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A remark on the note : "Partial Hölder continuity of the spatial derivatives of the solutions to nonlinear parabolic systems with quadratic growth"

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A Remark on the Note: «Partial Hölder Continuity of the Spatial Derivatives of the Solutions to Nonlinear Parabolic Systems with Quadratic Growth».

MARIO MARINO - ANTONIO MAUGERI (*)

Dedicated to Professor Sergio Campanato with our deepest esteem on his 65th birthday

SUNTO - In questa nota si dimostra che le soluzioni di classe

$$L^2(-T, 0, H^2(\Omega, \mathbb{R}^N)) \cap C^{0,\gamma}(\overline{Q}, \mathbb{R}^N), \qquad 0 < \gamma < 1,$$

del sistema (1.1) hanno derivate spaziali parzialmente hölderiane in Q. Si rimuove quindi la condizione richiesta in [1] che la soluzione sia anche di classe $H^1(-T, 0, L^2(\Omega, \mathbb{R}^N))$.

1. – Let Ω be a bounded open subset of $\mathbb{R}^n$ ($n \geq 2$), with sufficiently smooth boundary $\partial\Omega$, for instance of class C^3, and Q the cylinder $\Omega \times (-T, 0)$ ($T > 0$). In [1] we were concerned with the following second order nonlinear parabolic system of variational type (1):

$$(1.1) \qquad -\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du),$$

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(1) We follow the notations used in [1].

where $a^i(X, u, p)$, $i = 1, 2, \dots, n$, and $B^0(X, u, p)$ are vectors of $\mathbb{R}^N$ (N integer ≥ 1) measurable in X and continuous in (u, p). Under the assumptions:

(1.2) $a^i(X, u, p) \in C^1(\overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN})$, $\quad i = 1, 2, \dots, n$,

(1.3) *there exists a constant* $\nu > 0$ *such that*

$$\sum_{h,k=1}^{N} \sum_{i,j=1}^{n} \frac{\partial a_h^i(X, u, p)}{\partial p_k^j} \xi_h^i \xi_k^j \geq \nu \sum_{i=1}^{n} \|\xi^i\|^2$$

for every system $\{\xi^i\}_{i=1,2,\dots,n}$ *of vectors of* $\mathbb{R}^N$ *and for every* $(X, u, p) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$,

(1.4) *the vectors* $\partial a^i / \partial p_k^j$, $i, j = 1, 2, \dots, n$, $k = 1, 2, \dots, N$, *are uniformly continuous in* $\overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}$,

(1.5) $\forall (X, u, p) \in \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ *we have*:

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial a^i}{\partial p_k^j} \right\| \leq M, \qquad i = 1, 2, \dots, n,$$

$$\|a^i\| + \sum_{s=1}^{N} \left\| \frac{\partial a^i}{\partial x_s} \right\| + \sum_{k=1}^{N} \left\| \frac{\partial a^i}{\partial u_k} \right\| \leq M\left(1 + \sum_{j=1}^{n} \|p^j\|\right),$$

$$i = 1, 2, \dots, n,$$

(1.6) $\forall (X, u, p) \in \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ *we have*:

$$\|B^0(X, u, p)\| \leq g^0(X) + M \sum_{j=1}^{n} \|p^j\|^2,$$

$$g^0(X) \in L^q(Q), \quad q > n + 2,$$

we established the partial Hölder continuity in Q of the spatial gradient of the weak solutions u of class

$$L^2(-T, 0, H^2(\Omega, \mathbb{R}^N)) \cap$$

$$\cap H^1(-T, 0, L^2(\Omega, \mathbb{R}^N)) \cap C^{0,\gamma}(\overline{Q}, \mathbb{R}^N), \qquad 0 < \gamma < 1,$$

to the system (1.1) (see [1], Theorem 4.1).

The aim of this work is to find again the partial Hölder continuity of the spatial derivatives of the weak solutions to the system (1.1), under the assumption that these solutions belong merely to $L^2(-T, 0, H^2(\Omega, \mathbb{R}^N)) \cap C^{0,\gamma}(\overline{Q}, \mathbb{R}^N)$. In fact it is possible to remove the condition $u \in H^1(-T, 0, L^2(\Omega, \mathbb{R}^N))$ thanks to an interpolation theorem due to Niremberg [3] (see also Miranda [2]), which ensures the result $D_i u \in L^4(Q, \mathbb{R}^N)$ under the mere assumption $u \in L^2(-T, 0, H^2(\Omega, \mathbb{R}^N)) \cap C^{0,\gamma}(\overline{Q}, \mathbb{R}^N)$.

