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Almost Completely Decomposable Groups with Primary Cyclic Regulating Quotient.

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Dedicated to Hermann Heineken on his 60th birthday

ABSTRACT - The indecomposable, almost completely decomposable groups are described with primary cyclic quotient relative to some regulating subgroup. It is shown that for an indecomposable almost completely decomposable group with one primary cyclic regulating quotient all regulating quotients are cyclic, i.e. isomorphic. Moreover, an almost completely decomposable group with primary cyclic regulating quotient decomposes directly into a completely decomposable summand and a direct sum of indecomposable almost completely decomposable summands with primary cyclic regulating quotients.

1. - Introduction.

This paper is based on a manuscript [3] of Rolf Burkhardt in 1985, which was intended to continue [2]. There is an independent treatment of this subject by Mader and Vinsonhaler [6]. In contrast to this we consider almost completely decomposable groups as torsion-free extensions of completely decomposable groups (not necessarily regulating in the extension group) by finite groups. There is a certain natural overlap of the results, but the proofs and most of the statements, are different.

An *almost completely decomposable group* is a torsion-free abelian

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group which contains a completely decomposable group as a subgroup of finite index. It will always be assumed that our groups have finite ranks. Completely decomposable subgroups of minimal index are called *regulating*. The quotients relative to different regulating subgroups need not be isomorphic. Here we investigate the class of groups such that the cyclic group $\mathbb{Z}_{p^n}$ of order p^n is a quotient relative to some completely decomposable subgroup. Such a quotient is called a (*primary cyclic*) *regulating quotient* if the completely decomposable subgroup is regulating.

We fix some notation before continuing. For any torsion-free abelian group A let $A(t) = \{a \in A \mid tp(a) \geq t\}$, $A^*(t) = \sum_{s > t} A(s)$, and $A^\natural(t) = A^*(t)_*$ denote the usual type subgroups; and let $\mathrm{T}_{cr}(A) = \{t \mid A(t)/A^\natural(t) \neq 0\}$ denote the *critical typeset* of A. For $t \in \mathrm{T}_{cr}(A)$ let the torsion-free ranks $r_t(A) = \mathrm{rk}(A(t)/A^\natural(t))$ be called *critical ranks of* A. Let $W = \bigoplus_s W_s$ be a completely decomposable subgroup of finite index in A in *homogeneous decomposition*, i.e. s stands for a type, and W_s is a direct sum of rational groups of type s. The critical typesets of W and of A are $\mathrm{T}_{cr}(A) = \mathrm{T}_{cr}(W) = \{s \mid W_s \neq 0\}$. Note that $r_t(A) = r_t(W) = \mathrm{rk}\, W_t$ for the critical ranks of A. The regulating subgroups of the almost completely decomposable group A are exactly the subgroups $W = \sum_{s \in \mathrm{T}_{cr}(A)} A_s$ for arbitrary decompositions $A(s) = A_s \bigoplus_s A^\natural(s)$.

Our notation is standard and can be found in Arnold [1] and Fuchs [4].

2. – Partitions.

Let T be a finite set of types with a disjoint partition $T = T_0 \cup T_1 \cup \ldots \cup T_n$. Define $T(k) = T_0 \cup T_1 \cup \ldots \cup T_{k-1}$ for all $1 \leq k \leq n$, $T(0) = \emptyset$, $T_{>t} = \{s \in T(n) \mid s > t\}$ and $T_{\geq t} = \{s \in T(n) \mid s \geq t\}$ for $t \in T(n)$. A partition $T = T_0 \cup T_1 \cup \ldots \cup T_n$ is said to be a *p^n-admissible partition of T* if:

(1) $T(k+1) \not\subset T_{\geq t}$ for all $t \in T_k$, $0 \leq k < n$.

(2) $T_{>t} \subset T(k)$ for all $t \in T_k$, $k \neq n$.

(3) $T(k) \not\subset T_{>t}$ for all $t \in T_k$, $0 < k < n$.

A p^n-admissible partition is called *short* if $T_n = \emptyset$. Note that the properties (1) and (3) imply that $|T_0| \geq 2$, and that by property (2) nonempty subsets T_k, $k < n$, form antichains, i.e. the elements are pairwise incomparable. Moreover, if the properties (1) and (3) hold together then property (1) may be replaced by property (1) for $k = 0$ only. If the properties (1), (2) and (3) hold together then property (1) may be replaced by $|T_0| \geq 2$.

