RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

TAMOTU KINOSHITA

On the wellposedness in the Gevrey classes of the Cauchy problem for weakly hyperbolic equations whose coefficients are Hölder continuous in t and degenerate in $t = T$

Rendiconti del Seminario Matematico della Università di Padova, tome 100 (1998), p. 81-96
<http://www.numdam.org/item?id=RSMUP_1998__100__81_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 100 (1998)

On the Wellposedness in the Gevrey Classes of the Cauchy Problem for Weakly Hyperbolic Equations whose Coefficients are Hölder Continuous in t and Degenerate in $t = T$.

TAMOTU KINOSHITA (*)

1. Introduction.

In 1983 F. Colombini, E. Jannelli and S. Spagnolo got the result concerned with the relation between the Hölder continuity of the coefficients and the Gevrey well-posedness for weakly hyperbolic equations of second order (see [3]). In their paper the coefficients of the principle part are Hölder continuous and degenerate in infinite number of points. Therefore the order of the degeneration is automatically determined by the regularity of the coefficients. Pay attension to this fact, we shall restrict the coefficients which degenerate in only one (or finite number of) point(s) and distinguish the condition of the degeneration from the condition of the regularity. Our aim is to deduce the relation among the Hölder continuity of the coefficients and the order of the degeneration and the Gevrey well-posedness (see Theorem).

While many people studied the influence of the lower term for the Gevrey well-posedness(see [6], [7], [9], etc.). In this paper, imposing the condition that the lower terms degenerate in the same point with the principal part, we shall find the influence of the lower term (see Corollary). We remark that this result can be easily compared with Ivrii's result.

(*) Indirizzo dell'A.: Institute of Mathematics, University of Tsukuba, Ibaraki 305, Japan.

We shall consider in $[0, T] \times \boldsymbol{R}_x^n$,

$$\text{(P)} \qquad \begin{cases} u_{tt} - \sum_{i,j=1}^{n} a_{i,j}(t)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(t)\, u_{x_i} = 0\,, \\ u(0, x) = u_0(x)\,, \qquad u_t(0, x) = u_1(x)\,, \end{cases}$$

where $A(t) \equiv (a_{i,j})_{1 \leqslant i,j \leqslant n}$ is a real symmetric matrix whose components $a_{i,j}(t)$ $(1 \leqslant i, j \leqslant n)$ belong to $C^{k+\alpha}[0, T)$ $(k \in \boldsymbol{N}^1,\ 0 \leqslant \alpha \leqslant 1)$. $B(t) \equiv$ $\equiv (b_i)_{1 \leqslant i \leqslant n}$ is a real vector whose components $b_i(t)$ $(1 \leqslant i \leqslant n)$ belong to $C^0[0, T)$.

Now we assume the followings.

For $a(t, \xi) \equiv (A(t)\,\xi, \xi)$ $(= \sum a_{i,j} \xi_i \xi_j)$ and $b(t, \xi) \equiv (B(t), \xi)$ $(= \sum b_i(t)\, \xi_i)$, there exist $C_1, C_2 > 0$ independent of T, such that

$$\text{(1)} \quad C_1 \left(1 - \frac{t}{T}\right)^{\beta} |\xi|^2 \leqslant a(t, \xi) \leqslant C_1^{-1} \left(1 - \frac{t}{T}\right)^{\beta} |\xi|^2 \qquad (0 \leqslant \beta \leqslant \infty)$$

$$\text{(2)} \qquad |b(t, \xi)| \leqslant C_2 \left(1 - \frac{t}{T}\right)^{\gamma} |\xi| \qquad (-1 < \gamma \leqslant \infty)$$

for ${}^{\forall} t \in [0, T]$, ${}^{\forall} \xi \in \boldsymbol{R}_\xi^n \backslash \{0\}$.

In this paper we don't assume the conditions concerned with $\partial_t a(t, \xi)$, for example the condition such that that $\partial_t a(t, \xi)$ admits finite number of zeros for ${}^{\forall} \xi \in \boldsymbol{R}_\xi^n \backslash \{0\}$ (see [3], [5]).

THEOREM. *Let* $T > 0$. *The coefficients satisfy* (1), (2). *Then for any* u_0 *and* $u_1 \in G^s$, *the Cauchy problem* (P) *has a unique soltion* $u \in C^2([0, T], G^s)$, *provided*

$$\text{(3)} \qquad 1 \leqslant s < 1 + \frac{\alpha(\beta + 2)}{2(\beta - \alpha + 1)} \qquad \textit{if } k = 0\,,$$

$$\text{(4)} \quad 1 \leqslant s < 1 + \frac{(k + \alpha)(\beta + 2)}{2\beta} \qquad \textit{if } k = 1 \textit{ or } k = 2, 3, \dots, \ \beta - 2\gamma - 2 < 0\,,$$

$$\text{(5)} \quad 1 \leqslant s < 1 + \min\left\{ \frac{(k + \alpha)(\beta + 2)}{2\beta}, \frac{\beta + 2}{\beta - 2\gamma - 2} \right\}$$

$$\textit{if } k = 2, 3, \dots, \beta - 2\gamma - 2 \geqslant 0\,.$$

We remark that (3) when $k=0$, $\alpha=1$ coincides (4) when $k=1$, $\alpha=0$. We shall compare our theorem with the results of otheres.

i) When $\beta=0$, i.e., (P) is equivalent to the strictly hyperbolic equation, (3) becomes $1\leqslant s<1/(1-\alpha)$ which is quite same with the result of F. Colombini, E. De Giorgi, and S. Spagnolo (see [2] and see also [1]).

ii) When $\beta=\infty$, i.e., (P) is a weakly hyperbolic equation whose coefficients are identically equal to zero, (3) and (4) become $1\leqslant s<1+ +(k+\alpha)/2$ which is quite same with the result of F. Colombini, E. Jannelli, and S. Spagnolo (see [3] and see also [4], [10]). Hence we can see that our results connect with their result, in which the coefficients of the equations may be identically zero in some intervals.

iii) The coefficients of our equation degenerate in $t=T$. From the time-reversal of hyperbolic equations, we can see that our results also holds in the case that the coefficients degenerate in $t=0$. On the other hand Ivrii treated the equatins whose coefficients degenerate in $t=0$ and invistigated the influence of the lower terms (see [6]). Then we shall introduce the similar corollary as the Ivrii's result, which is obtained easily by our theorem(use (5)).

