Asymmetric bound states of differential equations in nonlinear optics

Rendiconti del Seminario Matematico della Università di Padova, tome 100 (1998), p. 231-247

<http://www.numdam.org/item?id=RSMUP_1998__100__231_0>
Asymmetric Bound States of Differential Equations in Nonlinear Optics (*).

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1. - Introduction.

Bound states of a nonlinear Schrödinger equation modelling propagation in a medium with dielectric function \( n^2 \) can be found as solutions of a differential equation of the type

\[
-u''(x) + \beta^2 u(x) = n^2(x, u^2(x)) u(x), \quad x \in \mathbb{R},
\]

that decay to zero at infinity, namely satisfying

\[
\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} u'(x) = 0.
\]

Actually, solutions \( u \) of (1)-(2) correspond to the eigenstate

\[
E(x, z) = e^{iz} u(x)
\]

propagating in the direction \( z \) and with waveguide index \( \beta > 0 \), see [7] (actually in such a paper the equations are Maxwell's). In particular, we


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are interested in the case considered in [1] when there is an internal layer with a linear response while the external medium is nonlinear and self-focusing. More precisely, the dielectric function $n^2$ is taken of the form

\begin{equation}
(3) \quad n^2(x, s) = \begin{cases} 
q^2 + c^2 & \text{if } |x| < d , \\
q^2 + s & \text{if } |x| > d ,
\end{cases}
\end{equation}

where $q, c \in \mathbb{R}$ and $d > 0$ denotes the thickness of the internal layer. In spite of the fact that the problem inherits a symmetry, it has been shown in [1] that at certain value $\beta = \beta_0$ a family of asymmetric solutions of (1)-(2) bifurcates from the the branch of the symmetric ones. The stability analysis has been carried out in [4,5]: the symmetric states become unstable for $\beta > \beta_0$, while the asymmetric states are the stable ones for $\beta$ greater than a certain $\beta_1 > \beta_0$, see figure 1 below. Both the preceding results rely on the fact that the nonlinearity $n^2$ in (3) is piece-wise linear and independent of $x$ and this specific feature permits to solve (1) explicitly.

The purpose of this Note is to investigate the same phenomenon described above for a class of equations (1) that, unlike the cited papers, cannot be integrated directly. We consider the case that the internal layer is thin and $n^2$ is still symmetric but has a rather general form and show the existence of asymmetric bound states of (1) provided $d$ is sufficiently small, see Theorem 1. To achieve this result we use a method, variational in nature, discussed in some recent papers, see [2,3], and related to the Poincaré-Melnikov theory of homoclinics. This abstract set up allows us also to discuss, for a slightly less general class of $n^2$ (but still including the model case (3)), the orbital stability of these bound states, see Theorem 8.

Fig. 1. – The curve in bold represents the asymmetric solutions.
2. – The main result.

Motivated by the preceding discussion, let us consider a thin layer of thickness \( d = \varepsilon \) and a dielectric function of the type

\[
  n^2(x, s) = n_{L}^2(x) + n_{NL}^2(x, s),
\]

with

\[
  \begin{cases}
    n_{L}^2(x) = q^2 + c^2 h(x/\varepsilon) \\
    n_{NL}^2(x, s) = s - \alpha(x/\varepsilon, s).
  \end{cases}
\]

We shall assume that \( h: \mathbb{R} \to \mathbb{R} \) and \( \alpha: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \) satisfy:

(a) \( h \) is an even function, with \( h(x) \geq 0, \ h \neq 0 \) and \( h(x) \in L^1(\mathbb{R}) \);

(b) \( \alpha \) is even, with respect to \( x \in \mathbb{R} \), with \( \alpha(x, \cdot) \in C^1(\mathbb{R}^+) \), \( \forall x \in \mathbb{R} \), and \( \alpha(x, 0) \equiv 0 \).

(c) There exists \( \sigma > 0 \) and \( k \in L^1(\mathbb{R}) \) such that \( |\alpha'(x, s)| \leq k(x)s^\sigma \), \( \forall s \geq 0 \). Moreover, letting

\[
  a(s) = \int_{-\infty}^{+\infty} \alpha(x, s) \, dx,
\]

one has that \( a(s) \) is increasing and \( a(s) \to +\infty \) as \( s \to +\infty \).

We remark here that it is possible to change in the hypothesis (c) the power \( s^\sigma \) by any continuous function in \( s \), and all the subsequent calculations remain valid.

To be consistent with the physical problem, \( h, \alpha \) should also be such that \( n_{L}^2 \) is non-increasing and \( n_{NL}^2(x, s) \) is non-decreasing in \( x > 0 \) and \( s > 0 \). However, we do not need such assumptions here. Letting \( \chi(x) \) denote the characteristic function of \([-1, 1]\), the dielectric function \( n^2 \) fits into the Akhmediev setting provided

\[
  h(x) = \chi(x), \quad \alpha(x, s) = \chi(x) \cdot s
\]

and corresponds to a layered medium with dielectric function given by (3), with \( d = \varepsilon \).