Let A be an almost completely decomposable group with critical typeset T and completely decomposable subgroup W with primary cyclic quotient $A/W \cong \mathbb{Z}_{p^n} \neq 0$. Let $W = \bigoplus_{s \in T} W_s$ be a homogeneous decomposition of W. Let $a \in A$ such that $A = \langle W, a \rangle$, then $p^n a = \sum_{s \in T} w_s$, $w_s \in W_s$. A partition $T = T_0 \cup T_1 \cup \ldots \cup T_n$ of the critical typeset T of A is defined by $t \in T_k$ if the p-height of w_t in W is $k < n$, i.e. $\chi_p^W(w_t) = k$, and $T_n = T \backslash (T_0 \cup T_1 \cup \ldots \cup T_{n-1})$. This partition is called a *W-partition of the critical typeset T of A*. Note that $T_0 = \emptyset$ would contradict $A/W \cong \mathbb{Z}_{p^n}$. Moreover, $V = \bigoplus_{s \in T_n} W_s$ is a direct summand of W and of A, i.e. $A = V \oplus \left\langle \bigoplus_{s \in T(n)} W_s \right\rangle_*^A$ and $A/W \cong \left\langle \bigoplus_{s \in T(n)} W_s \right\rangle_*^A \Big/ \left\langle \bigoplus_{s \in T(n)} W_s \right\rangle \cong \mathbb{Z}_{p^n}$. So, for simplicity we assume that $p^n a = \sum_{s \in T(n)} w_s$. In general the W-partition depends on the homogeneous decomposition of W.

Now we prove a technical lemma.

LEMMA 1. *Let A be an almost completely decomposable group with critical typeset T. Let W be a completely decomposable subgroup with primary cyclic quotient* $A/W \cong \mathbb{Z}_{p^n}$ *and homogeneous decomposition* $W = \bigoplus_{s \in T} W_s$. *Let* $T = T_0 \cup T_1 \cup \ldots \cup T_n$ *be the corresponding W-partition of T. Let* $S \subset T$.

Then $\left\langle \bigoplus_{s \in S} W_s \right\rangle_*^A / \bigoplus_{s \in S} W_s \cong \mathbb{Z}_{p^k}$ *if and only if k is maximal with respect to* $T(k) \subset S$.

PROOF. We may assume that $A = \langle W, a \rangle$, where $a = p^{-n}\left(\sum_{j=0}^{n-1} \sum_{s \in T_j} w_s\right)$ and $w_s \in W_s$. For a subset S of T define $W(S) = \bigoplus_{s \in S} W_s$. By hypothesis A/W is cyclic and we obtain that $W(S)_*^A / W(S) \cong (A \cap W(S)_*^A)/(W \cap W(S)_*^A)$ is cyclic by the modular law, since $W(S) = W \cap W(S)_*^A$.

Let first k be maximal such that $T(k) \subset S$. If $k = n$ then the claim is obvious. So we may assume that $k < n$. Let $h \in W(S)_*^A$ be of order p^b modulo $\bigoplus_{s \in S} W_s$ and be differently written $h = la + w = p^{-b}\left(\sum_{s \in S} \tilde{w}_s\right)$

with $w \in W$, $l \in \mathbb{Z}$ and $\tilde{w}_s \in W_s$. Then $p^{n-b}\left(\sum_{s \in S} \tilde{w}_s\right) = l\left(\sum_{s \in T} w_s\right) + p^n w$ or $\sum_{s \in S}(p^{n-b}\tilde{w}_s - lw_s) - \sum_{s \in T\setminus S} lw_s = p^n w \in p^n\left(\bigoplus_{s \in T} W_s\right)$. In particular, $\chi_p(lw_s) \geqslant n$ for $s \in T\setminus S$ and therefore p^{n-k} divides l, since $T_k \not\subset S$. Hence $p^k h \in W(S)$, i.e. $b \leqslant k$. On the other hand $p^{-k}\left(\sum_{i=0}^{k-1} \sum_{s \in T_i} w_s\right)$ is an element of $W(S)^A_*$ whose order modulo $W(S)$ is p^k.

Conversely let $W(S)^A_*/W(S) \cong \mathbb{Z}_{p^k}$. We now show that $T(k) \subset S$. There are complements L and M of the direct summands $W(S)$ and $W(T(k))$, respectively, such that $W = W(S) \oplus L = W(T(k)) \oplus M$. As

$$\mathbb{Z}_{p^k} \cong W(S)^A_*/W(S) \cong W(T(k))^A_*/W(T(k)) \tilde{\subset} A/W \cong \mathbb{Z}_{p^n}$$

we obtain $W(S)^A_* \oplus L = W(T(k))^A_* \oplus M$ and $p^{-k}\left(\sum_{s \in T(k)} w_s\right) \in W(S)^A_* \oplus L$, since $\mathbb{Z}_{p^n}$ contains a unique subgroup of order p^k. Thus $p^{-k}w_v \in L \subset W$ for $v \in T(k)\setminus S$, a contradiction, and $T(k) \subset S$.