COROLLARY. *Let* $T>0$, $\beta-2\gamma-2\geqslant 0$, *and* $a_{ij}(t)$ $(1\leqslant i,\ j\leqslant n)\in \in C^{2\beta/(\beta-2\gamma-2)}[0,\ T)$. *The coefficients satisfy* (1), (2). *Then the Cauchy problem* (P) *is well-posed in* G^s, *provided*

$$1\leqslant s<\frac{\beta+2}{\beta-2\gamma-2}.$$

We find that the Gevrey order of this corollary coicides the one of the Ivrii's result. Moreover the condition $\beta-2\gamma-2\geqslant 0$ also coincides the Ivrii's condition. In Ivrii's case the coefficients are given concretely as the power of t which naturally belong C^{∞}, while in our case the coefficients are admitted to belong $C^{2\beta/(\beta-2\gamma-2)}$.

Finally we mention that since generally the lower terms don't always degenerate in the only one (or finite number of) point(s) where the principal part also degenerates, the another type of condition is needed for the general lower terms (see [3], [7], [11]).

NOTATIONS. $C^{k+\alpha}[0,\ T)$ $(k\in N^1,\ 0\leqslant\alpha\leqslant 1)$ is the space of functions

f having k-derivatives continuous, and the k-th derivative locally Hölder continuous with exponent α on $[0, T]$.

G^s is the Gevrey functions f satisfying for any compact set $K \subset \boldsymbol{R}^n$,

$$\sup_{x \in K} |D^\alpha f(x)| \leq C_K \varrho_K^{|\alpha|} |\alpha|^s! \quad \text{for } \forall \alpha \in \boldsymbol{N}^n.$$

G_0^s is the space of γ^s with compact support.

2. Proof of Theorem.

When $s = 1$, the problem (P) is well-posed in G^1 with the coefficient $A(t)$ which belongs to $C^0[0, T]$ or even to $L^1[0, T]$ (see [2]). Therefore we can suppose $s > 1$ for the proof.

By Holmgren's theorem we get the uniqueness of solutions to (P) and can suppose that $u_0(x)$ and $u_1(x)$ belong to G_0^s. Moreover Ovciannikov theorem gives the existence of solutions (see [3]). Our task is to deduce the regularity for x of solutions.

We change (P) by Fourier transform.

$$\text{(6)} \qquad \begin{cases} v_{tt} + a(t, \xi)\, v + ib(t, \xi)\, v = 0\,, \\ v(0, \xi) = v_0(\xi)\,, \qquad v_t(0, \xi) = v_1(\xi)\,, \end{cases}$$

where

$$v = \frac{1}{\sqrt{2\pi}} \int e^{-ix\cdot\xi} u(x)\, dx\,, \qquad v_i = \frac{1}{\sqrt{2\pi}} \int e^{-ix\cdot\xi} u_i(x)\, dx \qquad (i = 0, 1)\,,$$

and define

$$A_\delta(t) = \begin{cases} A(0) & -\infty < t < 0\,, \\ A(t) & 0 \leq t \leq T - \delta\,, \\ A(T - \delta) & T - \delta < t < \infty\,, \end{cases}$$

$(0 < \delta \leq T)$. We shall first consider the case $k = 0$. Putting

$$A_\delta^\varepsilon(t) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi\left(\frac{s - t}{\varepsilon}\right) A_\delta(s)\, ds \left(= \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi\left(\frac{s}{\varepsilon}\right) A_\delta(t + s)\, ds \right)$$

$(0 < \varepsilon < 1)$, where $\varphi(t) \in C_0^\infty(\boldsymbol{R}_t^1)$ satisfies $0 \leq \varphi(t) < \infty$, $\int_{-\infty}^{\infty} \varphi(t)\, dt = 1$, we define

$$a_\delta^{(\varepsilon)}(t, \xi) \equiv (A_\delta^{(\varepsilon)}(t)\, \xi, \xi).$$

Moreover using this, we shall also define the approximate energy

$$E_\delta^{(\varepsilon)}(t, \xi) = e^{\varrho(t)\langle\xi\rangle_\nu^\kappa}(a_\delta^{(\varepsilon)}(t, \xi)\,|v|^2 + |v'|^2)^{1/2}, \tag{7}$$

where $\langle\xi\rangle_\nu = (|\xi|^2 + \nu^2)^{1/2}$ $(\nu > 0)$, $\varrho(t) \in C^1[0, T]$ is determined later. Defferentiating $(E_\delta^{(\varepsilon)})^2$ in t, by (6) we get

$$d(E_\delta^{(\varepsilon)})^2/dt = 2\varrho'\langle\xi\rangle_\nu^\kappa(E_\delta^{(\varepsilon)})^2 + e^{2\varrho\langle\xi\rangle_\nu^\kappa} \times$$

$$\times \{(da_\delta^{(\varepsilon)}/dt)\,|v|^2 + a_\delta^{(\varepsilon)}\cdot 2\,\mathrm{Re}\,(v, v') + 2\,\mathrm{Re}\,(v', v'')\} =$$

$$= 2\varrho'\langle\xi\rangle_\nu^\kappa(E_\delta^{(\varepsilon)})^2 + \frac{(da_\delta^{(\varepsilon)}/dt)}{a_\delta^{(\varepsilon)}}\, e^{2\varrho\langle\xi\rangle_\nu^\kappa} a_\delta^{(\varepsilon)}\,|v|^2 +$$

$$+ \frac{a_\delta^{(\varepsilon)} - a}{(a_\delta^{(\varepsilon)})^{1/2}}\, e^{2\varrho\langle\xi\rangle_\nu^\kappa}\cdot 2\,\mathrm{Re}\,((a_\delta^{(\varepsilon)})^{1/2} v, v') +$$

