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## Oscillation and Asymptotic Behaviour of Neutral Equations with Distributed Delay.

D. BAINOV(\*) - V. PETROV(\*\*)

ABSTRACT - Consider the neutral differential equation

$$\left[ x(t) + \int_0^{\sigma(t)} x(t-s) d_s r_1(t, s) \right]' + \int_0^{\sigma(t)} x(t-s) d_s r_2(t, s) = 0.$$

The asymptotic properties of the nonoscillatory solutions of the equation are studied. Sufficient conditions are also given to guarantee that all solutions oscillate.

### 1. - Introduction.

In the recent few years a considerable number of papers were published, devoted to the oscillatory properties of first order linear neutral differential equations. Up to now equations of the following form have been investigated

$$[x(t) + px(t - \tau)]' + qx(t - \sigma) = 0,$$

$$[x(t) + p(t)x(t - \tau)]' + q(t)x(t - \sigma) = 0,$$

$$\left[ x(t) + \sum_1^k p_i x(t - \tau_i) \right]' + \sum_1^m q_i x(t - \sigma_i) = 0,$$

$$\left[ x(t) + \sum_1^k p_i(t)x(t - \tau_i) \right]' + \sum_1^m q_i(t)x(t - \sigma_i) = 0.$$

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To these equations the papers [1]-[3], [6]-[10] were devoted. We shall note that nonautonomous neutral differential equations with distributed delay have not been studied up to now.

In the present paper the equation

$$(1) \quad \left[ x(t) + \int_0^{\sigma(t)} x(t-s) d_s r_1(t, s) \right]' + \int_0^{\sigma(t)} x(t-s) d_s r_2(t, s) = 0$$

is investigated with initial function  $\varphi(t) \in C([a, t_0], \mathbb{R})$ , ( $a = \inf_{t \geq t_0} \{t - \sigma(t)\}$ ), where the integrals in (1) are in the sense of Riemann-Stieltjes. Some ideas of [2] are developed, the asymptotic behaviour of (1) is investigated and sufficient conditions for oscillation of all solutions of (1) are obtained.

## 2. - Preliminary notes.

We shall say that conditions (A) are met if the following conditions hold:

$$A1) \quad \sigma(t) \in C([t_0, \infty), (0, \infty)),$$

$$A2) \quad \lim_{t \rightarrow \infty} (t - \sigma(t)) = \infty.$$

Conditions (A) imply that  $a = \inf_{t \geq t_0} (t - \sigma(t)) > -\infty$ .

Introduce conditions (B):

$$B1) \quad x \in C([a, \infty), \mathbb{R}),$$

$$B2) \quad x(t) + \int_0^{\sigma(t)} x(t-s) d_s r_1(t, s) \in C^1([t_0, \infty), \mathbb{R}).$$

Consider equation (1) with the initial condition

$$(2) \quad x(t) = \varphi(t), \quad t \in [a, t_0].$$

**DEFINITION 1.** The function  $x(t)$ , satisfying conditions (B) is said to be a solution of the initial value problem (1)-(2) if  $x(t)$  satisfies (1) for  $t \geq t_0$  and if the relation (2) holds.

Introduce the following conditions (C):

$$C1) \quad r_i(t, 0) = 0, \quad t \in [t_0, \infty), \quad i = 1, 2.$$

$$C2) \quad r_i(t, \sigma(t)) \in C([t_0, \infty), \mathbb{R}), \quad i = 1, 2.$$

$$C3) \quad r_1(t, s) \text{ is continuous at } s = 0 \text{ for any fixed } t \in [t_0, \infty).$$

C4) The functions  $v_i(t) = \sup_{s \in [0, \sigma(t)]} |r_i(t, s)|$ ,  $t \geq t_0$ ,  $i = 1, 2$  are bounded.

C5) For any fixed  $t \geq t_0$ ,  $r_i(t, s)$  are functions of bounded variation with respect to  $s$  in  $[0, \sigma(t)]$ .

$$C6) \lim_{t_1 \rightarrow t} \int_0^{\min\{\sigma(t_1), \sigma(t)\}} |r_i(t_1, s) - r_i(t, s)| ds = 0, \quad i = 1, 2.$$

LEMMA 1. Let conditions (A) and (C) hold. Then for any initial function  $\varphi(t) \in C([a, t_0], \mathbb{R})$  the initial value problem (1)-(2) has a unique solution.

Lemma 1 is obtained as a corollary of [4].

Introduce the following conditions (D):

D1)  $r_1(t, s)$  is nonincreasing with respect to  $s$  for  $s \in [0, \sigma(t)]$ .