Substituting (4) into (1) and setting \( \lambda = \beta^2 - q^2 \), we find the equation

\[
  -u'' + \lambda u = u^3 + c^2 h(x/\varepsilon)u - \alpha(x/\varepsilon, u^2)u.
\]
Solutions of (5) that decay at zero at infinity, namely satisfying (2), will be henceforth called bound states.

Equation (5) will be seen as a perturbation of

\[ -u'' + \lambda u = u^3. \]

For all \( \lambda > 0 \), (6) has the positive symmetric solution

\[ \phi_\lambda(x) = \sqrt{2\lambda} \cosh(\sqrt{\lambda} x), \]

together with all its translates

\[ \phi_\lambda(x + \theta), \quad \theta \in \mathbb{R}. \]

To state our main result some further notation is in order. From (a), we can define

\[ H = \int_{-\infty}^{+\infty} h(x) \, dx \in (0, +\infty). \]

From assumption (c) it follows that the equation

\[ a(2\lambda) \equiv \int_{-\infty}^{+\infty} a(x, 2\lambda) \, dx = c^2 H \]

has a unique solution \( \lambda_0 = \lambda_0(c) > 0. \)

**Theorem 1.** Suppose that \((a - c)\) hold and take \( \delta, \Lambda > 0 \) such that

\[ 0 < \delta < \lambda_0 - \delta < \lambda_0 + \delta < \Lambda. \]

Then there exists \( \varepsilon_0 = \varepsilon_0(\delta, \Lambda) > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), one has:

1) for all \( \lambda \in [\delta, \Lambda] \), equation (5) has a symmetric bound state \( \overline{u}_\varepsilon \), which satisfies

\[ \lim_{\varepsilon \to 0} \overline{u}_\varepsilon = \phi_\lambda \quad \text{in } H^1(\mathbb{R}). \]

2) for all \( \lambda \in [\lambda_0 + \delta, \Lambda] \), equation (5) has, in addition, a pair of asymmetric bound states \( \overline{v}_\varepsilon^\pm \) such that

\[ \lim_{\varepsilon \to 0} \overline{v}_\varepsilon^\pm(x) = \phi_\lambda(x \pm \theta_\lambda) \quad \text{in } H^1(\mathbb{R}) \]

for some \( \theta_\lambda > 0. \)

The existence of the symmetric solution is well known, even in a much greater generality, see [7]. The existence of the asymmetric sol-
utions will be proved in the sequel by means of some variational arguments introduced in [2,3].

3. - Poincaré-Melnikov method.

We will prove Theorem 1 by using the results discussed in [2, 3] which are concerned with the existence of critical points of perturbed functionals of the form

\[
f_\varepsilon(u) = \frac{1}{2} \|u\|^2 - F(u) + G(\varepsilon, u).
\]

We assume that the reader is familiar with the cited papers. To put our problem into the preceding abstract frame, let us consider the Hilbert space \( E = H^1(\mathbb{R}) \) equipped with scalar product

\[
(u|v) = \int_\mathbb{R} [u'v' + \lambda uv]dx
\]

and norm \( \|u\|^2 = (u|u) \) and define

\[
F(u) = \frac{1}{4} \int_\mathbb{R} u^4.
\]

Obviously, \( F \in C^\infty(E, \mathbb{R}) \). Critical points of \( f_0(u) = 1/2 \|u\|^2 - F(u) \) are the bound states of the unperturbed problem (6). As remarked before, the functional \( f_0 \) has, for any fixed \( \lambda > 0 \), a one parameter family of critical points \( Z = \{z_\theta = \phi_\lambda(\cdot + \theta)| \theta \in \mathbb{R}\} \). Such a \( Z \) is a smooth one dimensional manifold and the following non-degeneracy condition (see [6, p. 226]) is satisfied:

\[
\text{Ker} f''_0(z_\theta) = \text{span} \{z_\theta\}, \quad \forall z_\theta \in Z.
\]

Furthermore, since \( \phi_\lambda \) decays exponentially to zero at infinity, then it is easy to see that for all \( z \in Z \) the linear map \( F''(z) \) is compact. Here, as usual, \( F''(z) \) is defined by setting

\[
(F''(z)v|w) = D^2F(z)[v, w].
\]

In order to introduce the perturbation term \( G \) let us set

\[
W(y, u) = \int_0^u a(y, s) \, ds - c^2 h(y) \, u.
\]
Notice that $W(y, u^2(y))$ is in $L^1$ by hypotheses (a) and (c) and the inclusion $E \subset L^\infty(\mathbb{R})$. Furthermore, the change of variable $x = \varepsilon y$ yields:

$$
\int_{\mathbb{R}} W\left(\frac{x}{\varepsilon}, u^2(x)\right) dx = \varepsilon \int_{\mathbb{R}} W(y, u^2(\varepsilon y)) dy.
$$

We set

$$
\bar{G}(\varepsilon, u) = \frac{1}{2} \int_{\mathbb{R}} W(y, u^2(\varepsilon y)) dy
$$

and

$$
G(\varepsilon, u) = \begin{cases} 
\varepsilon \bar{G}(\varepsilon, u) & \text{if } \varepsilon \neq 0, \\
0 & \text{if } \varepsilon = 0.
\end{cases}
$$

With this notation, it turns out that bound states of (5) are the critical points of the Euler functional $f_{\varepsilon}$ defined in (8).