Assume that $T_k \subset S$, $k < i$. But now

$$p^{-(k+1)}\left(\sum_{i=0}^{k} \sum_{s \in T_i} w_s\right) \in W(S)^A_*$$

has order p^{k+1} modulo $W(S)$, which is a contradiction. Hence $T_k \not\subset S$.

Lemma 2. *Let A be an almost completely decomposable group with completely decomposable subgroup W and finite cyclic quotient A/W. Let t be a critical type of A such that $A(t)/A^\natural(t)$ is of rank > 1, then there is a decomposition $W = H \oplus L$ of W, where H is completely decomposable homogeneous of type t and*

$$A = H \oplus L^A_*.$$

In particular, if A has no rational direct summand or if A is indecomposable, then all critical ranks of A are 1.

Proof. The order of the cyclic group A/W is finite, say n. Let $W = \bigoplus_{s \in T} W_s$ be a homogeneous decomposition of W, where one of the homogeneous components W_t is assumed to be of rank > 1. Let $A = \langle W, a\rangle$ and $a = n^{-1}\left(\sum_{s\in T} w_s\right)$, where $w_s \in W_s$. Since W_t is homogeneous completely decomposable of finite rank, $\langle w_t\rangle_*^W$ is a direct summand of W_t and $W_t = \langle w_t\rangle_*^W \oplus H$, where $H \neq 0$ is homogeneous, completely decomposable by [4, 86.7 and 8]. Hence $A = H \oplus \left\langle \langle w_t\rangle_*^W \oplus \bigoplus_{s\neq t} W_s, a\right\rangle$ proving the lemma. ∎

LEMMA 3. *Let A be an almost completely decomposable group with critical typeset T and completely decomposable subgroup W with primary cyclic quotient A/W.*

If some W-partition of T is not short or fails to have property (2) *of an $|A/W|$-admissible partition, then there is a non-trivial direct decomposition $A = X \oplus H$ such that $W = X \oplus (H \cap W)$ and $H/(H \cap W) \cong A/W$, in particular A has a completely decomposable direct summand.*

PROOF. Let $A/W \cong \mathbb{Z}_{p^n}$ and let $W = \bigoplus_{t\in T} W_t$ be a homogeneous decomposition of W. Let $T = T_0 \cup \ldots \cup T_n$ be the corresponding W-partition of T and let $A = \langle W, a\rangle$ with $a = p^{-n}\left(\sum_{s\in T} w_s\right)$, where $w_s \in W_s$.

If this partition of T is not short then obviously $\bigoplus_{s\in T_n} W_s$ is a direct summand of A. So we may assume the partition to be short. In view of Lemma 2 all critical ranks may be assumed to be 1, since otherwise there would be rational summands. Moreover, assume that property (2) of a p^n-admissible partition is violated, i.e. there are $s, t \in T(n)$, $t < s$, such that $t \in T_l$ and $s \in T_k$, where $l \leqslant k$. There is a natural number $m \neq 0$ prime to p such that $\chi(mw_s) > \chi(w_t)$ and $\chi_p(mw_s) = \chi_p(w_s) = k \geqslant l = \chi_p(w_t)$. Since m and p are relatively prime there are integers x, y such that $xm + yp^n = 1$ and $a = p^{-n}\left(w_t + xmw_s + \sum_{u\in T\setminus\{s,t\}} w_u\right) + yw_s$. Define $w_t' = w_t + xmw_s$ and $a' = p^{-n}\left(w_t' + \sum_{u\in T\setminus\{s,t\}} w_u\right)$. Then $a = a' + yw_s$, where $a' \in \langle w_t', w_u \mid u \notin \{s,t\}\rangle_*^A$. Since $\chi^W(w_t') = \chi^W(w_t)$, by the choice of m, we get $W = \bigoplus_{u\in T}\langle w_u\rangle_*^W = \langle w_t'\rangle_*^W \oplus \bigoplus_{u\neq t}\langle w_u\rangle_*^W$ using a result of Mader [5], cf. [7, 2.2], and consequently $A = \langle W, a\rangle = \langle W, a'\rangle = \langle w_s\rangle_*^W \oplus \langle w_t', w_u \mid u \notin \{s,t\}\rangle_*^A$. ∎

Next we prove another technical lemma.

LEMMA 4. *Let A be an almost completely decomposable group with critical typeset T and completely decomposable subgroup W with primary cyclic quotient A/W.*

If for some W-partition of T there is a natural number $k \neq 0$ and a type $t \in T_k$ such that $T(k) \subset T_{>t}$, i.e. property (3) *of an $|A/W|$-admissible partition is violated for this t, then $A^{\natural}(t)$ is a direct summand of A and there is a completely decomposable subgroup L of A, such that $A = A^{\natural}(t) \oplus L_*^A$, L_*^A/L is primary cyclic and $|A/W| = |A/(W^{\natural}(t) \oplus L)|$.*

If for some W-partition of T there is a natural number $k < n$ and a type $t \in T_k$ such that $T(k+1) \subset T_{\geq t}$, then $W(t) + A^{\natural}(t) \neq A(t)$.