$$+ \frac{-ib}{(a_\delta^{(\varepsilon)})^{1/2}}\, e^{2\varrho\langle\xi\rangle_\nu^\kappa}\cdot 2\,\mathrm{Im}\,((a_\delta^{(\varepsilon)})^{1/2} v, v') \leqslant$$

$$\leqslant 2\varrho'\langle\xi\rangle_\nu^\kappa(E_\delta^{(\varepsilon)})^2 + \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}}(E_\delta^{(\varepsilon)})^2 +$$

$$+ \frac{|a_\delta^{(\varepsilon)} - a|}{(a_\delta^{(\varepsilon)})^{1/2}}(E_\delta^{(\varepsilon)})^2 + \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}}(E_\delta^{(\varepsilon)})^2,$$

here we used $|(x, y)| \leqslant |x||y| \leqslant (1/2)(|x|^2 + |y|^2)$.

Hence we get

$$dE_\delta^{(\varepsilon)}/dt \leqslant \frac{1}{2}\left\{2\varrho'\langle\xi\rangle_\nu^\kappa + \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}} + \frac{|a_\delta^{(\varepsilon)} - a|}{(a_\delta^{(\varepsilon)})^{1/2}} + \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}}\right\} E_\delta^{(\varepsilon)}.$$

Thus Gronwall's inequality yields

$$(8)\quad E_\delta^{(\varepsilon)}(t) \leqslant E_\delta^{(\varepsilon)}(0)\cdot$$

$$\cdot \exp\left\{\frac{1}{2}\int_0^t 2\varrho'\langle\xi\rangle_\nu^\kappa + \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}} + \frac{|a_\delta^{(\varepsilon)}-a|}{(a_\delta^{(\varepsilon)})^{1/2}} + \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}}\,ds\right\} \leqslant$$

$$\leqslant E_\delta^{(\varepsilon)}(0)\exp\left\{(\varrho(t)-\varrho(0))\langle\xi\rangle_\nu^\kappa + \right.$$

$$\left. + \frac{1}{2}\int_0^t \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}}\,ds + \frac{1}{2}\int_0^t \frac{|a_\delta^{(\varepsilon)}-a|}{(a_\delta^{(\varepsilon)})^{1/2}}\,ds + \frac{1}{2}\int_0^t \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}}\,ds\right\}$$

for $t \in [0, T]$.

Picking up each term, we shall estimate. From the definition of $a_\delta^{(\varepsilon)}$, we can get

$$(9)\quad |da_\delta^{(\varepsilon)}(t)/dt| = \left|((dA_\delta^{(\varepsilon)}(t)/dt)\,\xi,\,\xi)\right| \leqslant$$

$$\leqslant \left|\frac{1}{\varepsilon}\int_{-\infty}^{\infty} -\frac{1}{\varepsilon}\varphi'\left(\frac{s-t}{\varepsilon}\right)A_\delta(s)\,ds\right| |\xi|^2 =$$

$$= \left|-\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi'(s)\,A_\delta(t+\varepsilon s)\,ds\right| |\xi|^2 =$$

$$= \left|\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi'(s)(A_\delta(t)-A_\delta(t+\varepsilon s))\,ds\right| |\xi|^2 \leqslant$$

$$\leqslant C_3\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi'(s)(\varepsilon s)^\alpha\,ds\,|\xi|^2 \leqslant C_3C_4\varepsilon^{\alpha-1}|\xi|^2,$$

here we used $(1/\varepsilon)\int_{-\infty}^{\infty}\varphi'(s)\,ds = 0$ for $0<\forall\ \varepsilon<1$.

While from the definition of $a_\delta^{(\varepsilon)}$ and (1), we can also get

$$a_\delta^{(\varepsilon)}(t) \geqslant a(T-\delta) \geqslant C_1\left(1-\frac{T-\delta}{T}\right)^\beta |\xi|^2 = C_1\left(\frac{\delta}{t}\right)^\beta |\xi|^2 \tag{10}$$

for $0<\delta\leqslant T$.

By (9), (10) we get

$$\frac{1}{2}\int_0^t \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}}ds \leqslant \frac{1}{2}\int_0^t \frac{C_3 C_4 \varepsilon^{\alpha-1}|\xi|^2}{C_1(\delta/T)^\beta|\xi|^2}ds = C_5 T^\beta t\delta^{-\beta}\varepsilon^{\alpha-1}, \tag{11}$$

where $C_5=(1/2)\,C_1^{-1}C_3C_4$.

From the definition of $a_\delta^{(\varepsilon)}$ again, we can get

$$|a_\delta^{(\varepsilon)}(t)-a(t)| \leqslant \big|\big((A_\delta^{(\varepsilon)}(t)-A_\delta(t))\,\xi,\ \xi\big)\big| + |((A_\delta(t)-A(t))\,\xi,\ \xi)| \leqslant \tag{12}$$

$$\leqslant \left|\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi\left(\frac{s}{\varepsilon}\right)(A_\delta(t+s)-A_\delta(t))\,ds\right| |\xi|^2 + |A_\delta(t)-A(t)|\,|\xi|^2 \leqslant$$

$$\leqslant \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi\left(\frac{s}{\varepsilon}\right)s^\alpha ds\,|\xi|^2 + |A_\delta(t)-A(t)|\,|\xi|^2 \leqslant$$

$$\leqslant C_6\varepsilon^\alpha|\xi|^2 + |A_\delta(t)-A(t)|\,|\xi|^2.$$

Noting that $A_\delta(t)=A(t)$ on $0\leqslant t\leqslant T-\delta$, and $|A_\delta(t)-A(t)| \leqslant$ $\leqslant C_1^{-1}(\delta/T)^\beta$ on $T-\delta\leqslant t<T$, by (10), (12) we get