Let $G'(\varepsilon, u)$ and $G''(\varepsilon, u)$ be defined by setting

$$
(G'(\varepsilon, u) | v) = D_u G(\varepsilon, u)[v], \quad \forall v \in E,
$$

$$
(G''(\varepsilon, u) v | w) = D_{uw} G(\varepsilon, u)[v, w], \quad \forall v, w \in E.
$$

**Lemma 2.** $G \in C(\mathbb{R} \times E, \mathbb{R})$ and $G(0, u) = 0$ for all $u \in E$. Furthermore the following conditions hold:

(G₁) $G$ is of class $C^2$ with respect to $u \in E$, $G'(0, u) = 0$ and $G''(0, u) = 0$ for all $u \in E$;

(G₂) the maps $(\varepsilon, u) \mapsto G'(\varepsilon, u)$ and $(\varepsilon, u) \mapsto G''(\varepsilon, u)$ are continuous as maps from $\mathbb{R} \times E$ to $E$, respectively to $L(E, E)$;

(G₃) for all $z \in Z$ the map $\varepsilon \mapsto G(\varepsilon, z)$ (and hence $\varepsilon \mapsto G(\varepsilon, z)$) is $C^1$.

**Proof.** Let $\varepsilon_n \to \varepsilon$ in $\mathbb{R}$ and $u_n \to u$ in $E$. From the embedding of $E$ into $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we deduce that for every $y \in \mathbb{R},$

$$
|u_n(\varepsilon_n y) - u(\varepsilon y)| \leq |u_n(\varepsilon_n y) - u(\varepsilon_n y)| + |u(\varepsilon_n y) - u(\varepsilon y)| \to 0
$$

whence

$$
W(y, u^2_n(\varepsilon_n y)) \to W(y, u^2(\varepsilon y)).
$$
for all $y \in \mathbb{R}$. Since

$$|W(y, u_n^2(\varepsilon_n y)) - W(y, u^2(\varepsilon y))| \leq \frac{k(y)}{(\sigma + 1)(\sigma + 2)} \left[ |u_n(\varepsilon_n y)|^{2\sigma+4} + |u(\varepsilon y)|^{2\sigma+4} + c^2 h(y)[|u_n(\varepsilon_n y)|^2 + |u(\varepsilon y)|^2] \leq C_1 [k(y) + h(y)] \in L^1(\mathbb{R}),
$$

one immediately deduces that $G(\varepsilon_n, u_n) \rightarrow G(\varepsilon, u)$.

By straight calculation we find

$$D_u G(\varepsilon, u)[v] = \varepsilon \int_{\mathbb{R}} W'_u(y, u^2(\varepsilon y)) u(\varepsilon y) v(\varepsilon y) \, dy,$$

$$D_{uu} G(\varepsilon, u)[v, w] = 2\varepsilon \int_{\mathbb{R}} W''_u(y, u^2(\varepsilon y)) u^2(\varepsilon y) v(\varepsilon y) w(\varepsilon y) \, dy + \varepsilon \int_{\mathbb{R}} W'_u(y, u^2(\varepsilon y)) v(\varepsilon y) w(\varepsilon y) \, dy,$$

for every $v, w \in E$, and $(G_1)$ follows directly.

The proof of $(G_2)$ relies on the arguments of Lemma 4.1 of [3]. Let us prove the continuity of $(\varepsilon, u) \mapsto G'(\varepsilon, u)$. We have to show that

$$\|G'(\varepsilon_n, u_n) - G'(\varepsilon, u)\| = \sup_{\|v\| \leq 1} |D_u G(\varepsilon_n, u_n)[v] - D_u G(\varepsilon, u)[v]| \rightarrow 0.$$ 

Setting

$$S_n(y) = \varepsilon_n W_u(y, u_n^2(\varepsilon_n y)) u_n(\varepsilon_n y) \quad \text{and} \quad S(y) = \varepsilon W_u(y, u^2(\varepsilon y)) u(\varepsilon y),$$

there results

$$|S_n(y) v(\varepsilon_n y) - S(y) v(\varepsilon y)| \leq |S_n(y) v(\varepsilon_n y) - S_n(y) v(\varepsilon y)| + |S_n(y) v(\varepsilon y) - S(y) v(\varepsilon y)| \leq |S_n(y)| \cdot |v(\varepsilon_n y) - v(\varepsilon y)| + \|v\|_\infty |S_n(y) - S(y)|.$$
Hence we find, for all \( \|v\| \leq 1 \),
\[
|D_u G(\varepsilon, u_n)[v] - D_u G(\varepsilon, u)[v]| = \left| \int_{\mathbb{R}} (S_u(y) v(\varepsilon_n y) - S(y) v(\varepsilon y))dy \right| \leq \\
\leq \int_{\mathbb{R}} |S_u(y)| \cdot |v(\varepsilon_n y) - v(\varepsilon y)| \, dy + \|v\|_{\infty} \int_{\mathbb{R}} |S_u(y) - S(y)| \, dy.
\]

