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The Lattice of Very-Well-Placed Subgroups.

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1. - Introduction.

Every group will be finite and soluble. In this paper we study the well-placed subgroups of a soluble group. These subgroups are introduced by Hawkes in [6] and play an important role in the theory of finite soluble groups.

A natural question concerning the well-placed subgroups is the following: is the set of the well-placed subgroups of a group G a sublattice of the subgroup lattice of G? The answer is negative in general. We introduce a special type of well-placed subgroup called very-well-placed subgroup and study its properties. We prove that the set, denoted by $\boldsymbol{GE}_{\Sigma}(\boldsymbol{G})$ of the very-well-placed subgroups of a group G associated to a Hall system Σ of G is a sublattice of the subgroup lattice of G. Moreover, we describe completely all these sublattices. This allows us to obtain a new characterization of the $\underline{\underline{N}}^i$-normalizers of a group G, where i is a natural number smaller than or equal to the nilpotent length of G and $\underline{\underline{N}}^i$ the class of groups with nilpotent length at most i.

For basic definitions as well as notation we refer the reader to ([2], [3], [7]). We denote that U is maximal with $U \lessdot G$.

2. - Preliminaries.

We collect in this section some definitions and results we need

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in the sequel. First of all recall the definition of well-placed subgroup.

DEFINITION ([6], Def. 5.1). A subgroup U of G is called *well-placed* in G, if there exists a chain of subgroups $U = U_r < U_{r-1} < \dots < U_0 = G$, such that for $i = 1, \dots, r$:

a) U_i is maximal in U_{i-1};

b) U_i is critical in U_{i-1}, which means that $U_{i-1} = U_i \boldsymbol{F}_i(U_{i-1})$.

The $\underline{\underline{F}}$-normalizers of a soluble group associated to a saturated formation $\underline{\underline{F}}$ are an example of well-placed subgroups (see [2]).

The following proposition contains some remarkable facts about the well placed subgroups.

PROPOSITION 2.1. Let U be a well-placed subgroup of a group G.

a) U either covers or avoids the chief factors of G. Moreover if U covers the chief factor H/K of G, then $H \cap U/K \cap U$ is a chief factor of U and

$\mathrm{Aut}_G(H/K) \cong \mathrm{Aut}_U(H \cap U/K \cap U)$ (see [3], III, 6.6).

b) U belongs to the formation generated by G (see [1]).

c) If $\underline{\underline{H}}$ is a Schunck class and R is an $\underline{\underline{H}}$-projector of U, there exists an $\underline{\underline{H}}$-projector H of G such that $R \leq H$. Moreover, if $\underline{\underline{H}}$ is closed under well-placed subgroups (which is always true if $\underline{\underline{H}}$ is a saturated formation) then H may be chosen such that $R = H \cap U$ (see [3], III, 6.7).

The set of the well placed subgroups of a soluble group G is not a sublattice of the subgroup lattice of G, as the following example shows.

EXAMPLE 2.2. Let $H := \langle a, b \rangle$ be an elementary abelian group of order 9. There exists $c \in \mathrm{Aut}\, H$ such that $a^c = a^{-1}$ and $b^c = b^{-1}$. Let $M = [H]\langle c \rangle$ be the corresponding semidirect product.

Denote by $K = \langle ab \rangle$ a diagonal of H. Then M/K is isomorphic to the symmetric group of degree 3. Therefore M has an irreducible two-dimensional $GF(2)$-module N such that $\mathrm{Ker}(M \text{ on } N) = K$.

Set $G := [N]M$, $U := [N]\langle a, c \rangle$ and $V := [N]\langle b, c \rangle$. The subgroups U and V are critical, and therefore they are well-placed in G. However, $U \cap V = [N]\langle c \rangle$ is not well-placed in G.

However, by imposing extra conditions on the subgroups we consider, in particular by requiring that Hall systems reduce into them, we can produce sublattices.

