W. Cieślak
A. Miernowski
W. Mozgawa

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Isoptics of a Closed Strictly Convex Curve. - II.

W. Cieślak(*) - A. Miernowski(**) - W. Mozgawa(**)

1. - Introduction.

This article is concerned with some geometric properties of isoptics which complete and deepen the results obtained in our earlier paper [3]. We therefore begin by recalling the basic notions and necessary results concerning isoptics.

An $\alpha$-isoptic $C_{\alpha}$ of a plane, closed, convex curve $C$ consists of those points in the plane from which the curve is seen under the fixed angle $\pi - \alpha$.

We shall denote by $\mathcal{C}$ the set of all plane, closed, strictly convex curves. Choose an element $C \in \mathcal{C}$ and a coordinate system with the origin $O$ in the interior of $C$. Let $p(t), t \in [0, 2\pi]$, denote the support function of the curve $C$. It is well known [2] that the support function is differentiable and that $C$ can be parametrized by

$$z(t) = p(t)e^{it} + \hat{p}(t)ie^{it} \quad \text{for} \ t \in [0, 2\pi].$$

We recall that the equation of $C_{\alpha}$ has the form

$$z_{\alpha}(t) = p(t)e^{it} + \left( -p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t + \alpha) \right)ie^{it} =$$

$$= z(t) + \lambda(t, \alpha)ie^{it} = z(t + \alpha) + \mu(t, \alpha)ie^{i(t + \alpha)},$$


(**) Indirizzo degli AA.: U.M.C.S., Institute of Mathematics, pl. M. Curie-Skłodowskiej 1, 20-031 Lublin, Poland; e-mail: mierand@golem.umcs.lublin.pl; mozgawa@golem.umcs.lublin.pl.
where

\[ \lambda(\alpha, t) = \frac{1}{\sin \alpha} \left( p(t + \alpha) - p(t) \cos \alpha - \dot{p}(t) \sin \alpha \right), \]

\[ \mu(\alpha, t) = \frac{1}{\sin \alpha} \left( p(t + \alpha) \cos \alpha - \dot{p}(t + \alpha) \sin \alpha - p(t) \right), \]

and the tangent vector to \( C_\alpha \) is given by the formula

\[ \dot{z}_\alpha(t) = \left( -p(t) \cot \alpha + \frac{p(t + \alpha)}{\sin \alpha} - \dot{p}(t) \right) e^{i t} + \]

\[ + \left( \frac{p(t) - \dot{p}(t) \cot \alpha + \frac{\dot{p}(t + \alpha)}{\sin \alpha}}{\sin \alpha} \right) i e^{i t} \]

for \( t \in [0, 2\pi] \).

Moreover, the mapping \( F: ]0, \pi[ \times ]0, 2\pi[ \to \{ \text{the exterior of } C \} \setminus \{ \text{a certain support half-line} \} \) defined by \( F(\alpha, t) = z_\alpha(t) \) is a diffeomorphism and the jacobian determinant \( F'(\alpha, t) \) of \( F \) at \((\alpha, t)\) is equal to

\[ F'(\alpha, t) = \frac{-\lambda(\alpha, t) \mu(\alpha, t)}{\sin \alpha}. \]
2. – Crofton-type formulae for annuli.

In this section we take $C_\beta$ to be an arbitrary fixed isoptic, and we shall consider an annulus $CC_\beta$ formed by $C$ and $C_\beta$. Let $t_1(x, y)$ denote the distance between a point $(x, y) \in CC_\beta$ and a support point of $C$ determined by the first, with respect to the orientation of $C$, support line of $C$ passing by $(x, y)$, (see fig. 2).

**Theorem 2.1.** If $L$ is the length of $C \in \mathcal{C}$, then

$$\int \int_{C C_\beta} \frac{dx \, dy}{t_1(x, y)} = L \tan \frac{\beta}{2}. \tag{2.1}$$

**Proof.** Using the diffeomorphism $F$ we get

$$\int \int_{C C_\beta} \frac{dx \, dy}{t_1(x, y)} = \int_0^{2\pi} \int_0^\beta \frac{1}{\lambda(t, \alpha)} \frac{-\lambda(t, \alpha) \mu(t, \alpha)}{\sin \alpha} \, d\alpha \, dt =$$

$$= \int_0^\beta \frac{1}{\sin^2 \alpha} \left( p(t) - p(t + \alpha) \cos \alpha + \dot{p}(t + \alpha) \sin \alpha \right) \, dt \, d\alpha =$$

$$= \int_0^\beta \frac{1}{\sin \alpha} (L - L \cos \alpha) \, d\alpha = L \int_0^\beta \frac{1}{2 \cos^2 \frac{\alpha}{2}} \, d\alpha = L \tan \frac{\beta}{2}. \quad \blacksquare$$

An application of this formula will be given in the next paragraph.