From this and since
\[
|S_u(y)| \leq C_2[k(y) + h(y)] \equiv C_2 \gamma(y) \in L^1,
\]
we deduce:
\[
\|G'(\varepsilon_n, u_n) - G'(\varepsilon, u)\| = \sup_{\|v\| \leq 1} |D_u G(\varepsilon_n, u_n)[v] - D_u G(\varepsilon, u)[v]| \leq \\
\leq C_2 \sup_{\|v\| \leq 1} \int_{\mathbb{R}} |\gamma(y)| \cdot |v(\varepsilon_n y) - v(\varepsilon y)| \, dy + C_3 \int_{\mathbb{R}} |S_u(y) - S(y)| \, dy.
\]

Clearly, the latter integral tends to zero. As for the former, it can be uniformly estimated using the fact that \( E \subset C^{0, \nu} \) for any \( \nu \in (0, 1/2) \). Indeed, for any \( M > 0 \) and any \( \|v\| \leq 1 \) we find
\[
\int_{\mathbb{R}} |\gamma(y)| \cdot |v(\varepsilon_n y) - v(\varepsilon y)| \, dy \leq \\
\leq C_4 \|v\|_{C^{0, \nu}} \left| \varepsilon_n - \varepsilon \right|^\nu \int_{|y| \leq M} |y^\nu \gamma(y)| \, dy + C_5 \|v\|_{\infty} \int_{|y| \geq M} \gamma(y) \, dy \leq \\
\leq C_6 \left| \varepsilon_n - \varepsilon \right|^\nu \int_{|y| \leq M} |y^\nu \gamma(y)| \, dy + C_7 \int_{|y| \geq M} \gamma(y) \, dy.
\]

Taking limits as \( n \to \infty \) we infer
\[
\lim_{(\varepsilon_n, u_n) \to (\varepsilon, u)} \|G'(\varepsilon_n, u_n) - G'(\varepsilon, u)\| \leq C_7 \int_{|y| \geq M} \gamma(y) \, dy.
\]

Since \( M \) is arbitrary and \( \gamma \in L^1 \), it follows that
\[
\|G'(\varepsilon_n, u_n) - G'(\varepsilon, u)\| \to 0,
\]
as required. The continuity of \( G'' \) follows in a similar way.
Finally, to prove (G₃) it suffices to evaluate formally

\[ D_{\varepsilon} \tilde{G}(\varepsilon, u) = \int_{\mathbb{R}} W_u(y, u^2(\varepsilon y)) u(\varepsilon y) u'(\varepsilon y) y \, dy , \]

and to observe that for \( u = z_\theta \) we have from (a) and (c)

\[ |W_u(y, z_\theta^2(\varepsilon y)) z_\theta(\varepsilon y) z_\theta'(\varepsilon y) y| \leq \frac{k(y)}{\sigma + 1} z_\theta^{2\sigma + 3} |z_\theta' y| + e^2 h(y) z_\theta |z_\theta' y| \leq C_8[k(y) + h(y)] \in L^1(\mathbb{R}). \]

Thus, the theorem of derivation under the integral sign implies the assertion. ■

By Lemma 2, \( f \) can be factored by the abstract setting discussed in [2, 3]. For the reader convenience, let us sketch the procedure. First, we seek \( w \) orthogonal to \( z'_\theta \) satisfying

\[ f'_e(z_\theta + w) \in \text{span } \{ z'_\theta \}. \]

Considering the function

\[ \Phi: \mathbb{R} \times \mathbb{R} \times E \times \mathbb{R} \to E \times \mathbb{R}, \]

\[ \Phi(\varepsilon, \theta, w, \zeta) = (f'_e(z_\theta + w) - \zeta z'_\theta, (w|z'_\theta)) , \]

we are led to solve \( \Phi(\varepsilon, \theta, w, \zeta) = 0 \). An application of the Implicit Function Theorem yields

**Lemma 3.** For \( \varepsilon > 0 \) sufficiently small there exists a unique \( w = w(\varepsilon, \theta) \), orthogonal to \( z'_\theta \) and satisfying (11). Moreover there results

\[ w(\varepsilon, \theta) = \varepsilon w_0(\theta) + o(\varepsilon) , \]

and the symmetry property \( w(\varepsilon, \theta)(x) = w(\varepsilon, -\theta)(-x) \), \( \forall \theta, x \in \mathbb{R} \) (in particular, \( w(\varepsilon, 0) \) is an even function of \( x \in \mathbb{R} \)).