D2)  $r_2(t, s)$  is nondecreasing with respect to  $s$  for  $s \in [0, \sigma(t)]$ .

$$D3) \int_{t_0}^{\infty} r_2(t, \sigma(t)) dt = \infty.$$

DEFINITION 2. The solution  $x(t)$  of (1) is said to oscillate if there exists an increasing sequence  $\{t_n\}_1^{\infty}$ , such that  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $x(t_n) = 0$ ,  $n \in \mathbb{N}$ . Otherwise it is said to be nonoscillatory.

DEFINITION 3. The function  $x(t)$  is said to eventually have the property  $K$ , if there exists  $t_0$  such that for  $t \geq t_0$  the function has the property  $K$ .

By Definition 3 the nonoscillatory solutions of (1) are characterized as being eventually positive or eventually negative.

Let

$$(3) \quad z(t) = x(t) + \int_0^{\sigma(t)} x(t-s) d_s r_1(t, s).$$

Then

$$(4) \quad z'(t) = - \int_0^{\sigma(t)} x(t-s) d_s r_2(t, s).$$

We shall prove several lemmas which are essentially used in the proof of the main results.

LEMMA 2. Let conditions (A), (C) and (D) hold and let

$$(5) \quad r_1(t, \sigma(t)) \geq p > -1.$$

If  $x(t)$  is an eventually positive solution of (1), then  $x(t)$  is a bounded function.

PROOF. Let  $x(t)$  be an eventually positive solution of (1). From (4) it follows that  $z'(t) \leq 0$  eventually and  $z(t)$  is a nonincreasing function. D3) implies that  $z(t)$  is not an eventually constant function and thus either  $z(t) < 0$  or  $z(t) > 0$  eventually. Suppose that  $z(t) < 0$ . Then from (3), (5), C1) and D1) there follows the estimate

$$\begin{aligned} 0 > x(t) + \int_0^{\sigma(t)} x(t-s) d_s r_1(t, s) &\geq x(t) - \max_{[t-\sigma(t), t]} x(s) v_1(t) = \\ &= x(t) + \max_{[t-\sigma(t), t]} x(s) r_1(t, \sigma(t)) > x(t) - \max_{[t-\sigma(t), t]} x(s). \end{aligned}$$

From the above inequalities it follows that there exists  $t_1 > t_0$ , such that for  $t > t_1$  the inequality

$$(6) \quad x(t) < \max_{[t-\sigma(t), t]} x(s)$$

holds. By virtue of condition A2) we can choose  $\bar{t}$  such that  $t - \sigma(t) \geq t_0$  for  $t \geq \bar{t}$ . Then from (6) we have

$$(7) \quad x(t) < \max_{[t-\sigma(t), t]} x(s).$$

Suppose that  $x(t)$  is unbounded. Then  $\limsup x(t) = \infty$  and there exists a sequence  $\{t_n\}_{n=1}^{\infty}$ , such that  $\lim_{n \rightarrow \infty} t_n = \infty$ ,  $\lim_{n \rightarrow \infty} x(t_n) = \infty$  and  $\max_{[t_0, t_n]} x(s) = x(t_n)$ . The last inequality however contradicts (7).

Let  $z(t) > 0$  eventually. Since  $z(t)$  is a nonincreasing function, there exists the finite limit  $\lim_{t \rightarrow \infty} z(t) = c \geq 0$ . We shall prove that  $\liminf_{t \rightarrow \infty} x(t) = 0$ . Suppose that this is not true, i.e.  $d = \liminf_{t \rightarrow \infty} x(t) > 0$ . There exists  $\bar{t} \geq \bar{t}$  such that  $x(t) > d/2$  for  $t \geq \bar{t}$ . From (4) and D2) it follows that

$$z'(t) \leq - \min_{[t-\sigma(t), t]} x(s) r_2(t, \sigma(t)).$$

By virtue of A2) we can choose  $\tilde{t}$  such that  $t - \sigma(t) > \tilde{t}$  for  $t \geq \tilde{t}$ . Then

from the above estimate we have

$$z'(t) < -\frac{d}{2}r_2(t, \sigma(t)), \quad t \geq \tilde{t}.$$

Integrate the last inequality from  $\tilde{t}$  to  $t$  and obtain

$$z(t) \leq z(\tilde{t}) - \frac{d}{2} \int_{\tilde{t}}^t r_2(s, \sigma(s)) ds.$$