2. – It is well known the following interpolation result

LEMMA 2.1. *If* $u \in H^2(\Omega, \mathbb{R}^N) \cap C^{0,\gamma}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \gamma < 1$, *then*

$$D_i u \in L^s(\Omega, \mathbb{R}^N), \qquad i = 1, 2, \dots, n, \tag{2.1}$$

and

$$\int_\Omega \|D_i u\|^s dx \le c_1 [u]^{s-2}_{\gamma, \overline{\Omega}} \int_\Omega \sum_{i,j=1}^n \|D_{ij} u\|^2 dx + c_2 [u]^s_{\gamma, \overline{\Omega}}, \tag{2.2}$$

where $s = (2(2-\gamma))/(1-\gamma)$, c_1 *and* c_2 *are constants depending on* Ω, γ, n (2).

See [3], Theorem 1′ (with $\beta = \gamma$, $m = r = 2$, $j = 1$, $a = (1-\gamma)/(2-\gamma)$) (3).

LEMMA 2.2. *If* $u \in L^2(-T, 0, H^2(\Omega, \mathbb{R}^N)) \cap C^{0,\gamma}(Q, \mathbb{R}^N)$, $0<\gamma<1$, *then*

$$D_i u \in L^s(Q, \mathbb{R}^N), \qquad i = 1, 2, \dots, n, \tag{2.3}$$

and

$$\int_Q \|D_i u\|^s dX \le c_1 [u]^{s-2}_{\gamma, \overline{Q}} \int_Q \sum_{i,j=1}^n \|D_{ij} u\|^2 dX + c_2 T [u]^s_{\gamma, \overline{Q}}, \tag{2.4}$$

(2) $[u]_{\gamma, \overline{\Omega}} = \sup_{\substack{x, y \in \overline{\Omega} \\ x \ne y}} \dfrac{\|u(x) - u(y)\|}{\|x - y\|^\gamma}$.

(3) We would like to thank warmly Prof. Francesco Guglielmino for pointing to us this result.

where $s = (2(2-\gamma))/(1-\gamma)$, c_1 *and* c_2 *are the constants (depending on* Ω, γ, n*) that appear in the* (2.2) (4).

PROOF. If $u \in L^2(-T, 0, H^2(\Omega, \mathbf{R}^N)) \cap C^{0,\gamma}(\overline{Q}, \mathbf{R}^N)$, $0 < \gamma < 1$, for a.e. $t \in (-T, 0)$ it results:

$$u(x, t) \in H^2(\Omega, \mathbf{R}^N) \cap C^{0,\gamma}(\overline{\Omega}, \mathbf{R}^N),$$

then, by Lemma 2.1, we have:

$$D_i u(x, t) \in L^s(\Omega, \mathbf{R}^N), \qquad s = \frac{2(2-\gamma)}{1-\gamma}, \qquad i = 1, 2, \ldots, n,$$

and

$$\int_\Omega \|D_i u(x, t)\|^s dx \leq c_1 [u]_{\gamma, \overline{Q}}^{s-\frac{2}{\gamma}} \int_\Omega \sum_{i,j=1}^n \|D_{ij} u(x, t)\|^2 dx + c_2 [u]_{\gamma, \overline{Q}}^s . \tag{2.5}$$

Now, by integrating with respect to t both the sides of (2.5) in the interval $(-T, 0)$, we achieve the conclusion.

From the lemma above the partial Hölder continuity of the spatial derivatives of the weak solutions to system (1.1) easily follows.