PROOF. Let $|A/W| = p^n$ and assume that the type $t \in T_k$, $k \geq 1$, violates property (3) of a p^n-admissible partition of T, i.e. $\emptyset \neq T(k) \subset T_{>t}$. Then, since $t \leq \bigcup_{s \in T(k)} s$ and $t \in T_k$, there is a smallest natural number $m \neq 0$ such that $m^{-1}p^{-k}w_t \in W$ and

$$\chi^W(p^{-k}w_t) \leq \chi^W\left(m\left(\sum_{s \in T_{>t}} w_s\right)\right).$$

By $t \in T_k$ we conclude that m and p are relatively prime. Let $x, y \in \mathbb{Z}$ with $mx + p^{n-k}y = 1$, then $w_t' = p^{-k}\left(w_t + mx\left(\sum_{s \in T_{>t}} w_s\right)\right) \in A$, since $p^{-k}\left(\sum_{s \in T(k)} w_s\right) \in A$.

Now we determine $A^{\natural}(t)$. By Lemma 1 we have $A^{\natural}(t)/W^{\natural}(t) \cong \mathbb{Z}_{p^k}$ since $T(k) \subset T_{>t}$ and $T_k \not\subset T_{>t}$, hence

$$A^{\natural}(t) = \left\langle W^{\natural}(t),\ p^{-k}\left(\sum_{s \in T_{>t}} w_s\right)\right\rangle.$$

Define $X = W_t \oplus A^{\natural}(t)$. Certainly $W(t) \subset X$ and $w_t' \in X$. Since W_t is completely decomposable homogeneous of finite rank there is a direct decomposition $W_t = \langle w_t \rangle_*^W \oplus H$, cf. [4, 86.7 and 8]. Let $W_t' = \langle w_t' \rangle_*^X \oplus H$. So by Mader [7, 2.2] and the above estimate of the characteristics, $X = W_t' \oplus A^{\natural}(t)$. Since $(X + W)/W \cong A^{\natural}(t)/W^{\natural}(t) \cong \mathbb{Z}_{p^k}$ the quotient $A/(X+W) = \langle X + W, a\rangle/(X+W) \cong \mathbb{Z}(p^{n-k})$. Hence $a + X + W \in A/(X+W)$ has order p^{n-k}, i.e. $p^{-k}\left(\sum_{s \in T} w_s\right) = p^{n-k}a \in X + W$.

Now let us consider the following:

$$(*)\qquad p^{n-k}a = p^{-k}\left(\sum_{s\in T} w_s\right) = p^{-k}\left(w_t + \sum_{s\in T_{>t}} w_s\right) + p^{-k}\left(\sum_{s\notin T_{\geq t}} w_s\right) =$$

$$= p^{-k}\left(w_t + mx\left(\sum_{s\in T_{>t}} w_s\right)\right) + p^{-k}\left(\sum_{s\notin T_{\geq t}} w_s\right) + (1-mx)p^{-k}\left(\sum_{s\in T_{>t}} w_s\right) =$$

$$= w_t' + \sum_{s\notin T_{\geq t}} p^{-k}w_s + yp^{n-k}p^{-k}\left(\sum_{s\in T_{>t}} w_s\right).$$

Define $a' = a - yp^{-k}\left(\sum_{s\in T_{>t}} w_s\right)$, then $p^{n-k}a' = w_t' + \left(\sum_{s\notin T_{\geq t}} p^{-k}w_s\right)$. Since $p^{-k}\left(\sum_{s\in T_{>t}} w_s\right) \in A^{\natural}(t)$ we obtain finally

$$A = \langle W, a\rangle = \langle X + W, a'\rangle =$$

$$= \left\langle W_t' \oplus \bigoplus_{s\neq t} W_s,\ p^{-k}\left(\sum_{s\in T_{>t}} w_s\right),\ p^{-(n-k)}\left(w_t' + \sum_{s\notin T_{\geq t}} p^{-k}w_s\right)\right\rangle$$

and A has the non-trivial direct decomposition

$$A = \left\langle W^{\natural}(t),\ p^{-k}\left(\sum_{s\in T_{>t}} w_s\right)\right\rangle \oplus$$

$$\oplus\left\langle W_t' \oplus \bigoplus_{s\notin T_{\geq t}} W_s,\ p^{-(n-k)}\left(w_t' + \sum_{s\notin T_{\geq t}} p^{-k}w_s\right)\right\rangle,$$

with primary cyclic quotients of order p^k and p^{n-k}, respectively. Note that the first summand is $A^{\natural}(t)$.