D3) implies that  $\lim_{t \rightarrow \infty} z(t) = -\infty$ , which contradicts the inequality  $z(t) > 0$  eventually. Thus  $\liminf_{t \rightarrow \infty} x(t) = 0$ . Suppose that  $x(t)$  is an unbounded function. As above we choose a sequence  $\{t_n\}_1^\infty$  with the respective properties. Since  $\liminf_{t \rightarrow \infty} x(t) = 0$ , there exists a sequence  $\{\tau_k\}_1^\infty$ , such that  $\lim_{k \rightarrow \infty} \tau_k = \infty$  and  $\lim_{k \rightarrow \infty} x(\tau_k) = 0$ . Let  $n, k \in \mathbb{N}$  be large enough and such that  $t_n > \tau_k$ . Then the following estimate is valid:

$$\begin{aligned} z(t_n) - z(\tau_k) &> x(t_n) - x(\tau_k) + \int_0^{\sigma(t_n)} x(t_n - s) d_s r_1(t_n, s) \geq \\ &\geq x(t_n) - x(\tau_k) + \max_{[t_n - \sigma(t_n), t_n]} x(s) r_1(t_n, \sigma(t_n)) \geq x(t_n) - x(\tau_k) + px(t_n). \end{aligned}$$

Thus we have

$$z(t_n) - z(\tau_k) > x(t_n)(1 + p) - x(\tau_k).$$

The choice of the sequences  $\{t_n\}$  and  $\{\tau_k\}$  and (5) imply that  $z(t_n) - z(\tau_k) > 0$  and since  $t_n > \tau_k$ , we get to a contradiction with the fact that  $z(t)$  is an eventually nonincreasing function. ■

LEMMA 3 ([5]). Let  $p, \tau \in C([t_0, \infty), (0, \infty))$  and  $\lim_{t \rightarrow \infty} (t - \tau(t)) = \infty$ .  
If

$$\liminf_{t \rightarrow \infty} \int_{t - \tau(t)}^t p(s) ds > \frac{1}{e},$$

then the inequality

$$y'(t) + p(t)y(t - \tau(t)) \leq 0$$

has no eventually positive solutions.

### 3. - Main results.

**THEOREM 1.** Let conditions (A), (C), (D) and (5) hold. Then each non-oscillatory solution  $x(t)$  of (1) tends to 0 as  $t \rightarrow \infty$ .

**PROOF.** Let  $x(t)$  be an eventually positive solution of (1). From Lemma 2 it follows that the function  $x(t)$  is bounded. Then  $b = \limsup_{t \rightarrow \infty} x(t) < \infty$ . We shall prove that  $b = 0$ . Suppose that this is not true and choose  $\varepsilon > 0$  so that  $\varepsilon < (b(1+p))/(2-p)$ . There exists a sequence  $\{t_n\}_1^\infty$  such that  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $\lim_{n \rightarrow \infty} x(t_n) = b$ . Then we can choose  $\bar{t} \geq t_0$  and  $N$  so that  $|x(t_n) - b| < \varepsilon$  for  $n > N$  and  $x(t) - b < \varepsilon$  for  $t > \bar{t}$ . Since  $x(t)$  is a bounded function,  $z(t)$  is bounded too. Then the fact that  $z(t)$  is a nonincreasing function implies the existence of the finite limit  $\lim_{t \rightarrow \infty} z(t) = c$ . As in the proof of Lemma 2 it is shown that  $\liminf_{t \rightarrow \infty} x(t) = 0$  and then we can choose a sequence  $\{\tau_k\}_1^\infty$  such that  $\lim_{k \rightarrow \infty} \tau_k = \infty$  and  $\lim_{k \rightarrow \infty} x(\tau_k) = 0$ . There exists  $K$  such that for  $k > K$  we have  $x(\tau_k) < \varepsilon$ . From the sequences  $\{t_n\}_1^\infty$  and  $\{\tau_k\}_1^\infty$  choose the pair  $\tau_j, t_i$  so that  $j > K, i > N, \tau_j < t_i$  and  $t_i - \sigma(t_i) > t_0$ . Then the following estimate is valid:

$$\begin{aligned} z(t_i) - z(\tau_j) &= x(t_i) - x(\tau_j) + \\ &+ \int_0^{\sigma(t_i)} x(t_i - s) d_s r_1(t_i, s) - \int_0^{\sigma(\tau_j)} x(\tau_j - s) d_s r_1(\tau_j, s) \geq \\ &\geq x(t_i) - x(\tau_j) + \int_0^{\sigma(t_i)} x(t_i - s) d_s r_1(t_i, s) \geq \\ &\geq x(t_i) - x(\tau_j) + \max_{[t_i - \sigma(t_i), t_i]} x(s) r_1(t_i, \sigma(t_i)) \geq \\ &\geq b - \varepsilon - \varepsilon + p(b + \varepsilon) > 0. \end{aligned}$$

(The last inequality follows from the choice of  $\varepsilon$ .) Thus, for  $\tau_j < t_i$  we obtained that  $z(\tau_j) < z(t_i)$ , which contradicts the fact that the function  $z(t)$  is nonincreasing. Hence  $\limsup_{t \rightarrow \infty} x(t) = 0$  and  $\lim_{t \rightarrow \infty} x(t) = 0$ . If  $x(t)$  is an eventually negative solution of (1), then since (1) is a linear equation,  $-x(t)$  is an eventually positive solution of (1), which implies that in this case as well  $\lim_{t \rightarrow \infty} x(t) = 0$ . ■

In the same way the following theorem is proved:

**THEOREM 2.** Let conditions (A), (C), D2) and D3) hold and let  $r_1(t, s)$  be nondecreasing with respect to  $s$  for  $s \in [0, \sigma(t)]$ .