DEFINITION. Let $U \leq G$ and α an embedding property of G. A Hall system Σ of G *reduces via α into U*, if there exists a chain of subgroups $U = U_r < U_{r-1} < \ldots < U_0 = G$, such that

a) U_i is maximal in U_{i-1} for $i = 1, \ldots, r$.

b) Σ reduces into U_i for $i = 0, \ldots, r$.

c) U_i is α-subgroup of U_{i-1} for $i = 1, \ldots, r$.

Even, the set $W_\Sigma(G) = \{U \leq G \mid \Sigma$ reduces via critical into $U\}$ does not form a sublattice. We come back to Example 2.2. Let $\Sigma :=$ $:= \{\{1\}, L, H, G\}$ where $L = [N]\langle c \rangle$ and H as defined above. Clearly Σ reduces into U and into V, but $U \cap V$ is not well-placed in G.

LEMMA 2.3. Let L and M be maximal subgroups of G. Then

a) L and M are conjugate if and only if $\mathrm{Core}_G(L) = \mathrm{Core}_G(M)$ ([3], A, 16.1).

b) If L and M are not conjugate and $\mathrm{Core}_G(L) \nleq \mathrm{Core}_G(M)$, then $L \cap M$ is a maximal subgroup of M ([3], A, 16.5).

DEFINITION. Let $\underline{\underline{F}}$ be a formation. A maximal subgroup U of G is called ***F**-critical* in G if:

a) U is $\underline{\underline{F}}$-abnormal in G (that is to say $G/\mathrm{Core}_G(U) \notin \underline{\underline{F}}$), and

b) U is critical in G.

LEMMA 2.4 ([3], IV, 1.17). Let $\underline{\underline{F}}$ be a formation and $G = UN$ where $U \leq G$ and N is a normal sungroup of G. Then

a) $U^{\underline{\underline{F}}}N = G^{\underline{\underline{F}}}N$, and

b) if N is a nilpotent group, then $U^{\underline{\underline{F}}} \leq G^{\underline{\underline{F}}}$.

The notion of $\underline{\underline{F}}$-normalizer of G plays an important role in this work. The following proposition gives a useful characterization of $\underline{\underline{F}}$-normalizers.

PROPOSITION 2.5 [2]. Let $\underline{\underline{F}}$ be a staurated formation, where $\underline{\underline{N}} \subseteq \underline{\underline{F}}$. A subgroup D of G is an $\underline{\underline{F}}$-normalizer of a group G if and only if

a) $D \in \underline{\underline{F}}$ and *b)* D can be joined to G by an $\underline{\underline{F}}$-critical maximal chain, namely a chain of the form

$$(1) \qquad D = G_r \lessdot G_{r-1} \lessdot \dots \lessdot G_1 \lessdot G_0 = G,$$

where G_i is an $\underline{\underline{F}}$-critical subgroup of G_{i-1} $(i = 1, \dots, r)$.

We recall from [2] that each Hall system Σ of G gives rise to a unique $\underline{\underline{F}}$-normalizer $D_{\underline{\underline{F}}}(\Sigma)$ and from [8] that $D_{\underline{\underline{F}}}(\Sigma)$ can be characterized as the $\underline{\underline{F}}$-normalizer of G defined by the chain (1) with the additional condition that Σ reduces into each G_i for $i = 1, \dots, r$.

LEMMA 2.6. Let $\underline{\underline{F}}$ be a saturated formation such that $\underline{\underline{N}} \subseteq \underline{\underline{F}}$ and Σ a Hall system of G.

a) If M is a $\underline{\underline{F}}$-critical subgroup of G into which Σ reduces, then

$$D_{\underline{\underline{F}}}(\Sigma) = D_{\underline{\underline{F}}}(\Sigma \cap M) \quad ([3],\ \text{V},\ 3.7).$$

b) If W is a well-placed subgroup of G such that Σ reduces via critical into W, then

$$D_{\underline{\underline{F}}}(\Sigma \cap W) \leq D_{\underline{\underline{F}}}(\Sigma) \quad ([3],\ \text{V},\ 2.7).$$

3. - The lattice $GE_\Sigma(G)$.