$$\frac{1}{2}\int_0^t \frac{|a_\delta^{(\varepsilon)}-a|}{(a_\delta^{(\varepsilon)})^{1/2}}ds \leqslant \frac{1}{2}\int_0^t \frac{C_6\varepsilon^\alpha|\xi|^2}{\{C_1(\delta/T)^\beta|\xi|^2\}^{1/2}}ds + \tag{13}$$

$$+\begin{cases} \dfrac{1}{2}\displaystyle\int_{T-\delta}^t \frac{C_1^{-1}(\delta/T)^\beta|\xi|^2}{\{C_1(\delta/T)^\beta|\xi|^2\}^{1/2}}ds & \text{if } t\geqslant T-\delta\,, \\ 0 & \text{if } t<T-\delta\,, \end{cases}$$

$$\leqslant C_7T^{\beta/2}t\delta^{-\beta/2}\varepsilon^\alpha|\xi| + \begin{cases} C_8T^{-\beta/2}(t-T+\delta)\,\delta^{\beta/2}|\xi| & \text{if } t\geqslant T-\delta\,, \\ 0 & \text{if } t<T-\delta\,, \end{cases}$$

where $C_7=(1/2)\,C_1^{-1/2}C_6$, $C_8=(1/2)\,C_1^{-3/2}$.

We remark that

$$(14)\quad t-T+\delta=\frac{\delta t-\delta t+tT-T^2+\delta t}{T}=\frac{\delta t+(\delta-T)(T-t)}{T}\leqslant\frac{\delta t}{T}.$$

Hence by (13) we get

$$(15)\quad \frac{1}{2}\int_0^t\frac{|a_\delta^{(\varepsilon)}-a|}{(a_\delta^{(\varepsilon)})^{1/2}}ds\leqslant C_7T^{\beta/2}t\delta^{-\beta/2}\varepsilon^{\alpha}|\xi|+C_8T^{-1-\frac{\beta}{2}}t\delta^{1+\frac{\beta}{2}}|\xi|.$$

From the hypothesis (2), by (10) we get

$$(16)\quad \frac{1}{2}\int_0^t\frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}}ds\leqslant\frac{1}{2}\int_0^t\frac{C_2(1-s/T)^{\gamma}|\xi|}{\{C_1(\delta/T)^{\beta}|\xi|^2\}^{1/2}}ds=$$

$$=\frac{1}{2}C_1^{-1/2}C_2T^{\beta/2}\delta^{-\beta/2}\int_0^t\left(1-\frac{s}{T}\right)^{\gamma}ds=$$

$$=\frac{1}{2}C_1^{-1/2}C_2\frac{1}{\gamma+1}T^{\beta/2+1}\delta^{-\beta/2}\left\{1-\left(1-\frac{t}{T}\right)^{\gamma+1}\right\}.$$

It generally holds that for $0\leqslant x\leqslant 1$

$$(17)\quad (1-x)^a\geqslant\begin{cases}1-ax & \text{if } a\geqslant 1,\\ 1-x^a & \text{if } 0<a<1.\end{cases}$$

By (16), (17), we get

$$(18)\quad \frac{1}{2}\int_0^t\frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}}ds\leqslant\begin{cases}C_9T^{\beta/2}t\delta^{-\beta/2} & \text{if } \gamma\geqslant 0,\\ C_9\left(\dfrac{1}{\gamma+1}\right)T^{\beta/2-\gamma}t^{\gamma+1}\delta^{-\beta/2} & \text{if } -1<\gamma<0,\end{cases}$$

where $C_9=(1/2)\,C_1^{-1/2}C_2$.

By (8), (11), (15), (18), we have the estimate

$$E_\delta^{(\varepsilon)}(t)\leqslant E_\delta^{(\varepsilon)}(0)\exp\Big\{(\varrho(t)-\varrho(0))\langle\xi\rangle_\nu^{\kappa}+$$

$$+C_5T^{\beta}t\delta^{-\beta}\varepsilon^{\alpha-1}+C_7T^{\beta/2}t\delta^{-\beta/2}\varepsilon^{\alpha}|\xi|+$$

$$+C_8 T^{-1-\beta/2} t\delta^{1+\beta/2}|\xi| + \begin{cases} C_9 T^{\beta/2} t\delta^{-\beta/2} & \text{if } \gamma \geq 0\}, \\ C_9\left(\dfrac{1}{\gamma+1}\right) T^{\beta/2-\gamma} t^{\gamma+1} \delta^{-\beta/2} & \text{if } -1<\gamma<0\}. \end{cases}$$

Here, if we take

$$\delta = T\langle\xi\rangle^{-2\alpha/(\alpha\beta+2\beta+2)}, \qquad \varepsilon = \langle\xi\rangle^{(-2\beta-2)/(\alpha\beta+2\beta+2)}, \tag{19}$$

and put

$$\mu(t) = t + t^{\lambda}$$

with the parameter $\lambda = \min\{1, \gamma+1\}$, we have

$$\begin{aligned} E_\delta^{(\varepsilon)}(t) \leq E_\delta^{(\varepsilon)}(0) \exp\Big\{&(\varrho(t)-\varrho(0))\langle\xi\rangle_\nu^\kappa + \\ &+ C_5 t\langle\xi\rangle^{(-2\alpha+2\beta+2)/(\alpha\beta+2\beta+2)} + C_7 t\langle\xi\rangle^{(-2\alpha+2\beta+2)/(\alpha\beta+2\beta+2)} + \\ &+ C_8 t\langle\xi\rangle^{(-2\alpha+2\beta+2)/(\alpha\beta+2\beta+2)} + \\ &+ \begin{cases} C_9 t\langle\xi\rangle^{\alpha\beta/\alpha\beta+2\beta+2} & \text{if } \gamma \geq 0\}, \\ C_9\left(\dfrac{1}{\gamma+1}\right) T^{-\gamma} t^{\gamma+1} \langle\xi\rangle^{\alpha\beta/(\alpha\beta+2\beta+2)} & \text{if } -1<\gamma<0\} \end{cases} \\ \leq E_\delta^{(\varepsilon)}(0) \exp\Big\{&(\varrho(t)-\varrho(0))\langle\xi\rangle_\nu^\kappa + C_{10}\mu(t)\langle\xi\rangle^{(-2\alpha+2\beta+2)/(\alpha\beta+2\beta+2)}\Big\}, \end{aligned}$$

where $C_{10} = C_5 + C_7 + C_8 + C_9 \max\{1, (1/\gamma+1)\, T^{-1}\}$. Here we used

$$\frac{\alpha\beta}{\alpha\beta+2\beta+2} < \frac{-2\alpha+2\beta+2}{\alpha\beta+2\beta+2} \qquad \text{for } 0 \leq \alpha \leq 1,\ 0 \leq \beta \leq \infty.$$