**Proof.** For a complete proof we refer to section 2 of [2] or to section 2 of [3]. Here we only point out that (G₃) implies the differentiability of \( w \) at \( (0, \theta) \) and this gives rise to (20) with \( w_0(\theta) = (\partial w/\partial \varepsilon)(0, \theta) \). Moreover, taking into account that \( h \) and \( \alpha \) are even functions with respect to \( x \in \mathbb{R} \), one infers that the function \( x \mapsto w(\varepsilon, -\theta)(-x) \) satisfies also the requirements for \( w(\varepsilon, \theta) \), and the symmetry property follows. ■
Setting $Z_\varepsilon = \{z_\theta + w(\varepsilon, \theta)\}$, it turns out that $Z_\varepsilon$ is (locally) diffeomorphic to $Z$ and by (11) is a natural constraint for $f_\varepsilon$. This means that in a neighbourhood of $Z$ the critical points of $f_\varepsilon$ coincide with the critical points of $f_\varepsilon$ constrained on $Z_\varepsilon$.

Finally, let us evaluate $f_\varepsilon$ on $Z_\varepsilon$. Using (12) and recalling that $f_0(z_\theta) = b$ as well as $f_\varepsilon'(z_\theta) = 0$, for all $\theta \in \mathbb{R}$, there results:

$$f_\varepsilon(z_\theta + w) = f_0(z_\theta + w) + G(\varepsilon, z_\theta + w) =$$

$$= f_0(z_\theta) + \varepsilon f_0'(z_\theta) w_0 + o(\varepsilon) + \varepsilon[\tilde{G}(\varepsilon, z_\theta) + O(\varepsilon)] = b + \varepsilon \tilde{G}(\varepsilon, z_\theta) + o(\varepsilon).$$

As a consequence of $(G_3)$ we infer $\tilde{G}(\varepsilon, z_\theta) = I(\theta) + O(\varepsilon)$, where

$$I(\theta) = \tilde{G}(0, z_\theta) = \frac{1}{2} \int_\mathbb{R} W(y, z_\theta^2(0)) \, dy$$

and this yields

$$f_\varepsilon(z_\theta + w) = b + \varepsilon I(\theta) + o(\varepsilon).$$

In conclusion, we can state the following result:

**Theorem 4.** Suppose that there exist $r > 0$ and $\theta^* \in \mathbb{R}$ such that

(13) \( \min_{|\theta - \theta^*| = r} I(\theta), \quad \text{or} \quad \max_{|\theta - \theta^*| = r} I(\theta). \)

Then, for $\varepsilon > 0$ sufficiently small, there exists $\theta_\varepsilon$, with $|\theta_\varepsilon - \theta^*| \leq r$, such that $f_\varepsilon$ has a critical point $u_\varepsilon$ of the form $u_\varepsilon(x) = z_\theta^\varepsilon + O(\varepsilon)$.

**Remarks 5.** (i) Theorem 3.3 is prompted for the application to the specific problem discussed here. For more general abstract results, we refer to [2, 3].

(ii) If $I$ has a proper local minimum (or maximum) at $\theta^*$, then $\theta_\varepsilon \to \theta^*$ as $\varepsilon \to 0$.

(iii) The function $I$ is nothing but the primitive of the Melnikov function associated to (5).
4. – Proof of Theorem 1.

In order to apply Theorem 4 to our equation, we first recall that for the Melnikov primitive there results:

\[
\Gamma(\theta) = \Gamma_\lambda(\theta) = \frac{1}{2} \int_\mathbb{R} W(y, z_{\theta}^2(0)) \, dy = \frac{1}{2} \int_\mathbb{R} W(y, \phi_\lambda^2(\theta)) \, dy
\]

where we have used again the notation \( \phi_\lambda \) to indicate the solutions of (6). Observe that

\[
\Gamma''_\lambda(0) = \phi_\lambda(0) \phi''_\lambda(0) \left[ \int_{-\infty}^{+\infty} \alpha(y, \phi_\lambda^2(0)) \, dy - c^2 H \right] = -2 \lambda^2 [a(2\lambda) - c^2 H].
\]

Therefore \( \Gamma''_\lambda(0) < 0 \) whenever \( \lambda > \lambda_0 \). Observe also that

\[
\Gamma_\lambda(\theta) = \frac{1}{2} \left( \int_0^{\phi_\lambda(\theta)} \int_\mathbb{R} \alpha(y, s) \, ds - c^2 h(y) \phi_\lambda^2(\theta) \right) \, dy = \frac{1}{2} \left[ \int_0^{\phi_\lambda(\theta)} \int_\mathbb{R} \alpha(y, s) \, ds \, dy - c^2 \phi_\lambda^2(\theta) \int_\mathbb{R} h(y) \, dy \right] = \frac{1}{2} \phi_\lambda^2(\theta) \left[ \int_0^{1} \int_{\mathbb{R}} \alpha(y, \phi_\lambda^2(\theta) t) \, dt \, dy - c^2 H \right].
\]