![Fig. 2.](image-url)
3. – Area of the annulus.

We shall now consider the expression \( \{z_a, \dot{z}_a\} \), where \( \{a + bi, c + di\} = ad - bc \). From (1.2), (1.3) we get

\[
\{z_a(t), \dot{z}_a(t)\} = \frac{1}{\sin^2 \alpha} \left( p^2(t) + p^2(t + \alpha) - 2p(t)p(t + \alpha) \cos \alpha - \dot{\alpha} p(t) \sin \alpha + \dot{p}(t) \dot{p}(t + \alpha) \sin \alpha \right).
\]

Let \( A(\alpha) \) denote the area of the region bounded by \( C_a \). Using the Green formula

\[
A(\alpha) = \frac{1}{2} \int_0^{2\pi} \{z_a(t), \dot{z}_a(t)\} \, dt
\]

and next integrate by parts we get

\[
A(\alpha) \sin^2 \alpha = \int_0^{2\pi} \left( p^2(t) - p(t + \alpha)(\dot{p}(t) \sin \alpha + p(t) \cos \alpha) \right) \, dt.
\]

It follows that for an arbitrary strictly convex set \( C \) the function \( A \) is differentiable of class \( C^1 \).

**Theorem 3.1.** The function \( A \) satisfies the following differential equation

\[
A' \sin \alpha + 2A \cos \alpha = G(\alpha)
\]

and

\[
A'(0_+) = 0,
\]

where

\[
G(\tau) = \int_0^{2\pi} (p(t)p(t + \tau) - \dot{p}(t) \dot{p}(t + \tau)) \, dt \quad \text{for} \ \tau \in [0, 2\pi].
\]

**Proof.** Differentiating (3.2) we obtain

\[
(sin^2 \alpha A(\alpha))' = G(\alpha) \sin \alpha.
\]
Hence we get (3.3). The Crofton-type formula (2.1) implies

\[
(L \tan \frac{\beta}{2}) = \iint_{C_{\beta}} \frac{dx dy}{t_1(x, y)} \geq \frac{1}{\max_{0 \leq t \leq 2\pi} \lambda(t, \beta)} \iint_{C_{\beta}} dx dy = \frac{A(\beta) - A(0)}{\max_{0 \leq t \leq 2\pi} \lambda(t, \beta)}.
\]

Thus we have

\[
0 < \frac{A(\beta) - A(0)}{\beta} \leq L \frac{\tan \beta/2}{\beta} \max_{0 \leq t \leq 2\pi} \lambda(t, \beta).
\]

This inequality leads us to (3.4). \( \blacksquare \)

**Remark 3.1.** If a convex curve \( C \) contains a segment and \( A'(0) \) exists, then \( A'(0) > 0 \).

Indeed, assume that the length of this interval is \( m \). It is evident that the area bounded by the isoptic \( C_\alpha \) is greater than the area of \( C \) plus the area of the triangle (cf. fig. 3). Thus we have

\[
A(\alpha) - A(0) > \frac{m^2}{4} \tan \frac{\alpha}{2},
\]

that is

\[
A'(0) = \frac{m^2}{8} > 0.
\]

Fig. 3.
4. – Theorem on tangents to isoptic.

Let us fix an isoptic $C_\alpha$ of the curve $C$,

$$z_\alpha(t) = p(t)e^{it} + \left( -p(t)\cot\alpha + \frac{1}{\sin\alpha} p(t + \alpha) \right)ie^{it}.$$ 

We recall the following notations (cf. [3]):

$$\begin{align*}
    b(t, \alpha) &= p(t + \alpha)\sin\alpha + \dot{p}(t + \alpha)\cos\alpha - \dot{p}(t), \\
    B(t, \alpha) &= p(t) - p(t + \alpha)\cos\alpha + \dot{p}(t + \alpha)\sin\alpha, \\
    q(t, \alpha) &= z(t) - z(t + \alpha).
\end{align*}$$

We have

$$\begin{align*}
    q(t, \alpha) &= B(t, \alpha)e^{it} - b(t, \alpha)ie^{it}, \\
    \lambda(t, \alpha) &= b(t, \alpha) - B(t, \alpha)\cot\alpha, \\
    \mu(t, \alpha) &= -\frac{B(t, \alpha)}{\sin\alpha}
\end{align*}$$

and

$$\dot{z}_\alpha(t) = -\lambda(t, \alpha)e^{it} + q(t, \alpha)ie^{it},$$

where

$$\varphi(t, \alpha) = B(t, \alpha) + b(t, \alpha)\cot\alpha.$$ 

Let us fix $\tau \in (0, 2\pi)$. We denote by $h^\tau(t, \alpha)$ the function $h(t + \tau, \alpha)$. Let $\angle (v, w)$ denote the angle between $v$ and $w$.