Supposing

$$\kappa > \frac{-2\alpha+2\beta+2}{\alpha\beta+2\beta+2} (\equiv \kappa_0), \tag{20}$$

we get

$$\begin{aligned} E_\delta^{(\varepsilon)}(t) &\leq E_\delta^{(\varepsilon)}(0) \exp\left\{\langle\xi\rangle_\nu^\kappa \left(\varrho(t)-\varrho(0)+C_{10}\mu(t)\frac{\langle\xi\rangle^{\kappa_0}}{\langle\xi\rangle_\nu^\kappa}\right)\right\} \leq \\ &\leq E_\delta^{(\varepsilon)}(0) \exp\left\{\langle\xi\rangle_\nu^\kappa \left(\varrho(t)-\varrho(0)+C_{10}\mu(t)\nu^{\kappa_0-\kappa}\right)\right\}. \end{aligned} \tag{21}$$

Finally if we choose $\varrho(t)=\varrho(0)-C_{10}\mu(t)\,\nu^{\kappa_0-\kappa}$, we have the energy estimate

$$E_\delta^{(\varepsilon)}(t)\leqslant E_\delta^{(\varepsilon)}(0). \tag{22}$$

Noting that $\varrho(t)$ is the decreasing function, by (7), (10), the left side of (22) is changed as follows

$$E_\delta^{(\varepsilon)}(t,\xi)\geqslant e^{\varrho(T)\langle\xi\rangle_\nu^\kappa}\left(C_1\left(\frac{\delta}{T}\right)^\beta|\xi|^2|v(t,\xi)|^2+|v'(t,\xi)|^2\right)^{1/2}\geqslant \tag{23}$$

$$\geqslant\min\left\{1,\,C_1^{1/2}\left(\frac{\delta}{T}\right)^{\beta/2}\right\}e^{(\varrho(0)-C_{10}\mu(T)\nu^{\kappa_0-\kappa})\langle\xi\rangle_\nu^\kappa}\left(|\xi|^2|v(t,\xi)|^2+|v'(t,\xi)|^2\right)^{1/2}.$$

While by (1) and (7) the right side of (22) is changed as follows

$$E_\delta^{(\varepsilon)}(0,\xi)\leqslant e^{\varrho(0)\langle\xi\rangle_\nu^\kappa}\left(a_\delta^{(\varepsilon)}(0,\xi)|v_0(\xi)|^2+|v_1(\xi)|^2\right)^{1/2}= \tag{24}$$

$$=e^{\varrho(0)\langle\xi\rangle_\nu^\kappa}\left(a(0,\xi)|v_0(\xi)|^2+|v_1(\xi)|^2\right)^{1/2}\leqslant$$

$$\leqslant C_{11}e^{\varrho(0)\langle\xi\rangle_\nu^\kappa}\left(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2\right)^{1/2}.$$

By (22), (23), (24) we have

$$\min\left\{1,\,C_1^{1/2}\left(\frac{\delta}{T}\right)^{\beta/2}\right\}e^{(\varrho(0)-C_{10}\mu(T)\nu^{\kappa_0-\kappa})\langle\xi\rangle_\nu^\kappa}\left(|\xi|^2|v(t,\xi)|^2+|v'(t,\xi)|^2\right)^{1/2}\leqslant$$

$$\leqslant C_{11}e^{\varrho(0)\langle\xi\rangle_\nu^\kappa}\left(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2\right)^{1/2}.$$

Therefore we obtain

$$e^{\varrho_1\langle\xi\rangle_\nu^\kappa}\left(|\xi|^2|v(t,\xi)|^2+|v'(t,\xi)|^2\right)^{1/2}\leqslant$$

$$\leqslant\max\left\{1,C_1^{-1/2}\left(\frac{\delta}{T}\right)^{-\beta/2}\right\}C_{11}e^{(\varrho_1+C_{10}\mu(T)\nu^{\kappa_0-\kappa})\langle\xi\rangle_\nu^\kappa}\left(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2\right)^{1/2}\leqslant$$

$$\leqslant C_{12}e^{(\varrho_1-\varrho_0+C_{10}\mu(T)\nu^{\kappa_0-\kappa})\langle\xi\rangle_\nu^\kappa+\varrho_0(\langle\xi\rangle_\nu^\kappa-\langle\xi\rangle_{\nu_0}^\kappa)}\left(\frac{\delta}{T}\right)^{-\beta/2}\cdot$$

$$\cdot e^{\varrho_0\langle\xi\rangle_{\nu_0}^\kappa}\left(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2\right)^{1/2},$$

where $0<\varrho_1<\varrho_0$, $C_{12}=\max\{1,\,C_1^{-1/2}(\delta/T)^{-\beta/2}\}\,C_{11}$.