Then, one easily infers that

\[
\lim_{\theta \to \pm \infty} \Gamma_\lambda(\theta) = 0,
\]

with \( \Gamma_\lambda(\theta) < 0 \) for large values of \( |\theta| \). It follows that the Melnikov primitive \( \Gamma_\lambda \) has, for these values of \( \lambda \), 2 global minima \( \theta_\lambda > 0 \) and \( -\theta_\lambda \). If \( \lambda \in [\lambda_0 + \delta, \Lambda] \) there exists \( r > 0 \) independent of \( \lambda \), such that \( \Gamma_\lambda \) satisfies (13) with \( \theta^* = \pm \theta_\lambda \). Then such \( \theta_\lambda \) gives rise, through Theorem 4, to a critical point \( \theta_\lambda(\varepsilon) \) of \( f_\varepsilon \) on \( Z_\varepsilon \) and hence to a solution \( v_\varepsilon \) with

\[
v_\varepsilon(x) = \phi_\lambda(x + \theta_\lambda(\varepsilon)).
\]
Since we can also take \( r \) such that \( \theta_\lambda - r > 0 \), this solution is asymmetric. Similar argument for \(-\theta_\lambda\). For future reference, let us indicate how we can find in this frame the symmetric solution. Since \( \Gamma_\lambda \) is even, the value \( \theta = 0 \) is a critical point of \( \Gamma_\lambda \) for any \( \lambda > 0 \) and taking into account that \( w(\epsilon, 0) \) is even respect to \( x \), this critical point gives rise to a symmetric solution \( \bar{u}_\epsilon \) of (5). It turns out that \( \bar{u}_\epsilon \) corresponds to a minimum of \( \Gamma_\lambda \) for \( \lambda < \lambda_0 - \delta \), and a maximum of \( \Gamma_\lambda \) for \( \lambda > \lambda_0 + \delta \).

**Remarks.**

(i) When \( \alpha(x, s) = \alpha(x) s \) (that includes the Akhmediev model case) the Melnikov primitive becomes

\[
\Gamma_\lambda(\theta) = \frac{1}{4} A\phi^4_\lambda(\theta) - \frac{1}{2} c^2 H\phi^2_\lambda(\theta),
\]

where \( A = \int a(x) \, dx \). Then \( \lambda_0 = c^2 H/2A \) and for \( \lambda > \lambda_0 \) \( \Gamma_\lambda \) has precisely 3 nondegenerate critical points given by \( \theta = 0 \) and \( \pm \theta_\lambda \). The latters are global proper minima and thus \( \theta_\lambda(\epsilon) \to \theta_\lambda \) and \( v_\epsilon \to \phi_\lambda(\cdot + \theta_\lambda) \). Let us notice that in the model case one has \( \beta^2_\epsilon = \lambda_0 + q^2 + O(\epsilon) \). The graph of \( \Gamma_\lambda \) for different values of \( \lambda \) and the dependence of \( \theta_\lambda \) on \( \lambda \) are indicated in figures 2 and 3 below.

(ii) We also point out that the maximum value of the function \( \lambda \mapsto \theta_\lambda \) can be arbitrarily large, provided that \( \lambda_0 \) is sufficiently small. So, one can get «very asymmetric» bound states, by taking the data of the problem in such a way that \( \lambda_0 = c^2 H/2A \) be small.

(iii) The existence of asymmetric bound states depends on the combined effect of \( \alpha u^3 \) and \( c^2 hu \). Indeed, if either \( c = 0 \) or \( \alpha \equiv 0 \), the Melnikov primitive \( \Gamma_\lambda \) has for all \( \lambda > 0 \) a unique critical point at \( \theta = 0 \). Therefore the preceding arguments show that (5) has, near \( Z \), only symmetric solutions. These bound states turn out to be unstable (if \( c = 0 \)), or stable (if \( \alpha \equiv 0 \)), for all \( \lambda > 0 \), see Remark below.

![Fig. 2. - Graphs of \( \Gamma_\lambda(\theta) \) for different values of \( \lambda \).](image-url)
5. – Remarks on stability.

Here we shortly discuss the orbital stability of solitary waves $e^{i\lambda x} u_\epsilon(x)$ corresponding to solutions found in Theorem 1. By «orbital stability» we mean that a solution $\psi(z, x)$ of the Schrödinger equation exists for all $z \geq 0$ and remains $H^1$-close to the solitary wave $e^{i\lambda x} u_\epsilon(x)$ provided $\psi(0, x)$ is sufficiently near $u_\epsilon(x)$ in $H^1$. See, for example, [4]. Since the results will depend on the value of $\lambda$, we will emphasize the dependence on $\lambda$ by writing $u_{\epsilon, \lambda}$ instead of $u_\epsilon$.

We shall take $\alpha(x, s) = \alpha(x) s$. Our discussion relies on some results of [4] which, in the present setting, can be formulated as follows.