**Theorem 4.1.** Let $C_\alpha$ be the $\alpha$-isoptic of $C \in \mathbb{C}$. The following relation holds

$$\angle (\dot{z}_\alpha, \dot{z}_\alpha^\tau) + \angle (q, q^\tau) = 2\tau.$$ 

**Proof.** We have

$$\dot{z}_\alpha^\tau = -(\lambda^\tau + q^\tau\sin\tau)e^{it} + (q^\tau\cos\tau - \lambda^\tau\sin\tau)ie^{it}$$

and

$$\langle \dot{z}_\alpha, \dot{z}_\alpha^\tau \rangle = \frac{bb^r + BB^r}{\sin^2\alpha} \cos\tau + \frac{BB^r - b^rB}{\sin^2\alpha} \sin\tau,$$

where $\langle , \rangle$ is the canonical euclidean scalar product.
On the other hand
\[ q^r = B^r (\cos \tau + i \sin \tau) e^{it} - b^r (i \cos \tau - \sin \tau) ie^{it} \]

and
\[ \langle q, q^r \rangle = (BB^r + bb^r) \cos \tau - (bB^r - Bb^r) \sin \tau. \]  

By the above consideration we get
\[ \{ q, q^r \} = (bB^r - Bb^r) \cos \tau + (bb^r + BB^r) \sin \tau. \]  

By the above formulae we have
\[ \sin^2 \alpha (\hat{z}_a, \hat{z}_a^r) = \{ q, q^r \} \sin 2\tau + \langle q, q^r \rangle \cos \tau. \]

Taking into account that \( |\hat{z}_a| \sin \alpha = |q| \) we get
\[ \cos \angle (\hat{z}_a, \hat{z}_a^r) = \cos (2\tau - \angle (q, q^r)). \]

This shows that either
\[ \angle (\hat{z}_a, \hat{z}_a^r) + \angle (q, q^r) = 2\tau \]  
or
\[ \angle (\hat{z}_a, \hat{z}_a^r) + 2\tau = \angle (q, q^r). \]  

Fig. 4.
or

\[ \angle (\dot{z}_a, \dot{z}_a^r) = 2\pi - 2\tau + \angle (q, q^r). \]

If \( \tau \to 0 \), then \( \angle (\dot{z}_a, \dot{z}_a^r) \to 0 \) and \( \angle (q, q^r) \to 0 \), on the other hand if \( \tau \to 2\pi \), then \( \angle (\dot{z}_a, \dot{z}_a^r) \to 2\pi \) and \( \angle (q, q^r) \to 2\pi \). This implies relation (4.8).

If \( \tau = \pi \), then we get

**COROLLARY 4.1.**

\[ \angle (\dot{z}_a(t), \dot{z}_a(t+\pi)) + \angle (q(t, \alpha), q(t+\pi, \alpha)) = 2\pi. \]

**COROLLARY 4.2.** Vector \( \dot{z}_a \) is parallel to \( \dot{z}_a^r \) if and only if \( q \) is parallel to \( q^r \).

5. - Isoptics of curves of constant width.

Let \( C: z(t) = p(t)e^{it} + \dot{p}(t)ie^{it} \) be a curve of constant width \( d \). Then its width is given by \( d = p(t) + p(t+\pi) \). If \( t \mapsto z_a(t) \) is the parametrization of its \( \alpha \)-isoptic then

\[ z_a(t) - z_a(t+\pi) = de^{it} + \frac{d}{\sin \alpha} (1 - \cos \alpha) ie^{it}. \]

It follows that

\[ |z_a(t) - z_a(t+\pi)| = \frac{d}{\cos(\alpha/2)}. \]

Thus we get

**THEOREM 5.1.** If \( C \in \mathcal{C} \) is of constant width \( d \) then the distance between the points \( z_a \) and \( z_a(t+\pi) \) of its \( \alpha \)-isoptic \( C_a \) is constant and equal to \( d/\cos(\alpha/2) \).