Moreover noting that for $\nu_1 \geqslant \nu_2$

$$\langle\xi\rangle_{\nu_1}^{\kappa} - \langle\xi\rangle_{\nu_2}^{\kappa} = (\nu_1 - \nu_2) \int_0^1 \partial_\nu \langle\xi\rangle_\nu^\kappa \,|_{\nu = \nu_2 + \theta(\nu_1 - \nu_2)} d\theta =$$

$$= (\nu_1 - \nu_2) \int_0^1 \kappa\big(\nu_2 + \theta(\nu_1 - \nu_2)\big) \langle\xi\rangle_{\nu_2 + \theta(\nu_1 - \nu_2)}^{\kappa - 2} d\theta = \kappa(\nu_1 - \nu_2)\, \nu_2^{\kappa - 2} \leqslant \kappa \nu_1^2 \nu_2^{\kappa - 2},$$

and taking

$$\nu = \max\left\{\nu_0, \left(\frac{\varrho_0 - \varrho_1}{2C_{10}\mu(T)}\right)^{1/(\kappa_0 - \kappa)}\right\},$$

by (19) we get

$$e^{(\varrho_1 - \varrho_0 + C_{10}\mu(T)\, \nu^{\kappa_0 - \kappa})\langle\xi\rangle_\nu^\kappa + \varrho_0(\langle\xi\rangle_\nu^\kappa - \langle\xi\rangle_{\nu_0}^\kappa)} \left(\frac{\delta}{T}\right)^{-\beta/2} \leqslant$$

$$\leqslant e^{(\varrho_1 - \varrho_0 + C_{10}\mu(T)((\varrho_0 - \varrho_1)/(2C_{10}\mu(T))))\langle\xi\rangle_\nu^\kappa + \varrho_0 \kappa \nu^2 \nu_0^{\kappa - 2}} \langle\xi\rangle^{\alpha\beta/(\alpha\beta + 2\beta + 2)}$$

$$\leqslant e^{\varrho_0 \kappa \nu^2 \nu_0^{\kappa - 2}} (e^{((\varrho_1 - \varrho_0)/2)\langle\xi\rangle_\nu^\kappa} \langle\xi\rangle^{\alpha\beta/(\alpha\beta + 2\beta + 2)}) \leqslant C_{13}\, e^{\varrho_0 \kappa \nu^2 \nu_0^{\kappa - 2}}.$$

At last we have

$$(25) \qquad e^{\varrho_1 \langle\xi\rangle_\nu^\kappa} (|\xi|^2 \,|v(t, \xi)|^2 + |v'(t, \xi)|^2)^{1/2} \leqslant$$

$$\leqslant \text{const}\, e^{\varrho_0 \langle\xi\rangle_{\nu_0}^\kappa} (|\xi|^2 \,|v_0(\xi)|^2 + |v_1(\xi)|^2)^{1/2}$$

$$\text{for } \kappa_0 <^\forall \kappa < 1, \quad 0 <^\forall \varrho_1 < \varrho_0.$$

Since $u_0(x)$ and $u_1(x)$ belong to G_0^s, there exist ϱ_0, $\nu_0 > 0$ such that

$$(26) \qquad e^{\varrho_0 \langle\xi\rangle_{\nu_0}^{1/s}} (|\xi|^2 \,|v_0(\xi)|^2 + |v_1(\xi)|^2)^{1/2} < \infty.$$

Thus by (P), (20), (25) we find that the solution $u \in C^2([0, T], G^s)$, where $s = \kappa^{-1}$ satisfies

$$1 \leqslant s < 1 + \frac{\alpha(\beta + 2)}{2(\beta - \alpha + 1)}.$$

This implies (3).

We shall next consider the case of $k = 1, 2, \ldots,$. We also define the approximate energy

(27) $$E_\delta(t, \xi) = e^{\varrho(t)\langle\xi\rangle_\nu^\kappa}(a_\delta(t, \xi)|v|^2 + |v'|^2)^{1/2},$$

where $a_\delta(t, \xi) \equiv (A_\delta(t)\xi, \xi)$.

Similarly we get (see (8))

(28) $$E_\delta(t) \leq E_\delta(0)\exp\left\{(\varrho(t) - \varrho(0))\langle\xi\rangle_\nu^\kappa + \right.$$

$$\left. + \frac{1}{2}\int_0^t \frac{|da_\delta/dt|}{a_\delta}ds + \frac{1}{2}\int_0^t \frac{|a_\delta - a|}{a_\delta^{1/2}}ds + \frac{1}{2}\int_0^t \frac{|b|}{a_\delta^{1/2}}ds\right\} \quad \text{for } t \in [0, T].$$

In order to estimate the coefficients which belong to C^1 at least, we need the following lemma.

LEMMA (Colombini, Jannelli, Spagnolo). *Let $f(t)$ be a real function of class $C^{k+\alpha}$ on some compact interval $I \subset \boldsymbol{R}$, with k integer ≥ 1 and $0 \leq \alpha \leq 1$, and assume that $f(t) \geq 0$ on I. Then the function $f(t)^{1/(k+\alpha)}$ is absolutely continuous on I. Moreover*

$$\|(f^{1/(k+\alpha)})'\|_{L^1(I)}^{k+\alpha} \leq C(k, \alpha, I)\|f\|_{C^{k+\alpha}(I)}.$$

For the proof, refer to [3].

Hence by (1) we get

(29) $$\frac{1}{2}\int_0^t \frac{|da_\delta/dt|}{a_\delta}ds = \frac{1}{2}\int_0^t \frac{|da_\delta/dt|}{a_\delta^{1/(k+\alpha)}a_\delta^{1-1/(k+\alpha)}}ds$$

$$\leq \frac{1}{2}\left(C_1\left(\frac{\delta}{T}\right)^\beta|\xi|\right)^{-1/(k+\alpha)}\int_0^t \frac{|da_\delta/dt|}{a_\delta^{1-1/(k+\alpha)}}ds \leq \psi(t)\,T^{\beta/(k+\alpha)}\delta^{-\beta/(k+\alpha)},$$

where $\psi(t)$ is the increasing function satisfying $\psi(0) = 0$.