In this section, we introduce the concept *very-well-placed* and prove that the set, denoted by $GE_\Sigma(G)$, of the very-well-placed subgroups of a group G associated to a Hall system Σ of G forms a sublattice of the subgroup lattice of G.

DEFINITIONS. Let G be a group with nilpotent length n and denote by $L_{-1}(G)$ the $\underline{\underline{N}}^{n-1}$-residual of G (i.e. the smallest normal subgroup N of G such that $G/N \in \underline{\underline{N}}^{n-1}$). A subgroup U of G is said to be *strongly critical* if $UL_{-1}(G) = G$.

A subgroup U of G is said to be *very-well-placed* in G, if there exists a chain $U = U_r \lessdot U_{r-1} \lessdot \dots \lessdot U_0 = G$, such that for $i = 1, \dots, r$:

a) U_i is maximal in U_{i-1};

b) U_i is strongly critical in U_{i-1}.

The next counterexample shows that the set of all very-well-placed subgroups of a group G is not closed under intersections.

EXAMPLE 3.1. Let $V := S_3$ the symmetric group of degree 3 and $K := GF(3)$.

Let A_3 be the normal Sylow 3-subgroup of S_3. Let P_1 be the principal indecomposable projective KV-module such that $P_1/P_1 J(KV) \cong \cong K \cong \mathrm{Soc}(P_1)$.

Set $G := [P_1]V$ the semidirect product of V with P_1. Since $F(G) = = A_3 \times P_1$, it follows that the nilpotent length of G is 2.

Set $U := P_1 H$, where H is a Sylow 2-subgroup of V. Clearly $UG^{\underline{N}} = G$.

Hence U is a strongly critical maximal subgroup of G. Since $G/P_1 \cong \cong S_3$, there exists $g \in G$ such that $U \cap U^g = P_1$. Clearly U^g is a strongly critical maximal subgroup of G, but P_1 is not very-well-placed in G.

Therefore, we restrict our discussion to the set

$$GE_\Sigma(G) = \{U \leqslant G \mid \Sigma \text{ reduces via strongly critical into } U\}.$$

REMARKS 3.2. *a*) The embedding property very-well-placed is transitive.

b) If G is a nilpotent group, then all subroups of G are very-well-placed.

c) If U is a strongly critical maximal subgroup of G and $R := := \mathrm{Core}_G(U)$, then the nilpotent length of G and G/R are equal. Hence U is a $N^{n(G)-1}$-critical subgroup of G.

d) If Σ is a Hall system of G and U, V are subgroups of G such that $U \leqslant V \leqslant G$ and Σ reduces into V, then Σ reduces into U if and only if the Hall system $\Sigma \cap V$ of V reduces into U.

e) Let $U \leqslant G$ and Σ a Hall system of G. Then

$$U \in GE_\Sigma(G) \text{ implies } GE_{\Sigma \cap U}(U) \subseteq GE_\Sigma(G).$$

LEMMA 3.3. *Let $G = UN$ with N a nilpotent normal subgroup of G, and Σ a Hall system of G which reduces into U. If $V \leqslant G$ is such that $U \leqslant V \leqslant G$, then Σ reduces into V.*

PROOF. Since a Hall system Σ reduces into a product of permutable subgroups, into which Σ reduces, (see [3], I, 4.22 *b*)), then Σ reduces into V because $U(V \cap N) = V$, U and $V \cap N$ permute, and Σ reduces into U and into the subnormal subgroup $V \cap N$ of G.

LEMMA 3.4. If U and V are strongly critical maximal subgroups of the group G, such that $U \neq V$, and Σ is a Hall system reducing into U and V, then $U \cap V$ is strongly critical maximal in U and V.

PROOF. If G is a nilpotent group, then the result is trivial.

Suppose $n(G) > 1$.

We prove first that $U \cap V \lessdot V$ as well as $U \cap V \lessdot U$.