If, for the second part, $T_0 = \{t\}$, then $p^{n-1}a = p^{-1}w_t + p^{-1}\left(\sum_{s\neq t} w_s\right)$ and $p^{-1}w_t \in A(t)\setminus W_t$. Hence W_t is not pure in A, i.e. $A(t) \neq W_t \oplus \oplus A^{\natural}(t)$.

So we may assume $0 < k < n$, i.e. $\{t\} \neq T(k+1) \subset T_{\geq t}$. The second hypothesis is stronger than the first and we may continue with formula $(*)$. Dividing by p we obtain

$$p^{n-(k+1)}a = p^{-1}w_t' + p^{-(k+1)}\left(\sum_{s\notin T_{\geq t}} w_s\right) + yp^{n-k-1}p^{-k}\left(\sum_{s\in T_{>t}} w_s\right).$$

Now $yp^{n-k-1}p^{-k}\left(\sum_{s\in T_{>t}} w_s\right) \in A$, since $k < n$, and also

$p^{-(k+1)}\left(\sum_{s \notin T_{\geqslant t}} w_s\right) \in A$, since $T(k+1) \subset T_{\geqslant t}$. So we conclude finally $p^{-1} w_t' \in A$, i.e. $W(t) + A^{\natural}(t) \neq A(t)$. ■

In general the W-partition of the critical typeset T of an almost completely decomposable group A for a completely decomposable subgroup W with primary cyclic quotient depends on the homogeneous decomposition of W, but for a group A without rational direct summands it is an invariant as the following proposition shows.

PROPOSITION 5. *Let W be a completely decomposable subgroup of an almost completely decomposable group A with primary cyclic quotient and without rational direct summands. Then all W-partitions of the critical typeset of A coincide.*

PROOF. Let $A/W \cong \mathbb{Z}_{p^n} \neq 0$ and let $W = \bigoplus_{s \in T} W_s$ be a homogeneous decomposition of W. Let $A = \langle W, a \rangle$ and let $T = T_0 \cup T_1 \cup \ldots \cup T_n$ be the corresponding W-partition of the critical typeset. Suppose $A = \langle W, a \rangle = \langle W, b \rangle$, then $b + W = ma + W$ where $m \in \mathbb{N}$ and p are relatively prime. Hence the W-partitions corresponding to a and b, respectively, coincide.

It remains to show that the partition is independent of the choice of the homogeneous decomposition of W. Let $W = W_t' \oplus \bigoplus_{s \neq t} W_s$ be another homogeneous decomposition of W, where only one summand is changed. Let $p^n a = w_t + \sum_{s \neq t} w_s = w_t' + \sum_{s \neq t} w_s'$, where $w_t \in W_t$, $w_t' \in W_t'$ and $w_s, w_s' \in W_s$ for all $s \neq t$. Hence $w_t' = w_t + \sum_{s \neq t} (w_s - w_s')$ and $\chi_p^W(w_t') = \min\{\chi_p^W(w_t), \chi_p^W(w_s - w_s') \mid s \neq t\}$. Thus $\chi_p^W(w_t') \leqslant \chi_p^W(w_t)$ and equality by symmetry. The identity $w_t' = w_t + \sum_{s \neq t} (w_s - w_s')$ also implies $w_s = w_s'$ for all $s \geqslant t$. Thus, if for some $s > t$ we have $\chi_p^W(w_s) \neq \chi_p^W(w_s')$, then $\chi_p^W(w_s), \chi_p^W(w_s') \geqslant \chi_p^W(w_t) = \chi_p^W(w_t')$ violating property (2) of a p^n-admissible partition. So by Lemma 3 we get a rational direct summand contradicting the hypothesis. This shows that all W-partitions are equal and the proposition is proved. ■

3. - Decompositions.

Now we establish a necessary and sufficient condition for a completely decomposable subgroup to be regulating.

PROPOSITION 6. *Let A be an almost completely decomposable* *group with critical typeset T, completely decomposable subgroup W and primary cyclic quotient A/W. W is a regulating subgroup if and only if some (or each) W-partition of T has property* (1) *of an* $|A/W|$-*admissible partition.*

More precisely, $A(t) = W(t) + A^{\natural}(t)$ *if and only if* $T(k+1) \not\subset T_{\geq t}$, *where* $t \in T_k$, $k < n$, *is a critical type and* $T = T_0 \cup T_1 \cup \ldots \cup T_n$ *is some W-partition of T.*

PROOF. Let $W = \bigoplus_{s \in T} W_s$ be a homogeneous decomposition of W, let $T = T_0 \cup \ldots \cup T_n$ be the corresponding W-partition of T and let $A = \langle W, a \rangle$ with $A/W \cong \mathbb{Z}_{p^n}$. We may assume that $p^n a = \sum_{s \in T(n)} w_s$, where $w_s \in W_s$. We prove the specified statement.