In the same way with the second term of (15), we get

$$\frac{1}{2}\int_0^t \frac{|a_\delta - a|}{a_\delta^{1/2}}\,ds \leqslant C_8 T^{-1-\beta/2} t\delta^{1+\beta/2}|\xi|\,. \tag{30}$$

When $1 \leqslant k+\alpha \leqslant 2$, in the same way with (18), we get

$$\frac{1}{2}\int_0^t \frac{|b|}{(a_\delta)^{1/2}}\,ds \leqslant \begin{cases} C_9 T^{\beta/2} t\delta^{-\beta/2} & \text{if } \gamma \geqslant 0\,, \\ C_9\left(\dfrac{1}{\gamma+1}\right) T^{\beta/2-\gamma} t^{\gamma+1}\delta^{-\beta/2} & \text{if } -1<\gamma<0\,, \end{cases} \tag{31}$$

$$\leqslant \begin{cases} C_9 T^{\beta/2} t\delta^{-\beta/(k+\alpha)} & \text{if } \gamma \geqslant 0\,, \\ C_9\left(\dfrac{1}{\gamma+1}\right) T^{\beta/(2-\gamma)} t^{\gamma+1}\delta^{-\beta/(k+\alpha)} & \text{if } -1<\gamma<0\,. \end{cases}$$

By (28),(29),(30),(31) we have the estimate

$$\begin{aligned} E_\delta(t) \leqslant E_\delta(0)\exp\{&(\varrho(t)-\varrho(0))\langle\xi\rangle_\nu^\kappa + \psi(t)\,T^{\beta/(k+\alpha)}\delta^{-\beta/(k+\alpha)} + \\ &+ C_8 T^{-1-\beta/2} t\delta^{1+\beta/2}|\xi| + \\ &+\begin{cases} C_9 T^{\beta/2} t\delta^{-\beta/(k+\alpha)}\} & \text{if } \gamma \geqslant 0\,, \\ C_9\left(\dfrac{1}{\gamma+1}\right) T^{\beta/2-\gamma} t^{\gamma+1}\delta^{-\beta/(k+\alpha)}\} & \text{if } -1<\gamma<0\,. \end{cases}\end{aligned}$$

Here, if we take

$$\delta = T\langle\xi\rangle^{-1/(1+\beta/2+\beta/(k+\alpha))},$$

and put

$$\tilde{\mu}(t) = t + t^\lambda + \psi(t)$$

with the parameter $\lambda = \min\{1, \gamma+1\}$, we have

$$E_\delta(t) \leqslant E_\delta(0)\exp\left\{(\varrho(t)-\varrho(0))\langle\xi\rangle_\nu^\kappa + C_{14}\tilde{\mu}(t)\langle\xi\rangle^{2\beta/(2\beta+(k+\alpha)(\beta+2))}\right\},$$

where $C_{14} = 1 + C_8 + C_9\max\left\{1, (1/(\gamma+1))\,T^{-1}\right\}$.

Supposing

$$\kappa > \frac{2\beta}{2\beta+(k+\alpha)(\beta+2)}\,(\equiv \kappa_0)\,, \tag{32}$$

we get

$$E_\delta(t) \leqslant E_\delta(0)\exp\left\{(\varrho(t)-\varrho(0))\langle\xi\rangle_\nu^\kappa + C_{14}\tilde{\mu}(t)\,\nu^{\kappa_0-\kappa}\right\}. \tag{33}$$

When $2 \leqslant k+\alpha \leqslant \infty$, we remark that if $\beta/2-\gamma-1<0$, the following holds

$$\frac{1}{2}\int_0^t \frac{|b|}{a_\delta^{1/2}}ds \leqslant \frac{1}{2}\int_0^t \frac{|b|}{a^{1/2}}ds \leqslant \frac{1}{2}\int_0^t \frac{C_2(1-s/T)^\gamma|\xi|}{\{C_1(1-s/T)^\beta|\xi|^2\}^{1/2}}ds =$$

$$= \frac{1}{2}C_1^{-1/2}C_2\int_0^t\left(1-\frac{s}{T}\right)^{\gamma-\beta/2}ds \leqslant \text{const}.$$

This implies that the lower term does'nt influence the well-posedness of the problem. Hence it is sufficient to invistigate the case $\beta/2-\gamma-1\geqslant 0$. With the parameter $\tau<1$, we get

$$\frac{1}{2}\int_0^t \frac{|b|}{a_\delta^{1/2}}ds \leqslant \frac{1}{2}\int_0^t \frac{|b|}{a_\delta^{1/2-(\gamma+\tau)/\beta}a_\delta^{(\gamma+\tau)/\beta}}ds \leqslant$$

$$\leqslant \frac{1}{2}\left\{C_1\left(\frac{\delta}{T}\right)^\beta|\xi|^2\right\}^{-1/2+(\gamma+\tau)/\beta}\int_0^t\frac{|b|}{a_\delta^{(\gamma+\tau)/\beta}}ds \leqslant$$

$$\leqslant \frac{1}{2}\left\{C_1\left(\frac{\delta}{T}\right)^\beta|\xi|^2\right\}^{-1/2+(\gamma+\tau)/\beta}\int_0^t\frac{C_2(1-s/T)^\gamma|\xi|}{\{C_1(1-s/T)^\beta|\xi|^2\}^{(\gamma+\tau)/\beta}}ds =$$

$$= \frac{1}{2}C_1^{-1/2}C_2T^{\beta/2-(\gamma+\tau)}\delta^{\gamma+\tau-\beta/2}\int_0^t\left(1-\frac{s}{T}\right)^{-\tau}ds =$$

$$= \frac{1}{2}C_1^{-1/2}C_2\frac{1}{1-\tau}T^{\beta/2-(\gamma+\tau)+1}\delta^{\gamma+\tau-\beta/2}\left\{1-\left(1-\frac{t}{T}\right)^{1-\tau}\right\}.$$

By (17) we get

$$\frac{1}{2}\int_0^t\frac{|b|}{a_\delta^{1/2}}ds \leqslant \begin{cases} \frac{1}{2}C_1^{-1/2}C_2T^{\beta/2-(\gamma+\tau)}t\delta^{\gamma+\tau-\beta/2} & \text{if } \tau\leqslant 0, \\ \frac{1}{2}C_1^{-1/2}C_2\frac{1}{1-\tau}T^{\beta/2-\gamma}t^{1-\tau}\delta^{\gamma+\tau-\beta/2} & \text{if } 0<\tau<1. \end{cases} \tag{34}$$