Since Σ reduces into the maximal and therefore pronormal subgroups U and V, it follows from ([3], I, 6.6) that U and V are not conjugate subgroups of G. Therefore by Lemma 2.3 *a*), $R := \mathrm{Core}_G(U) \neq \neq \mathrm{Core}_G(V) =: R^*$.

Assume $R \nleq R^*$ without loss of generality. Hence from Lemma 2.3 *b*), we have $U \cap V \lessdot V$.

We show now that $U \cap V \lessdot U$.

Since $L_{-1}(G)$ is a nilpotent group and $V \lessdot G$, it follows that $L_{-1}(G) \cap V \trianglelefteq G$. Hence $L_{-1}(G) \cap V \leqslant R^*$, and therefore $V/R^* \in \in \underline{\underline{N}}^{n(G)-1}$ because $V/(V \cap L_{-1}(G)) \cong G/L_{-1}(G) \in \underline{\underline{N}}^{n(G)-1}$.

Now assume that $R^* \leqslant R$. Hence $V/V \cap R \in \underline{\underline{N}}^{n(G)-1}$ and since $G/R \cong VR/R \cong V/V \cap R$ we have $G/R \notin \underline{\underline{N}}^{n(G)-1}$, a contradiction to Remark 3.2 *c*). Therefore $R^* \nleq R$ and again from Lemma 2.3 *b*) it follows $U \cap V \lessdot U$. Now we prove that $U \cap V$ is a strongly critical subgroup of U. The affirmation $U \cap V$ is strongly critical in V follows with the same arguments.

We prove tht $n(U) = n(G)$.

Assume for a contradiction that $n(U) < n(G)$. By Proposition 2.5, U is a $\underline{\underline{N}}^{n(G)-1}$-normalizer of G, because $U \in \underline{\underline{N}}^{n(G)-1}$ and U is a $\underline{\underline{N}}^{n(G)-1}$-critical subgroup of G (see Remark 3.2 *c*)). Since V is a $\underline{\underline{N}}^{n(G)-1}$-critical subgroup of G, V is a $\underline{\underline{N}}^{n(G)-1}$-normalizer of G too. This implies that U and V must be conjugate, a contradiction.

Now we have $UR^* = G$ and $n(U) = n(G)$. By Lemma 2.4, $U^{\underline{\underline{N}}^{n(G)-1}} R^* = G^{\underline{\underline{N}}^{n(G)-1}} R^*$. Finally, the desired conclusion follows from

$$U = G \cap U = VL_{-1}(G)R^* \cap U = VL_{-1}(U)R^* \cap U =$$

$$VL_{-1}(U) \cap U = (V \cap U)L_{-1}(U).$$

With the next theorem we show that $GE_\Sigma(G)$ forms a lattice.

THEOREM 3.5. Let Σ be a Hall system of the group G, and U, V subgroups belonging to $\boldsymbol{GE}_\Sigma(G)$. Then $U \cap V$ and $\langle U, V \rangle$ belong to $\boldsymbol{GE}_\Sigma(G)$.

PROOF. Since $U, V \in \boldsymbol{GE}_\Sigma(G)$, there exist chains

$$U = U_r \lessdot U_{r-1} \lessdot \ldots \lessdot U_0 = G$$

and

$$V = V_m' \lessdot V_{m-1} \lessdot \ldots \lessdot V_0 = G,$$

where Σ reduces into U_i ($i = 0, \ldots, r$) and V_j ($j = 0, \ldots, m$).
We consider two cases:

If $U \leq V_i$, then it follows trivially that $U \in \boldsymbol{GE}_{\Sigma \cap V_1}(V_1)$. Moreover, clearly $V \in \boldsymbol{GE}_{\Sigma \cap V_1}(V_1)$. We have then by induction on $|G|$ that $U \cap V$ and $\langle U, V\rangle \in \boldsymbol{GE}_{\Sigma \cap V_1}(V_1)$, and therefore $U \cap V$ and $\langle U, V\rangle$ belong to $\boldsymbol{GE}_\Sigma(G)$.