Assume $t \in T_k$, $k < n$, and $T(k+1) \not\subset T_{\geq t}$. We show that $A(t) = W_t \oplus A^{\natural}(t)$. Let $t \in T(n)$ and $h \in A(t) = \left\langle \bigoplus_{s \geq t} W_s \right\rangle_*^A$. We have to prove that $h \in W_t \oplus A^{\natural}(t)$. By Dedekind's modular law it is enough to show $h \in W + A^{\natural}(t)$, since then $h \in (W + A^{\natural}(t)) \cap A(t) = W_t \oplus A^{\natural}(t)$. We may write $h = la + w$, where $w \in W$, l is a natural number, and so

$$(**) \qquad p^n h = l \sum_{s \in T} w_s + p^n w \in W(t) = \bigoplus_{s \geq t} W_s .$$

Since $T(k+1) \not\subset T_{\geq t}$ for some $0 \leq k < n$, then there is a natural number $i = \min\{j \mid T_j \not\subset T_{>t}\}$ and $i \leq k$. Note that $T(i) \subset T_{>t}$ and $T_i \not\subset T_{>t}$ by definition of i. If $i < k$ then $T_i \not\subset T_{\geq t}$ since $t \notin T_i$. If $i = k$ then $T(k) \subset T_{>t}$, $T_k \not\subset T_{>t}$ and $T_k \not\subset T_{\geq t}$ using $T(k+1) \not\subset T_{\geq t}$. By formula (**) we conclude in both cases that p^{n-i} divides l. Thus

$$h = p^{i-n} l \left[p^{-i} \left(\sum_{j=0}^{i-1} \sum_{s \in T_j} w_s \right) \right] + p^{-n} l \left(\sum_{j=i}^{n-1} \sum_{s \in T_j} w_s \right) + w .$$

Now since $p^{i-n} l \in \mathbb{N}$, $p^{-n} l \left(\sum_{j=i}^{n-1} \sum_{s \in T_j} w_s \right) + w \in W$ and since the first summand on the right hand side is in $A^{\natural}(t)$ we get $h \in W + A^{\natural}(t)$ as desired.

Conversely, if $t \in T_k$ is such that $k < n$ and $T(k+1) \subset T_{\geq t}$, then $W_t \oplus A^{\natural}(t) \neq A(t)$ by Lemma 4. Since $V = \bigoplus_{s \in T_n} W_s$ is a direct summand of A the Butler equation $A(t) = W_t \oplus A^{\natural}(t)$ holds automatically if $t \in T_n$. Hence W is regulating if and only if some (or each) W-partition of T has property (1) of an $|A/W|$-admissible partition. ∎

Now we establish a necessary and sufficient indecomposability criterion.

PROPOSITION 7. *Let A be an almost completely decomposable group with critical typeset T and completely decomposable subgroup W with primary cyclic quotient A/W. The group A is indecomposable if and only if all critical ranks are* 1, *some* (*or each*) *W-partition $T = T_0 \cup T_1 \cup \ldots \cup T_n$ of T is short, has the properties* (2) *and* (3) *of an $|A/W|$-admissible partition, and $T_0 \neq \emptyset$.*

PROOF. Let A be indecomposable and let $A/W \cong \mathbb{Z}_{p^n}$, $A = \langle W, a\rangle$ and $a = p^{-n}\left(\sum_{s \in T} w_s\right)$, where $W_s = \langle w_s \rangle_*^W$ for all $s \in T$. Lemma 2 shows that the critical ranks are 1. By Proposition 5 all partitions coincide and by Lemmata 3 and 4 this partition of T is short and has properties (2) and (3) of a p^n-admissible partition. Certainly $|T_0| \neq \emptyset$, since otherwise $|A/W| < p^n$.

Conversely, let $W = \bigoplus_{s \in T} W_s$ be a homogeneous decomposition of W with corresponding W-partition $T = T_0 \cup T_1 \cup \ldots \cup T_n$. Now assume that A is decomposable, i.e. $A = B_1 \oplus B_2$. Property (2) of a p^n-admissible partition implies that the types in T_0 are maximal in the critical typeset T. Since $A(s) = B_1(s) \oplus B_2(s)$ and since the critical ranks are all 1 we have for all types $s \in T_0$ that $W_s \subset \langle w_s \rangle_*^A = A(s)$ and either $W_s \subset B_1$ or $W_s \subset B_2$.