Now we prove the following

**THEOREM 5.2.** Let \( C \in \mathcal{C} \) and let \( \alpha \) be linearly independent of \( \pi \) over \( \mathbb{Q} \). If the distance between the points \( z_a(t) \) and \( z_a(t+\pi) \) on the \( \alpha \)-isoptic \( C_a \) is constant then \( C \) is a curve of constant width.
PROOF. First, we note that
\[ z_a(t) - z_a(t + \pi) = d(t) e^{it} + \left\{ -d(t) \cot \alpha + \frac{d(t + \alpha)}{\sin \alpha} \right\} ie^{it}, \]
where \( d(t) = p(t) + p(t + \pi) \). Let
\[ D = |z_a(t) - z_a(t + \pi)|. \]
Then there exists a function \( t \mapsto \xi(t) \), \( 0 < \xi(t) < \pi \) such that
\[ d(t) = D \sin \xi(t), \]
\[ -d(t) \cot \alpha + \frac{d(t + \alpha)}{\sin \alpha} = D \cos \xi(t). \]
From these formulae it follows that
\[ d(t + \alpha) = D \sin (\alpha + \xi(t)). \]
On the other hand we have
\[ d(t + \alpha) = D \sin \xi(t + \alpha). \]
Thus we can write
\[ \xi(t + \alpha) = \xi(t) + \alpha + 2\pi j \]
or
\[ \xi(t + \alpha) = \pi - (\xi(t) + \alpha) + 2\pi k \]
for some \( k, j \in \mathbb{Z} \). Since \( 0 < \xi(t) < \pi \), then
(5.3) \[ \xi(t + \alpha) = \xi(t) + \alpha \]
or
(5.4) \[ \xi(t + \alpha) + \xi(t) + \alpha = \pi. \]
The function \( d(t) = p(t) + p(t + \pi) \) is periodic of period \( 2\pi \). Thus
(5.5) \[ \xi(t + 2\pi) = \xi(t) + 2\pi m, \]
but since \( 0 < \xi(t) < \pi \) then
(5.6) \[ \xi(t + 2\pi) = \xi(t). \]
The conditions (5.3) and (5.6) are contradictory because
\[ \xi(t) = \xi(t + 2\pi) = \xi \left( t + 4 \cdot \frac{\pi}{2} \right) = \xi(t) + 4 \cdot \frac{\pi}{2}. \]

This means that (5.4) and (5.6) must hold. By (5.4) we have
\[ (5.7) \quad \xi(t + 2\alpha) + \xi(t + \alpha) = \pi. \]

Thus subtracting (5.4) from (5.7) we get
\[ \xi(t + 2\alpha) = \xi(t). \]

This means that the function \( \xi \) has two periods \( 2\pi \) and \( 2\alpha \). Since \( \alpha \) is linearly independent of \( \pi \) over \( \mathbb{Q} \), then \( \xi \) has to be constant.

In the above theorem \( \alpha \) has to be necessarily linearly independent of \( \pi \) over \( \mathbb{Q} \). This condition can not be removed as shows the example of an ellipse and its \((\pi/2)\)-isoptic which is a circle.

6. **Differential equations related to isoptics.**

In this paragraph we shall consider a curve \( C \in \mathcal{C} \) satisfying the following condition:

\[
\begin{cases}
    p \in C^2, \\
    R(t) = p(t) + \dot{p}(t) > 0,
\end{cases}
\]

where \( R \) is the radius of curvature. The curve \( C \) will be then called an oval.

Let us fix an oval \( C \) and consider a family of its isoptics \( \{ C_\alpha : 0 < \alpha < \pi \} \), where \( C_\alpha \) is an isoptic given by \( z_\alpha(t) = z(t, \alpha) = z(t) + \lambda(t, \alpha) e^{it} \). We shall now find a differential equation which is satisfied by the function \( \lambda \). Let us note that
\[
\begin{cases}
    \frac{\partial b}{\partial \alpha} = R(t + \alpha) \cos \alpha, \\
    \frac{\partial B}{\partial \alpha} = R(t + \alpha) \sin \alpha,
\end{cases}
\]
THEOREM 6.1. Let $C$ be an oval and let $p$ denote its support function. Let $\mathbf{A}(t, \alpha)$ be an $\alpha$-isoptic of the oval $C$. Then the function $A(t, \alpha) > 0$ satisfies the partial differential equation

\begin{align*}
\frac{\partial b}{\partial t} &= B(t, \alpha) + R(t + \alpha) \cos \alpha - R(t), \\
\frac{\partial B}{\partial t} &= -b(t, \alpha) + R(t + \alpha) \sin \alpha.
\end{align*}

Moreover

\begin{align*}
A(t, 0) &= 0 \quad \text{and} \quad A(t, -) \quad \text{is an increasing function.}
\end{align*}

PROOF. We have $A = b - B \cot \alpha$. Using (6.2) and (6.3) we get

\begin{align*}
\frac{\partial \lambda}{\partial \alpha} - \frac{\partial \lambda}{\partial t} + \lambda(t, \alpha) \cot \alpha &= R(t).
\end{align*}

Then formula (6.4) is easy to check. The condition (6.5) is obvious.

