $PSL(2, q)$ (see [6, p. 646] and [12, p. 257]), we have $G = NL$ for a suitable subgroup $L \neq G$. Also (L, M) is a counterexample, but this is against the minimality of (G, M).

6) N is not cyclic.

Otherwise N would be central, but this contradicts 5).

7) M is an induced module.

Let $\overline{M}$ be a homogeneous component of M, considered as FN-module. Suppose that $\overline{M} = M$. Since N is not cyclic, the kernel of the action of G on M is nontrivial against the faithfulness of the action of G on M. So $\overline{M} \neq M$ and [6; p. 565] implies that M is induced.

8) Final contradiction.

Let I be the stabilizer of $\overline{M}$ in G. So 7) implies $M = (\overline{M})^G$. We have $I/N < G/N \cong PSL(2, q)$. By Lemma 2.5, there exists $SN/N \in \mathrm{Syl}_2(G/N)$ with $SN/N \cap I/N = N/N$, i.e. $SN \cap I = N$. Then $N(S \cap I) = N$. Since

$(|S|, |N|) = 1$ it follows that $S \cap I = 1$. Let T be a set of double coset representatives with respect to S and I in G. We may assume $1 \in T$. By Mackey's theorem [6, p. 557], we have:

$$M_{|S} = \bigoplus_{t \in T} (\overline{M} \otimes t_{|I^t \cap S})^S = (\overline{M} \otimes 1_{|I \cap S})^S \oplus \left\{ \bigoplus_{t \neq 1} (\overline{M} \otimes t_{|I^t \cap S})^S \right\}.$$

Since $I \cap S = 1$, we have that $(\overline{M} \otimes 1_{|I \cap S})^S$ is direct sum of regular FS-modules. Therefore the above implies that $C_{\overline{M}}(S) \neq 0$ and so $C_M(S) \neq 0$. ∎

The following deals with groups having A_5 as nonsoluble chief factor:

PROPOSITION 2.7. *Let $\mathcal{B}$ be the class of all groups G for which there exists a nontrivial normal subgroup N such that:*

a) $G/N \cong A_5$;

b) N is abelian of odd order;

c) If S is a Sylow 2-subgroup of G, then $C_N(S) = 1$.

If $G \in \mathcal{B}$ and M is a faithful and irreducible FG-module where F is of odd characteristic, then $C_M(S) \neq 0$.

PROOF. Let $G \in \mathcal{B}$ and let $S \in \mathrm{Syl}_2(G)$. Assume that the FG-module M is a counterexample, that is $C_M(S) = 0$. We then have:

1) M is an induced module.

Since $C_N(S) = 1$, we see that N is not cyclic. As in part 7) of the proof of Proposition 2.6, it follows that if $\overline{M}$ is a homogeneous component of M, restricted to N, then $\overline{M} \neq M$. So the stabilizer I of $\overline{M}$ is properly contained in G and $M = \overline{M}^G$.

Let $L = G/N \cong A_5$ and let H be the normalizer in L of a Sylow 5-subgroup P of L.

2) We have $I/N \cong H$.

An inspection of the proper subgroups of A_5 shows that $I/N \cong H$, because otherwise, there would exist a Sylow 2-subgroup S of G such that $I/N \cap SN/N = N/N$. As in part 8) of the proof of Proposition 2.6, we get $C_M(S) \neq 0$. But this is a contradiction.

3) For all involutions $\sigma \in I$, we have $C_{\overline{M}}(\sigma) = 0$. In particular σ is not contained in the kernel K of the representation of I on $\overline{M}$.

In fact, suppose $C_{\overline{M}}(\sigma) \neq 0$. Let T be a right transversal of I in G.

Then 1) implies that $M = \bigoplus_{t \in T} \overline{M} \otimes t$. We may assume that $1, \tau \in T$, where $\langle \sigma, \tau \rangle = S$. We then have

$$0 \neq \langle x \otimes 1 + x \otimes \tau \,|\, x \in C_{\overline{M}}(\sigma) \rangle \leq C_M(S),$$

a contradiction.

4) $[N, \sigma] \subseteq K$.

Otherwise, the group $[[N, \sigma]/([N, \sigma] \cap K)]\langle \sigma \rangle$ is a Frobenius group. But here, [7, p. 411] implies $C_{\overline{M}}(\sigma) \neq 0$, against 3).

5) $|N|$ is not divisible by 5.

In fact, otherwise there would exist a minimal normal 5-subgroup R of G with $R \leq N$. Now R is a faithful and irreducible module for L, and S acts fixed point freely on R. So, from Lemma 2.1 and [7, p. 38 ff] it follows that $|R| = 5^3$. Moreover, considering the action of L on R it can be seen that $|[R, \sigma]| = 5^2$. By 4) we have $[R, \sigma] \subseteq K$. Since M is a faithful module, we get $[R, \sigma] = K \cap R \trianglelefteq I$. But this is a contradiction because $[R, \sigma]$ is not normalized by P.

6) $C_N(P) \cap C_N(\sigma) \neq 1$.

Since $(|N|, |H|) = 1$, it follows that $N = (N \cap K) \oplus N_0$ where N_0 is H-invariant. Moreover N_0 must be cyclic. Also we have $N_0 = [N_0, P] \oplus \oplus C_{N_0}(P)$. Since $[N_0, P]$ is invariant for the nonabelian group H, we get $[N_0, P] = 1$, so $N_0 \subseteq C_N(P)$. On the other hand, by 4), we have $N_0 \subseteq C_N(\sigma)$.