If $U \nleq V_1$, then it follows using Lemma 3.4 and induction on $|G|$ that $U \cap V_1 \in \boldsymbol{GE}_\Sigma(G)$ and therefore $U \cap V_1 \in \boldsymbol{GE}_{\Sigma \cap V_1}(V_1)$. Again, since $V \in \boldsymbol{GE}_{\Sigma \cap V_1}(V_1)$ it follows by induction on the order of G that $U \cap V \in \boldsymbol{GE}_{\Sigma \cap V_1}(V_1)$, and thus $U \cap V \in \boldsymbol{GE}_\Sigma(G)$.
We prove now that $\langle U, V\rangle \in \boldsymbol{GE}_\Sigma(G)$.
Assume $\langle U. V\rangle \neq G$ without loss of generality.
We show first that $n(U) = n(G)$. Assume for a contradiction that $n(U) < n(G)$. We choose $k \in \{0, \ldots, r\}$ so that $n(U_k) < n(G)$ and $n(U_t) = n(G)$ for all $t = 0, \ldots, k-1$. By Proposition 2.5, U_k is a $\underline{\underline{N}}^{n(G)-1}$-normalizer of G and therefore of U_{k-1}. Since $U_{k-1} \cap V_1$ is $\underline{\underline{N}}^{n(G)-1}$-critical in U_{k-1} it follows that $U_{k-1} \cap V_1$ must be a $\underline{\underline{N}}^{n(G)-1}$-normalizer of U_{k-1}. Hence U_k and $U_{k-1} \cap V_1$ are conjugate in U_{k-1}. This implies that $U_k = U_{k-1} \cap V_1$ because $\Sigma \cap U_{k-1}$ reduces into U_k and $U_{k-1} \cap V_1$.
Therefore $U \leq U_k \leq V_1$, a contradiction to our assumption. The fact $n(U) = n(G)$ implies trivially $n(U_i) = n(G)$ for $i = 0, \ldots, r-1$. Hence by Lemma 2.4 *b*),

$$L_{-1}(U) \leq L_{-1}(U_{r-1}) \leq \ldots \leq L_{-1}(G).$$

Therefore

$$G = U_1 L_{-1}(G) = U_2 L_{-1}(U_1) L_{-1}(G) = U_2 L_{-1}(G) = \ldots = U L_{-1}(G),$$

and then $\langle U, V\rangle L_{-1}(G) = G$.
If $\langle U, V\rangle \lessdot G$ then the result follows.
If $\langle U, V\rangle$ is not maximal in G, then choose $L \leq G$ such that $\langle U, V\rangle < L \lessdot G$. Clearly, L is a strongly critical maximal subgroup of G. Otherwise, Σ reduces into L by Lemma 3.3. Therefore, $U, V \in \boldsymbol{GE}_{\Sigma \cap L}(L)$. By

induction on the order of G, $\langle U, V\rangle \in \boldsymbol{GE}_{\Sigma\cap L}(L)$ and thus $\langle U, V\rangle \in$ $\in \boldsymbol{GE}_{\Sigma}(G)$.

4. – Description of the lattice $GE_{\Sigma}(G)$.

In this section we describe the sublattice $GE_{\Sigma}(G)$ by determining the saturated formations for which the $\underline{\underline{F}}$-normalizers belong to $GE_{\Sigma}(G)$.

DEFINITION. Let $\underline{\underline{F}}$ be a saturated formation. The maximal subgroup U of G is called *strongly* $\underline{\underline{F}}$*-critical* if:

a) U is strongly critical in G, and

b) U is $\underline{\underline{F}}$-abnormal in G.

THEOREM 4.1. Let $\underline{\underline{F}}$ be a saturated formation such that $\underline{\underline{N}} \subseteq \underline{\underline{F}}$. Then the following conditions are equivalent.

a) Every $G \notin \underline{\underline{F}}$ contains a strongly $\underline{\underline{F}}$-critical subgroup.

b) $\underline{\underline{F}} = \underline{\underline{S}}$ or there exists $n' \in \mathbb{N}$ such that $\underline{\underline{N}}^{n'-1} \subseteq \underline{\underline{F}} \subseteq \underline{\underline{N}}^{n}$.