If  $r_1(t, \sigma(t)) \leq p < 1$ , then each nonoscillatory solution of (1) tends to 0 as  $t \rightarrow \infty$ .

**REMARK 1.** Theorem 1 generalizes or generalizes and extends a number of known results, for instance Theorem 3 iv) ([2]), Theorem 1 ([10]), Theorem 5 ([7]), Corollary 3b ([1]).

**REMARK 2.** The condition  $r_1(t, \sigma(t)) \leq p < 1$  in Theorem 2 is essential. We shall illustrate this fact with the following example:

**EXAMPLE 1** [8]. Consider the equation

$$(8) \quad [x(t) + x(t-2)]' + q(t)x(t) = 0,$$

where

$$q(t) = \frac{(1/t^2) + 1/(t-2)^2}{\psi(t) + 1/t}$$

and  $\psi(t)$  is the 4-periodic function

$$\psi(t) = \begin{cases} 0, & t \in [0, 1], \\ t-1, & t \in (1, 2], \\ 1, & t \in (2, 3], \\ 4-t, & t \in (3, 4]. \end{cases}$$

Clearly, equation (8) is a particular case of (1), moreover  $r_1(t, \sigma(t)) \equiv 1$ . It is immediately verified that  $x(t) = \psi(t) + 1/t$  is a nonoscillatory solution of (8), yet the limit  $\lim_{t \rightarrow \infty} x(t)$  does not exist. On the other hand all

conditions of Theorem 2, except the condition  $r_1(t, \sigma(t)) \leq p < 1$  are met.

The question whether the assertion of Theorem 2 is still valid without the condition  $r_1(t, \sigma(t)) \leq p < 1$ , if D3) is replaced with the more restrictive condition  $r_2(t, \sigma(t)) \geq m > 0$ ,  $t \geq t_0$ , is open. For example, this is true (Theorem 2 ([2])) for the equation

$$[x(t) + p(t)x(t-\tau)]' + q(t)x(t-\sigma) = 0,$$

where  $p, q$  are continuous, nonnegative functions and  $\tau, \sigma \geq 0$ . (In



this case  $D3$ ) reduces to  $\int_{t_0}^{\infty} q(t) dt = \infty$  and the condition  $r_2(t, \sigma(t)) \geq m > 0$  reduces to  $q(t) \geq q > 0$ .)

Define the function  $\tau(t)$

$$\tau(t) = \sup \{s \in [0, \sigma(t)] / r_2(t, s) = 0\}, \quad t \geq t_0.$$

**THEOREM 3.** Let conditions (A), (C), (D) and (5) hold and let  $\tau(t) \in C([t_0, \infty), (0, \infty))$ . If

$$(9) \quad \liminf_{t \rightarrow \infty} \int_{t - \tau(t)}^t r_2(s, \sigma(s)) ds > \frac{1}{e},$$

then each solution of (1) is oscillatory.

**PROOF.** Suppose that (1) has at least one nonoscillatory solution  $x(t)$ . Without loss of generality, let  $x(t)$  be eventually positive. From Theorem 1 it follows that  $\lim_{t \rightarrow \infty} x(t) = 0$ . (3), (5) and  $D1$ ) imply that  $\lim_{t \rightarrow \infty} z(t) = 0$ . On the other hand,  $z(t)$  is an eventually nonincreasing nonconstant function. Hence  $z(t) > 0$  eventually. (3) implies that  $z(t) < x(t)$  and then from (1) we have

$$z'(t) + \int_0^{\sigma(t)} z(t-s) d_s r_2(t, s) \leq 0.$$

By virtue of the definition of  $\tau(t)$ , the last inequality takes the form

$$0 \geq z'(t) + \int_{\tau(t)}^{\sigma(t)} z(t-s) d_s r_2(t, s) \geq z'(t) + z(t - \tau(t)) r_2(t, \sigma(t)).$$

Thus we obtained that the eventually positive function  $z(t)$  is a solution of the inequality

$$y'(t) + r_2(t, \sigma(t)) y(t - \tau(t)) \leq 0.$$

Then, by condition (9) we get to a contradiction with the assertion of Lemma 3. ■

**REMARK 3.** Theorem 3 generalizes Theorem 5 ([2]).

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