Let $u_{\epsilon, \lambda}$ be a solution of (5) and consider the eigenvalues $l$ of the linearized equation

\begin{equation}
-v'' + \lambda v - \left(3 u_{\epsilon, \lambda}^2 + c^2 \, h \left( \frac{x}{\epsilon} \right) - 3 \alpha \left( \frac{x}{\epsilon} \right) u_{\epsilon, \lambda}^2 \right) v = lv.
\end{equation}

Let $N = N(u, \epsilon, \lambda)$ denote the number of negative eigenvalues of (14) and let

$$\mu(\lambda) := \frac{\partial}{\partial \lambda} \int_R \left| u_{\epsilon, \lambda}(x) \right|^2 dx.$$ 

Then one has:

(A) $N = 1$ and $\mu(\lambda) > 0$ implies stability;

(B) $N = 1$ and $\mu(\lambda) < 0$ implies instability;

(C) $N = 2$ and $\mu(\lambda) > 0$ implies instability.
In all the cases, the rest of the spectrum of (14) is assumed to be positive and bounded away from zero. See Theorem 2 and Section 6. D of [4]-I for statements (A), (B) and the Instability Theorem in [4]-II for the statement (C).

In the model case, namely when \( \alpha(x) = h(x) = \chi(x/d) \), the characteristic function of the interval \([-d, d]\), the solitary wave corresponding to the symmetric mode becomes unstable for \( \lambda > \lambda_0 \). Moreover, there exists \( \lambda_1 > \lambda_0 \) such that the solitary wave corresponding to the asymmetric bound state is stable for \( \lambda > \lambda_1 \) and unstable for \( \lambda \in (\lambda_0, \lambda_1) \). See [4, 5]. Actually, one shows by a direct calculation that \( \mu(\lambda) > 0 \) for all \( \lambda > 0 \) but when \( u_{\varepsilon, \lambda} \) is asymmetric and \( \lambda_0 < \lambda < \lambda_1 \), see figure 1, where we have used the parameter \( \beta \) such that \( \lambda = 3^2 - q^2 \). As for the spectral analysis, it is carried out by a phase plane analysis. This is no more possible in the more general case when \( \alpha(x, s) = \alpha(x) s \) and it will be investigated by taking advantage of the variational approach discussed before.

We will use in the sequel the notation \( \overline{u}_{\varepsilon, \lambda} \) for the symmetric solution, \( v_{\varepsilon, \lambda} \) for the asymmetric one, and \( z_{\lambda, 0} \) for \( \phi_\lambda(\cdot + \theta) \). According to Remark 6-(i) we know that

\[
\overline{u}_{\varepsilon, \lambda} = z_{\lambda, 0} + O(\varepsilon), \quad v_{\varepsilon, \lambda} = z_{\lambda, \theta} + O(\varepsilon).
\]

**Lemma 7.** Take \( \delta, \Lambda \) like in Theorem 1. Then there exists \( \varepsilon_0' = \varepsilon_0' (\delta, \Lambda) > 0 \) (\( \varepsilon_0' \leq \varepsilon_0 \)) such that for all \( \varepsilon \in (0, \varepsilon_0'] \) one has

1) if \( u_{\varepsilon, \lambda} = \overline{u}_{\varepsilon, \lambda} \),

(a) \( \lambda \in [\delta, \lambda_0 - \delta] \Rightarrow N = 1 \);

(b) \( \lambda \in [\lambda_0 + \delta, \Lambda] \Rightarrow N = 2 \);

2) if \( u_{\varepsilon, \lambda} = v_{\varepsilon, \lambda} \) and \( \lambda \in [\lambda_0 + \delta, \Lambda] \) then \( N = 1 \).

In all the cases, the rest of the spectrum is positive and bounded away from zero.

**Proof.** In the proof of this Lemma we let \( \theta \) denote either 0 or \( \pm \theta_\lambda \). The number of negative eigenvalues of (14), \( N(u, \varepsilon, \lambda) \) equals the dimension of the subspace where \( D^2 f_{\varepsilon, \lambda}(u_{\varepsilon, \lambda}) \) is negative defined. Let first take \( \varepsilon = 0 \) and the corresponding family of solutions \( z_{\lambda, \theta} \). By a straight
calculation there results

\[ D^2 f_{\lambda, \theta}(z_{\lambda, \theta}, z_{\lambda, \theta}) < 0 , \]
\[ D^2 f_{\lambda, \theta}(z_{\lambda, \theta}, z'_{\lambda, \theta}) = 0 , \]
\[ D^2 f_{\lambda, \theta}(v, v) > 0 , \quad \forall v \perp \text{span} \{ z_{\lambda, \theta}, z'_{\lambda, \theta} \}, \quad v \neq 0 , \]

for every \( \lambda, \theta \). By the way, these relationships are related to the fact that 
\( z_{\lambda, \theta} \) can be found as Mountain-Pass critical point of \( f_{\lambda, \theta} \) and is degenerate  
because it appears together its translates. Let \( J = [\delta, \lambda_0 - \delta] \cup \cup \lambda_0 + \delta, \Lambda \). Since the preceding inequalities are uniform for \( \lambda \in J \) 
then, after a small perturbation, one has for all \( \lambda \in J \):

\[ D^2 f_{\epsilon, \lambda}(u_{\epsilon, \lambda}, z_{\lambda, \theta}, z_{\lambda, \theta}) < 0 , \]

as well as

\[ D^2 f_{\epsilon, \lambda}(u_{\epsilon, \lambda}, v) > 0 , \quad \forall v \perp \text{span} \{ z_{\lambda, \theta}, z'_{\lambda, \theta} \}, \quad v \neq 0 . \]