We shall find a partial differential equation for the function $v = |q| = \sqrt{b^2 + B^2}$. It follows from (6.2) and (6.3) that

\begin{align*}
\frac{\partial b}{\partial t} &= B(t, \alpha) + \frac{\partial b}{\partial \alpha} - R(t), \\
\frac{\partial B}{\partial t} &= -b(t, \alpha) + \frac{\partial B}{\partial \alpha}.
\end{align*}

Differentiating the first equation with respect to $\alpha$ and then using the second one we get

\begin{align*}
\frac{\partial^2 b}{\partial \alpha^2} - \frac{\partial^2 b}{\partial \alpha \partial t} &= R(t + \alpha) \sin \alpha, \\
\frac{\partial^2 B}{\partial \alpha^2} - \frac{\partial^2 B}{\partial \alpha \partial t} &= -R(t + \alpha) \cos \alpha.
\end{align*}
Moreover, we have
\[
\frac{\partial^2 B}{\partial t^2} - \frac{\partial^2 B}{\partial a \partial t} + B(t, \alpha) = R(t) - R(t + \alpha) \cos \alpha.
\]

We first find a differential equation for the function \( u = (1/2)(b^2 + B^2) \).

In view of the above calculation we get
\[
\begin{align*}
\frac{\partial^2 u}{\partial \alpha^2} &= R^2(t + \alpha) + b(t, \alpha) \frac{\partial^2 b}{\partial \alpha^2} + B(t, \alpha) \frac{\partial^2 B}{\partial \alpha^2}, \\
\frac{\partial^2 u}{\partial \alpha \partial t} &= \frac{\partial b}{\partial t} \frac{\partial b}{\partial \alpha} + b(t, \alpha) \frac{\partial^2 b}{\partial \alpha^2} + \frac{\partial B}{\partial t} \frac{\partial B}{\partial \alpha} + B(t, \alpha) \frac{\partial^2 B}{\partial \alpha^2}.
\end{align*}
\]

These equations imply

**Proposition 6.1.** The function \( u = (1/2)(b^2 + B^2) \) satisfies the following differential equation

\[
(6.8) \quad \frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial t \partial \alpha} = R(t) R(t + \alpha) \cos \alpha.
\]

In a similar way we find an equation for the function \( v = \sqrt{2} u \). The function \( v \) satisfies the following partial differential equation

\[
(6.9) \quad v \left( \frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial^2 v}{\partial t \partial \alpha} \right) + \left( \frac{\partial v}{\partial \alpha} \right)^2 - \frac{\partial v}{\partial t} \frac{\partial v}{\partial \alpha} = R(t) R(t + \alpha) \cos \alpha.
\]

Now we consider the function \( F(\alpha) = \int_0^{2\pi} R(t) p(t + \alpha) \, dt \). Then

\[
F''(\alpha) = \int_0^{2\pi} R(t) \hat{p}(t + \alpha) \, dt,
\]

\[
F(\alpha) + F''(\alpha) = \int_0^{2\pi} R(t) R(t + \alpha) \, dt.
\]

By (6.8) we have

\[
(6.10) \quad (F(\alpha) + F''(\alpha)) \cos \alpha = \int_0^{2\pi} \left( \frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial t \partial \alpha} \right)(t, \alpha) \, dt =
\]

\[
= \frac{d^2}{da^2} \int_0^{2\pi} u(t, \alpha) \, dt = \frac{1}{2} \frac{d^2}{da^2} \int_0^{2\pi} |q(t, \alpha)|^2 \, dt.
\]
If we put

\[ Q(\alpha) = \frac{1}{2} \int_0^{2\pi} |q(t, \alpha)|^2 \, dt, \]

then \( F \) satisfies the following differential equation

\[ (F + F'') \cos \alpha = Q''. \]

This formula implies that if \( C \in \mathcal{C} \) and \( F \in C^2 \), then

\[ Q'' \left( \frac{\pi}{2} \right) = 0. \]

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