PROOF. *a)* $\Rightarrow$ *b)* We show first that for every n either $(\underline{\underline{N}}^{n} \cap F) \subseteq$ $\subseteq \underline{\underline{N}}^{n-1}$ or $\underline{\underline{N}}^{n-1} \subseteq \underline{\underline{F}}$.

Assume for a contradiction that there is a natural number m such that $(\underline{\underline{N}}^{m} \cap \underline{\underline{F}}) \not\subseteq \underline{\underline{N}}^{m-1}$ as well as $\underline{\underline{N}}^{m} \not\subseteq \underline{\underline{F}}$. Let $G \in (\underline{\underline{N}}^{m} \cap \underline{\underline{F}})\backslash \underline{\underline{N}}^{m-1}$ and $H \in \underline{\underline{N}}^{m-1} \backslash \underline{\underline{F}}$ be minimal counter-examples. Clearly G and H are primitive groups.

Set $X = G \times H$. Since for every saturated formation $\underline{\underline{H}}$ we have $(G \times H)^{\underline{\underline{H}}} = G^{\underline{\underline{H}}} \times H^{\underline{\underline{H}}}$, then $L_{-1}(X) = L_{-1}(G)$. Let U be a stabilizer of H. Since $GU \in \underline{\underline{F}}$ and GU is a $\underline{\underline{F}}$-critical subgroup of X, then GU is a $\underline{\underline{F}}$-normalizer of X by Proposition 2.5. Hence all $\underline{\underline{F}}$-normalizers of X contain $L_1(X)$ because they are conjugate to GU.

By hypothesis, X contains a strongly $\underline{\underline{F}}$-critical subgroup V, since $X \notin \underline{\underline{F}}$. Using the characterization of $\underline{\underline{F}}$-normalizers, we deduce that V contains a $\underline{\underline{F}}$-normalizer of X. Furthermore, V contains $L_{-1}(X)$ too, a contradiction to the choice of V.

Then let n' be maximal such that $\underline{\underline{N}}^{n'-1} \subseteq \underline{\underline{F}}$ (if $\underline{\underline{N}}^{i} \subseteq \underline{\underline{F}}$ for all i, then $\underline{\underline{F}} = \underline{\underline{S}}$). Hence $\underline{\underline{N}}^{n'} \not\subseteq \underline{\underline{F}}$ and it follows that $(\underline{\underline{N}}^{n'+1} \cap \underline{\underline{F}}) \subseteq \underline{\underline{N}}^{n'}$. This implies $\underline{\underline{F}} \subseteq \underline{\underline{N}}^{n'}$. Assume for a contradiction that $\underline{\underline{F}} \not\subseteq \underline{\underline{N}}^{n'}$. Then we can choose $G \in \underline{\underline{F}} \setminus \underline{\underline{N}}^{n'}$ of minimal order and thus we have $G \in (\underline{\underline{N}}^{n'+1} \cap \cap \underline{\underline{F}}) \subseteq \underline{\underline{N}}^{n'}$.

$b) \Rightarrow c)$ If $\underline{\underline{F}} = \underline{\underline{S}}$, then the result is trivial.

Assume $\underline{\underline{F}} \neq \underline{\underline{S}}$. Let then m be the natural number such that $\underline{\underline{N}}^{m-1} \subseteq \subseteq \underline{\underline{F}} \subset \underline{\underline{N}}^{m}$. This implies that for any $n \in \mathbb{N}$ either $\underline{\underline{N}}^{n-1} \subseteq \underline{\underline{F}}$ $(n \leqslant m)$ or $(\underline{\underline{F}} \cap \underline{\underline{N}}^{n}) \subseteq \underline{\underline{N}}^{n-1}$ $(n > m)$.

Let $G \notin \underline{\underline{F}}$.