Next, using the properties of \( G \) and the fact that \( \theta_*(\epsilon) \rightarrow \theta^* \) as \( \epsilon \rightarrow 0 \), one can show, see Lemma 3.2 of [3]:

\[ \lim_{\epsilon \rightarrow 0} \epsilon^{-1} D^2 f_{\epsilon, \lambda}(u_{\epsilon, \lambda}, z_{\lambda, \theta}, z_{\lambda, \theta}) = \Gamma''_\lambda(\theta^*) . \]

According to Remark 6-(i), the critical points \( \theta^* \) are nondegenerate for \( \lambda \in J \) and hence (15) yields

\[ \Gamma''_\lambda(\theta^*) > 0 \Rightarrow D^2 f_{\epsilon, \lambda}(u_{\epsilon, \lambda}, z_{\lambda, \theta}, z_{\lambda, \theta}) > 0 , \]
\[ \Gamma''_\lambda(\theta^*) < 0 \Rightarrow D^2 f_{\epsilon, \lambda}(u_{\epsilon, \lambda}, z_{\lambda, \theta}, z_{\lambda, \theta}) < 0 , \]

provided \( \epsilon \) is sufficiently small. Recalling that \( \bar{u}_{\epsilon, \lambda} \) corresponds to a non-

degenerate minimum (maximum) of \( \Gamma_\lambda \) provided that \( \lambda \in [\delta, \lambda_0 - \delta] \) \( \lambda \in [\lambda_0 + \delta, \Lambda] \), while \( v_{\epsilon, \lambda} \) always corresponds to nondegenerate mini-

mum of \( \Gamma_\lambda \) for \( \lambda \in [\lambda_0 + \delta, \Lambda] \), the Lemma follows. ■

**Theorem 8.** Let \( \alpha(x, s) = \alpha(x) s \) and \( h \) satisfy hypotheses (a – c). Take \( \delta, \Lambda \) like in Theorem 1 and suppose, like in the model case, 
that

\[ \frac{\partial}{\partial \lambda} \int_R |\bar{u}_{\epsilon, \lambda}(x)|^2 dx > 0 , \quad \forall \lambda > 0 , \]
while

\[ \frac{\partial}{\partial \lambda} \int |v_{\epsilon, \lambda}(x)|^2 dx < 0, \quad \forall \lambda \in [\lambda_0 + \delta, \lambda_1), \]

\[ \frac{\partial}{\partial \lambda} \int |v_{\epsilon, \lambda}(x)|^2 dx > 0, \quad \forall \lambda \in (\lambda_1, \Lambda], \]

for some \( \lambda_1 = \lambda_1(\epsilon) \in (\lambda_0 + \delta, \Lambda) \). Then:

1) The solitary waves corresponding to symmetric bound states \( \tilde{u}_{\epsilon, \lambda} \) are stable for \( \lambda \in [\delta, \lambda_0 - \delta] \), and unstable for \( \lambda \in [\lambda_0 + \delta, \Lambda] \).

2) The solitary waves corresponding to asymmetric bound states \( v_{\epsilon, \lambda} \) are unstable for \( \lambda \in [\lambda_0 + \delta, \lambda_1) \) and stable for \( \lambda \in (\lambda_1, \Lambda] \).

**Proof.** If \( u_{\epsilon, \lambda} = \tilde{u}_{\epsilon, \lambda} \) we have that \( \mu(\lambda) > 0 \ \forall \lambda > 0 \). Moreover, by Lemma 7-1) we infer

\[ N = \begin{cases} 
1 & \text{if } \lambda \in [\sigma, \lambda_0 - \delta], \\
2 & \text{if } \lambda \in [\lambda_0 + \delta, \Lambda]. 
\end{cases} \]

Thus \( (A) \), resp. \( (C) \), implies stability, resp. instability. If \( u_{\epsilon, \lambda} = v_{\epsilon, \lambda} \), Lemma 7-2) yields \( N = 1 \). Moreover, one has

\[ \begin{cases} 
\mu(\lambda) < 0 & \text{if } \lambda \in [\lambda_0 + \delta, \lambda_1), \\
\mu(\lambda) > 0 & \text{if } \lambda \in (\lambda_1, \Lambda_1). 
\end{cases} \]

In the former case \( (B) \) implies instability, while in the latter stability follows from \( (A) \). ■

**Remark 9.** Completing Remark 6-(iii), we point out that if either \( c = 0 \) or \( \alpha \equiv 0 \), the unique critical point \( \theta = 0 \) of \( \Gamma_\lambda \) is a maximum, respectively a minimum, and hence the corresponding (symmetric) solution is unstable, respectively stable. ■

**REFERENCES**


Manoscritto pervenuto in redazione il 24 marzo 1997.