7) Final contradiction.

Let $N = N_1 > N_2 > \ldots > N_h = 1$ be part of a chief series of G. Since $C_N(H) \neq 1$ and $(|N|, |H|) \neq 1$ it follows that there exists a chief factor N_e/N_{e+1} of G such that $C_{N_e/N_{e+1}}(H) \neq 1$. Without loss of generality, we may assume that N is a minimal normal subgroup of G. So N can be viewed as an irreducible and faithful G/N-module and we will use the additive notation. Let N_0 be the trivial H-module. Then by 6), N_0 is a submodule of $N_{|H}$, so that $\mathrm{Hom}\,(N_0, N_{|H}) \neq 0$. Therefore by Nakayama's reciprocity law [7, p. 50] we get $\mathrm{Hom}\,(N_0^L, N) \neq 0$ and so there exists a non-trivial homomorphism from N_0^L to N, which is an epimorphism because N is irreducible.

It follows that $\dim N \leq 6$, and Lemma 2.1 implies $\dim N \in \{3, 6\}$. If $\dim N = 6$ then $N \cong N_0^L$, so $C_N(S) \neq 0$ and we have a contradiction. If $\dim N = 3$, then $[N, P]$ decomposes into a direct sum of regular modules for $\langle\sigma\rangle$ by Proposition 2.2. Hence we have $\dim[N, P] = 2$. Since $C_N(P) \cap C_N(\sigma) \neq 0$, it follows by 6) that $\dim C_N(\sigma) = 2$. But this is against Lemma 2.1. ■

3. – Conclusion.

3.1 PROOF OF THE THEOREM. Let S be a Sylow 2-subgroup of G. Since G is non soluble, the result of Gorenstein and Walter [3] implies that $G/O(G) \cong PSL(2, q)$ with $q \equiv 3, 5 \bmod 8$. Let $N = O(G)$. By [2, p. 348], we know that N' is nilpotent. We split the proof into two cases:

$q > 5$: Let t be a prime dividing $|N/N'|$ and let $t \neq p$. By Proposition 2.3 we have $C_{N/N'}(S) \neq 1$. So $C_N(S) \neq 1$, a contradiction. Hence N/N' is a p-group. By way of contradiction suppose that N' is not a p-group. As N' is nilpotent, we can choose a chief factor N'/K of G which is a p'-group. But then Proposition 2.6, applied to $M = N'/K$, yields $C_{N'}(S) \neq 1$, a contradiction.

$q = 5$: Suppose by way of contradiction that N is not nilpotent. Then there exists a chief factor $\widehat{N}$ of G below N, which is central in N', but not in N. So $\widehat{N}$ is a faithful and irreducible module for $G/C_G(\widehat{N})$ which satisfies the conditions of Proposition 2.7. Then $C_{\widehat{N}}(S) \neq 1$ and so $C_N(S) \neq 1$, but this is the final contradiction. ■

If G is assumed to be soluble in the statement of the theorem, then $O(G)$ need not be nilpotent. In fact, it is easy to construct examples in which G is 2-nilpotent with a self-centralizing Sylow 2-subgroup of order 4 such that the normal 2-complement is of Fitting length two. For the convenience of the reader we give an example of a group G such that $G/N \cong A_4$ where $N = O(G)$ is of Fitting length two and the Sylow 2-subgroups of G are self-centralizing.

EXAMPLE 3.2. *The group $H = A_4$ can act faithfully and irreducibly on a vector space V of dimension 3 over $GF(3)$ (see Example 2.4). Let V_1 be a subspace of dimension 1, invariant for $S \in \mathrm{Syl}_2(H)$. Let V_2 be an S-invariant complement of V_1 in V. Set $T = [V]S$ and $G = [V]H$. There exists an irreducible T-module M_1 of dimension 2 over $GF(5)$ with kernel V_2. Let $M = M_1^G$ be the induced module. Then it is easy to verify that $C_M(S) = \{0\}$. Set $N = MV$, then we have $C_N(S) = \{0\}$ and N is not nilpotent.*

Finally, for every prime $p \geq 5$, we construct a finite group G with a self-centralizing Sylow 2-subgroup such that $G/O_p(G) \cong PSL(2, p)$ and $O_p(G)$ is of prescribed derived length. We are indebted to the referee for greatly improving upon our original example.

EXAMPLE 3.3. *Let p be a prime, $p \equiv 3, 5 \bmod 8$. Let $\boldsymbol{Z}_p$ be the ring of p-adic integers and consider the group $SL(2, \boldsymbol{Z}_p)$ and its normal subgroup N consisting of all matrices of the form* $\begin{pmatrix} 1+pa & pb \\ pc & 1+pd \end{pmatrix}$. *Note that N is the group $\mathfrak{M}_{1,1,1}$ of* [6, p. 387]. *Let $\Gamma = PSL(2, \boldsymbol{Z}_p)$. We identify N with a subgroup of Γ, so that we have $\Gamma/N \cong PSL(2, p)$. Moreover (see* [6, p. 387 ff.]) *it is known that for every positive integer d, the factor group $N/N^{(d)}$ is a p-group of derived length precisely d. It is easy to check that $G = \Gamma/N^{(d)}$ is a finite group with a self-centralizing Sylow 2-subgroup of order 4 in which $O_p(G)$ is of derived length d.*

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