Now we show that either $T_0 \subset \mathrm{T}_{cr}(B_1)$ or $T_0 \subset \mathrm{T}_{cr}(B_2)$. Let $W(T_0) = \bigoplus_{s \in T_0} W_s$ then $W(T_0) = W_1 \oplus W_2$, where $W_1 = W(T_0) \cap B_1$ and $W_2 = W(T_0) \cap B_2$. Define $A(T_0) = W(T_0)_*^A$. Then $A(T_0)$ is fully invariant, since the types in T_0 are maximal, and we have $A(T_0) = (B_1 \cap A(T_0)) \oplus (B_2 \cap A(T_0))$, i.e. $A(T_0)/W(T_0) \cong (B_1 \cap A(T_0))/W_1 \oplus (B_2 \cap A(T_0))/W_2$. But $W(T_0) = W \cap A(T_0)$ since $W(T_0)$ is pure in W, hence $A(T_0)/W(T_0)$ is isomorphic to a subgroup of the cyclic group A/W. This implies that either $W_1 = B_1 \cap A(T_0)$ or $W_2 = B_2 \cap A(T_0)$. For a suitable element $z \in W$ we have $p^{n-1}a = p^{-1}\left(\sum_{s \in T_0} w_s\right) + z$. Let $w = \sum_{s \in T_0} w_s = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$. Then $A(T_0) = \langle W(T_0), p^{-1}w\rangle = \langle W_1, p^{-1}w_1\rangle \oplus \langle W_2, p^{-1}w_2\rangle$. Since the order of $p^{-1}w_i$ modulo W_i is p if $w_i \neq 0$ we obtain either $w_1 = 0$ or $w_2 = 0$. So either $W(T_0) \subset B_1$ or $W(T_0) \subset B_2$, i.e. either $T_0 \subset \mathrm{T}_{cr}(B_1)$ or $T_0 \subset \mathrm{T}_{cr}(B_2)$.

So we may assume $T_0 \subset \mathrm{T}_{cr}(B_1)$. This starts an induction on i. The hypothesis is $T(i) \subset \mathrm{T}_{cr}(B_1)$ and we want to show that $T_i \subset \mathrm{T}_{cr}(B_1)$, where $i \geqslant 1$. Let $A_i = \left\langle \bigoplus_{s \in T(i)} W_s \right\rangle_*^A$, i.e. $A_i = \left\langle \sum_{s \in T(i)} A(s) \right\rangle_*^A$ by property (2) of a p^n-admissible partition. Thus A_i is fully invariant and $A_i = (A_i \cap \cap B_1) \oplus (A_i \cap B_2)$ for all i. If $A_{i+1} \subset B_1$ then $T_i \subset \mathrm{T}_{cr}(B_1)$. So we may assume that $A_{i+1} \cap B_2 \neq 0$. We know that $A_i \subset A_{i+1} \cap B_1$. Moreover, by [4, 88.3], $A_{i+1}/A_i = \left\langle \bigoplus_{s \in T_i} (W_s + A_i),\ p^{-(i+1)} \left(\sum_{s \in T_i} (w_s + A_i) \right) \right\rangle$ is an indecomposable group since the types in T_i are pairwise incomparable due to property (2) of a p^n-admissible partition and since all these elements w_s are of p-height i in W. Because of $A_{i+1}/A_i = [(A_{i+1} \cap \cap B_1)/A_i] \oplus [((A_{i+1} \cap B_2) \oplus A_i)/A_i]$ we deduce $A_i = A_{i+1} \cap B_1$ and therefore $A_{i+1} = A_i \oplus (A_{i+1} \cap B_2)$. Thus $A_{i+1} \cap B_2$ contains a regulating subgroup $W_i' = \bigoplus_{s \in T_i} W_s'$ of index p. Property (1) of a p^n-admissible partition is not part of the hypothesis, i.e. W is not necessarily a regulating subgroup, cf. Proposition 6. But only for types s in T_0 it is possible that $A(s) \neq W_s \oplus A^{\natural}(s)$, since property (3) of a p^n-admissible partition implies property (1) for all $s \notin T_0$. Let $s \in T_i$, $i \geqslant 1$. We obtain $A_{i+1}(s) = = W_s \oplus A^{\natural}_{i+1}(s) = W_s' \oplus A^{\natural}_{i+1}(s)$ for all $s \in T_i$. For all $w_s' \in W_s'$ there is an element $x_s \in A^{\natural}_{i+1}(s)$ such that $w_s' = \mu w_s + x_s$, where $x_s \equiv \equiv \lambda_s p^{-k_s} \left(\sum_{j=0}^{k_s - 1} \sum_{s \in T_j} w_s \right)$ modulo W with integers λ_s and k_s, and with $\mu w_s \in \in W_s$. Since $x_s \in A^{\natural}_{i+1}(s)$ implies $T(k_s) \subset T_{>s}$, we obtain, since $s \in T_i$, and by property (3) of a p^n-admissible partition that $k_s \leqslant i-1$. But then $p(A_{i+1} \cap B_2) \subset W_i'$ and $p^{i-1} W_i' \subset W$, since $p^{i-1} x_s \in W$. Hence $p^i(A_{i+1} \cap \cap B_2) \subset W$, i. e. $p^i \left(p^{-(i+1)} \left(\sum_{j=0}^{i} \sum_{s \in T_j} w_s \right) \right) \in W$, which is a contradiction. ■