THEOREM 2.1. *If* $u \in L^2(-T, 0, H^2(\Omega, \mathbf{R}^N)) \cap C^{0,\gamma}(\overline{Q}, \mathbf{R}^N)$, $0 < \gamma < 1$, *is a weak solution in* Q *to system* (1.1) (5) *and if conditions* (1.2)-(1.6) *are fulfilled, then there exists a set* $Q_0 \subset Q$, *closed in* Q, *such that*

$$D_i u \in C^{0,\mu}(Q \backslash Q_0, \mathbf{R}^N), \qquad \forall \mu < 1 - \frac{n+2}{q}, \quad i = 1, 2, \ldots, n,$$

(4) $[u]_{\gamma, \overline{Q}} = \sup\limits_{\substack{X, Y \in \overline{Q} \\ X \neq Y}} \dfrac{\|u(X) - u(Y)\|}{d^\gamma(X, Y)}$, $\quad d(X, Y) = \max\{\|x - y\|, |t - \tau|^{1/2}\}$, $X = (x, t)$, $Y = (y, \tau)$.

(5) In the sense that it results:

$$\int_Q \left\{ \sum_{i=1}^n (a^i(X, u, Du) | D_i \varphi) - \left(u \,\middle|\, \frac{\partial \varphi}{\partial t} \right) \right\} dX = \int_Q (B^0(X, u, Du) | \varphi) \, dX,$$

$$\forall \varphi \in C_0^\infty(Q, \mathbf{R}^N).$$

and

$$\mathcal{M}_{n+2-r}(Q_0) = 0\,,$$

for every $r \in (2, (2/\gamma_0) \wedge (2(n+2))/(n+2-2\gamma))$ (6).

PROOF. From the assumption

$$u \in L^2(-T, 0, H^2(\Omega, \mathbb{R}^N)) \cap C^{0,\gamma}(\overline{Q}, \mathbb{R}^N)$$

and from the estimate $(2(2-\gamma))/(1-\gamma) > 4$, it follows, by Lemma 2.2:

$$D_i u \in L^4(Q, \mathbb{R}^N), \qquad i = 1, 2, \ldots, n\,;$$

then, taking into account that $B^0(X, u, p)$ satisfies (1.6), we have:

$$B^0(X, u, Du) \in L^2(Q, \mathbb{R}^N)\,. \tag{2.6}$$

On the other hand from assumption (1.5) on $a^i(X, u, p)$ we obtain

$$\|D_i a^i(X, u, Du)\| \leqslant \left\|\frac{\partial a^i}{\partial x_i}\right\| + \sum_{k=1}^{N}\left\|\frac{\partial a^i}{\partial u_k}\right\| \|D_i u_k\| + \\ + \sum_{j=1}^{n}\sum_{k=1}^{N}\left\|\frac{\partial a^i}{\partial p_k^j}\right\| \|D_{ij} u_k\| \leqslant c\left(1 + \sum_{j=1}^{n}\|D_j u\|^2 + \sum_{i,j=1}^{n}\|D_{ij} u\|\right)$$

from which

$$D_i a^i(X, u, Du) \in L^2(Q, \mathbb{R}^N)\,, \qquad i = 1, 2, \ldots, n\,. \tag{2.7}$$

Now let us recall that u is a solution in Q of system (1.1); then (see footnote (5)), $\forall \varphi \in C_0^\infty(Q, \mathbb{R}^N)$, it results:

$$\int_Q \left(u \,\Big|\, \frac{\partial \varphi}{\partial t}\right) dX = -\int_Q \left(\sum_{i=1}^{n} D_i a^i(X, u, Du) + B^0(X, u, Du) \,|\, \varphi\right) dX\,,$$

(6) $\mathcal{M}_{n+2-r}$ is the $(n+2-r)$-dimensional Hausdorff measure with respect to the parabolic metric:

$$d(X, Y) = \max\{\|x - y\|, |t - \tau|^{1/2}\}, \qquad X = (x, t),\ Y = (y, \tau)\,.$$

γ_0 is the real number in the interval $((n-2)/n, 1)$ that appears in the statement of Theorem 3.1 in [1].

from which, by means of (2.6) and (2.7), we reach

$$\exists \, \frac{\partial u}{\partial t} \in L^2(Q, \mathbb{R}^N).$$

Then u verifies all the assumptions of Theorem 4.1 in [1] and therefore the conclusion follows by this theorem.

REMARK 2.1. The Theorem 2.1 can be proved in a «direct way», following the technique used in [1], that is without applying the Theorem 4.1 of [1]. We preferred to make use of Theorem 4.1 in [1] for the sake of shortness.

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