If $\Phi(G) \neq 1$, then $G/\Phi(G)$ contains by induction on $|G|$ a strongly $\underline{\underline{F}}$-critical subgroup $M/\Phi(G)$. Hence M is a strongly $\underline{\underline{F}}$-critical of G, because $L_{-1}(G/\Phi(G)) = L_{-1}(G)\Phi(G)/\Phi(G)$.

Assume then $\Phi(G) = 1$ and set $n' = n(G)$. Hence, by hypothesis, either $(\underline{\underline{F}} \cap \underline{\underline{N}}^{n'}) \subseteq \underline{\underline{N}}^{n'-1}$ or $\underline{\underline{N}}^{n'-1} \subseteq \underline{\underline{F}}$.

If $(\underline{\underline{F}} \cap \underline{\underline{N}}^{n'}) \subseteq \underline{\underline{N}}^{n'-1}$, then a maximal complement M to $L_{-1}(G)$ is $\underline{\underline{F}}$-abnormal in G and therefore strongly $\underline{\underline{F}}$-critical in G. M would be a $\underline{\underline{F}}$-normal subgroup of G, then $G/\mathrm{Core}_G(M) \in \underline{\underline{F}} \cap \underline{\underline{N}}^{n'} \subseteq \underline{\underline{N}}^{n'-1}$ and thus $L_{-1}(G) \leqslant M$, a contradiciton to the choice of M.

Assume then that $\underline{\underline{N}}^{n'-1} \subseteq \underline{\underline{F}}$.

Since $\Phi(G) = 1$, the Fitting subgroup of G can be decomposed as follows: $F(G) = \mathrm{Soc}(G) = N_1 \times \ldots \times N_t$, where N_i is a minimal normal subgroup of G for all $i = 1, \ldots, t$.

Set $N_i^* = N_1 \ldots N_{i-1} N_{i+1} \ldots N_t$ for all $i = 1, \ldots t$; and let M_i be a complement to $F(G)/N_i^*$.

Then $F(G) \cap (\cap \mathrm{Core}_G(M_i)) \leqslant \cap N_i = 1$. Hence $\cap \mathrm{Core}_G(M_i) = 1$.

Now suppose that M_i is $\underline{\underline{F}}$-normal in G for all $i = 1, \ldots, t$. Therefore, $G/\mathrm{Core}_G(M_i) \in \underline{\underline{F}}$ and $G \in \underline{\underline{F}}$ because $\underline{\underline{F}}$ is a formation. This is a contradiction to the choice of G.

Let then M_j be a $\underline{\underline{F}}$-abnormal subgroup of G. Hence M_j is $\underline{\underline{F}}$-abnormal and therefore strongly $\underline{\underline{F}}$-critical in G.

Using the same argument as Carter and Hawkes in [2], a characterization of $\underline{\underline{F}}$-normalizers may be given.

LEMMA 4.2. Let $\underline{\underline{F}}$ be a saturated formation such that $\underline{\underline{N}}^{n-1} \subseteq \underline{\underline{F}} \subseteq \subseteq \underline{\underline{N}}^{n}$ for some $n \in \mathbb{N}$, $n > 1$. The subgroup D is a $\underline{\underline{F}}$-normalizer of G if and only if

$a)$ $D \in \underline{\underline{F}}$ and

$b)$ there exists a chain $D = G < G_{s-1} < \ldots < G_0 = G$, where G_{i+1} is a strongly $\underline{\underline{F}}$-critical subgroup of G_i $(i = 1, \ldots, s-1)$.

Moreover, we have $D = D_{\underline{\underline{F}}}(\Sigma)$ for a Hall system Σ of G if and only if $D \in \underline{\underline{F}}$ and Σ reduces via strongly $\underline{\underline{F}}$-critical into D. This may be proved by using the same arguments as A. Mann in ([8], Theorem 6).

COROLLARY 4.3. The $\underline{\underline{N}}^i$-normalizers of a group G, where $i = 1, \ldots, n(G)$, are very-well-placed in G.