COROLLARY 8. *Let A be an almost completely decomposable group with critical typeset T, completely decomposable subgroup W and primary cyclic quotient A/W. Then the following are equivalent:*

(1) *A is indecomposable and W is a regulating subgroup.*

(2) *All critical ranks are* 1 *and some* (*or each*) *W-partition of T is short $|A/W|$-admissible.*

PROOF. The Propositions 6 and 7 show both directions. ■

THEOREM 9. *An indecomposable almost completely decomposable group with primary cyclic regulating quotient has only primary cyclic regulating quotients.*

PROOF. Let $W = \bigoplus_s W_s$ be a regulating subgroup of A with critical typeset T and quotient $A/W \cong \mathbb{Z}_{p^n}$. Let $T = T_0 \cup T_1 \cup \ldots \cup T_{n-1}$ be the W-partition of T. By Corollary 8 or Lemmata 2, 3 and 4 we know that all critical ranks are 1 and that this partition is short p^n-admissible. Let $W' = W_t' \oplus \bigoplus_{s \neq t} W_s$ be another regulating subgroup, where W_t' is any complement of $A^\natural(t)$ in $A(t)$ and $t \in T_k$. Without loss of generality $k \neq 0$ since $W_t = W_t'$ for maximal types, and those in T_0 are maximal. Let $A = \langle W, a \rangle$ where $a = p^{-n}\left(\sum_{s \in T} w_s\right)$ is of order p^n modulo W. It is enough to show that the order of a modulo W' is also p^n. Assume $p^{n-1}a \in W'$, i.e. $p^{n-1}a = w_t' + \sum_{s \neq t} \tilde{w}_s$ with $\tilde{w}_s \in W_s$ and $w_t' \in W_t'$. Thus $w_t' = p^{n-1}a - \sum_{s \neq t} \tilde{w}_s \in p^{-1}(W_t \oplus W^\natural(t))$. Hence let $w_t' = qw_t + p^{-1}\left(\sum_{s > t} \overline{w}_s\right)$ where $\overline{w}_s \in W_s$ and $q \in \mathbb{Q}$. Then $p^n a = \sum_{s \in T} w_s = pqw_t + \sum_{s \not> t} p\tilde{w}_s + \sum_{s > t} (p\tilde{w}_s + \overline{w}_s)$. In particular, $q = p^{-1}$, $w_s = p\tilde{w}_s$ for all $s \not> t$ and $p\tilde{w}_s + \overline{w}_s = w_s$ for all $s > t$. Since $A(t) = W_t \oplus A^\natural(t) = W_t' \oplus A^\natural(t)$ and $w_t' = p^{-1}w_t + p^{-1}\left(\sum_{s > t} \overline{w}_s\right)$ we have $\chi_p^{A(t)}(w_t') + 1 = \chi_p^{A(t)}(w_t) = \chi_p^W(w_t) = k$ and $\chi_p^{A(t)}\left(\sum_{s > t} \overline{w}_s\right) \geq k$. Moreover, by Lemma 1 we have $A^\natural(t)/W^\natural(t) \cong \mathbb{Z}_{p^l}$, where l is a natural number, maximal with respect to $T(l) \subset T_{>t}$. Note that $T_0 \subset T_{>t}$ by $w_s = p\tilde{w}_s$ for all $s \not> t$, thus $l \geq 1$. We now show that $l = k$. Therefore let us assume that $l < k$. By $A^\natural(t)/W^\natural(t) \cong \mathbb{Z}_{p^l}$ and $\chi_p^{A(t)}\left(\sum_{s > t} \overline{w}_s\right) \geq k$ we obtain $p^{l-k}\overline{w}_s \in W_s$ for all $s > t$, i.e. in particular for all $s \in T_0$. By $p\tilde{w}_s + \overline{w}_s = w_s$ for all $s > t$ including all $s \in T_0$ we get a contradiction. So $l = k$ and $T(k) \subset T_{>t}$. Since $t \in T_k$ we know that A must be decomposable by Lemma 4 which is again a contradiction.

COROLLARY 10. *An almost completely decomposable group A with primary cyclic regulating quotient has a direct decomposition*

$$A = V \oplus H_1 \oplus \ldots \oplus H_m,$$