THEOREM 4.4. Let Σ be a Hall system of G and $n := n(G)$. Set $D^i(\Sigma) = D_{\underline{\underline{N}}^i}(\Sigma)$ for $i = 1, \ldots, n$, and

$$M_i = \{U \leq G \mid D^i(\Sigma) \leq U \leq D^{i+1}(\Sigma) \quad \text{for} \quad i \in \{1, \ldots, n-1\}\}.$$

Then

$$GE_\Sigma(G) = \left(\bigcup_{i=1}^{n-1} M_i\right) \cup \{U \leq G \mid U \leq D^1(\Sigma)\}.$$

PROOF. «$\subseteq$». Let $U \in GE_\Sigma(G)$ and $r = n(U)$.
If $r = 1$, $U \leq D^1(\Sigma)$ from Lemma 2.6 *b*).
Thus, we assume $r > 1$ and prove that $U \in M_{r-1}$. Again by Lemma 2.6 *b*) we have that $U \leq D^r(\Sigma)$.
We show now that $D^{r-1}(\Sigma) \leq U$.
Let U_i be the penultimate link of a chain of strongly critical maximal subgroups from U to G.
By Remark 3.2 *c*) the subgroup U_i is $\underline{\underline{N}}^{n(G)-1}$-critical in G and therefore U_1 is $\underline{\underline{N}}^{r-1}$-critical in G. Hence $D^{r-1}(\Sigma \cap U_1) = D^{r-1}(\Sigma)$ by Lemma 2.6 *a*).
Finally, by induction on $|U_1|$ it follows that $D^{r-1}(\Sigma) = D^{r-1}(\Sigma \cap \cap U_1) \leq U$.

«$\supseteq$». If $U \leq D^1(\Sigma)$, then U is very-well-placed in $D^1(\Sigma)$ (Remark 3.2 *b*)). By Lemma 4.2, $D^1(\Sigma) \in GE_\Sigma(G)$. Hence clearly $U \in GE_\Sigma(G)$.
Now we assume that $D^i(\Sigma) \leq U \leq D^{i+1}(\Sigma)$ for $i \in \{1, \ldots, n-1\}$. Since $D^{i+1}(\Sigma) \in GE_\Sigma(G)$ by Lemma 4.2, it is enough to show that $U \in GE_{\Sigma \cap D^{i+1}}(D^{i+1}(\Sigma))$.
Let $U = U_t \lessdot U_{t-1} \lessdot \ldots \lessdot U_0 = D^{i+1}(\Sigma)$ be a chain of subgroups, such that U_j is maximal in U_{j-1} for $j = 1, \ldots, t$.
By Proposition V, 3.13 from [3], $D^i(\Sigma)$ is an $\underline{\underline{N}}^i$-normalizer of $D^{i+1}(\Sigma)$ and therefore $D^i(\Sigma)$ is an $\underline{\underline{N}}^i$-projector of $D^{i+1}(\Sigma)$ (see [2], Theorem 5.6). Hence $D^i(\Sigma)$ is an $\underline{\underline{N}}^i$-projector of U_j $(j = 1, \ldots, t)$ by the persistence of projector in intermediate subgroups. Therefore,

$D^i(\Sigma)L_{-1}(U_j) = U_j$ and thus $U_{j+1}L_{-1}(U_j) = U_j$ for all $j = 0, \ldots, t-1$; which means that U_{j+1} is strongly critical in U_j.

Finally, since $\Sigma \cap D^{i+1}(\Sigma)$ reduces into $D^i(\Sigma)$, we conclude by Lemma 3.3 that $\Sigma \cap D^{i+1}(\Sigma)$ reduces into U_j and therefore Σ reduces into U_j for all $j = 0, \ldots, t-1$.

COROLLARY 4.6. Let n be the nilpotent length of G. The subgroup U is an $\underline{\underline{N}}^i$-normalizer of G, $i \leq n$, if and only if U is a very-well-placed $\underline{\underline{N}}^i$-maximal subgroup of G.

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