By (28), (29), (30), (34) we have the estimate

$$E_\delta(t) \leqslant E_\delta(0) \exp\{(\varrho(t) - \varrho(0))\langle\xi\rangle_\nu^\kappa + \psi(t)\, T^{\beta/(k+\alpha)}\delta^{-\beta/(k+\alpha)} + $$
$$+ C_8 T^{-1-\beta/2} t\delta^{1+\beta/2}|\xi| +$$
$$+\begin{cases} \frac{1}{2} C_1^{-1/2} C_2 T^{\beta/2-(\gamma+\tau)} t\delta^{\gamma+\tau-\beta/2}\} & \text{if } \tau \leqslant 0\,, \\ \frac{1}{2} C_1^{-1/2} C_2 \frac{1}{1-\tau} T^{\beta/2-\gamma} t^{1-\tau}\delta^{\gamma+\tau-\beta/2}\} & \text{if } 0 < \tau < 1\,. \end{cases}$$

Here, if we take

$$\delta = T\langle\xi\rangle^{\max\left\{\frac{-1}{1+(\beta/2)+\beta/(k+\alpha)},\ \frac{-1}{1-\tau+\beta-\gamma}\right\}},$$

and put

$$\tilde{\mu}(t) = t + t^\gamma + \psi(t)$$

with the parameter $\lambda = \min\{1, 1-\tau\}$, we have

$$E_\delta(t) \leqslant E_\delta(0) \exp\{(\varrho(t) - \varrho(0))\langle\xi\rangle_\nu^\kappa +$$
$$+ C_{15}\tilde{\mu}(t)\langle\xi\rangle^{\max\{2\beta/(2\beta+(k+\alpha)(\beta+2)),\,(\beta/2-\gamma-\tau)/(1+\beta-\gamma-\tau)\}}\}\,,$$

where $C_{15} = 1 + C_8 + (1/2)\, C_1^{-1/2} C_2 \max\{1, 1/(1-\tau)\, T^\tau\}$.

We remark that for any fixed

$$(35) \qquad \kappa > \max\left\{\frac{2\beta}{2\beta+(k+\alpha)(\beta+2)},\ \frac{\beta/2-\gamma-1}{\beta-\gamma}\right\}\ (\equiv \kappa_0),$$

there exists $\tau < 1$ such that

$$\kappa > \max\left\{\frac{2\beta}{2\beta+(k+\alpha)(\beta+2)},\ \frac{\beta/2-\gamma-\tau}{1+\beta-\gamma-\tau}\right\}\ (\equiv \kappa_1).$$

Then we get

$$(36) \qquad E_\delta(t) \leqslant E_\delta(0) \exp\{\langle\xi\rangle_\nu^\kappa(\varrho(t) - \varrho(0) + C_{15}\tilde{\mu}(t)\, \nu^{\kappa_1-\kappa})\}\,.$$

Taking place of (21) by (33),(36), and noting that if $k = 1, 2, \ldots, \beta - 2\gamma - 2 < 0$, by (32) it holds that

$$s\ (= \kappa^{-1}) < \frac{1}{\kappa_0} = 1 + \frac{(k+\alpha)(\beta+2)}{2\beta}$$

and if $k=2, 3, \ldots, \beta-2\gamma-2 \geqslant 0$, by (35) it holds that

$$s\ (=\kappa^{-1}) < \frac{1}{\kappa_0} = \frac{1}{\max\{2\beta/(2\beta+(k+\alpha)(\beta+2)), (\beta/2-\gamma-1)/(\beta-\gamma)\}} =$$

$$= 1 + \min\left\{\frac{(k+\alpha)(\beta+2)}{2\beta}, \frac{\beta+2}{\beta-2\gamma-2}\right\},$$

we can conclude the theorem.

Acknowledgments. The author wishes to thank heartily Prof. K. Kajitani, T. Hoshiro, and K. Yagdjian for many useful advices.

REFERENCES

[1] M. Cicognani, *On the strictly hyperbolic equations which are Hölder continuous with respect to time*, preprint.

[2] F. Colombini - E. De Giorgi - S. Spagnolo, *Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps*, Ann. Scuola Norm. Sup. Pisa, **6** (1979), pp. 511-559.

[3] F. Colombini - E. Jannelli - S. Spagnolo, *Wellposedness in the Gevrey classes of the Cauchy problem for a non strictly hyperbolic equation with coefficients depending on time*, Ann. Scuola Norm. Sup. Pisa, **10** (1983), pp. 291-312.

[4] P. D'Ancona, *Gevrey well posedness of an abstract Cauchy problem of weakly hyperbolic type*, Publ. RIMS Kyoto Univ., **24** (1988), pp. 433-449.

[5] P. D'Ancona, *Local existence for semilinear weakly hyperbolic equations with time dependent coefficients, Nonlinear Analysis. Theory, Methods and Applications*, Vol **21**, No. 9 (1993), pp. 685-696.

[6] V. Ya. Ivrii, *Cauchy problem conditions for hyperbolic operators with characteristics of variable multiplicity for Gevrey classes*, Siberian. Math., **17** (1976), pp. 921-931.

[7] K. Kajitani, *The well posed Cauchy problem for hyperbolic operators*, Exposé au Séminaire de Vaillant du 8 février (1989).

[8] T. Kinoshita, *On the wellposedness in the Gevrey classes of the Cauchy problem for weakly hyperbolic systems with Hölder continuous coefficients in t*, preprint.

[9] M. Reissig - K. Yagdjian, *Levi conditions and global Gevrey regularity for the solutions of quasilinear weakly hyperbolic equations*, Mathematische Nachrichten, **178** (1996), pp. 285-307.

[10] T. Nishitani, *Sur les équations hyperboliues à coefficients hölderiens en t et de classes de Gevrey en x*, Bull. Sci. Math., **107** (1983), pp. 739-773.

[11] H. Odai, *On the Cauchy problem for a hyperbolic equation of second order*, Doctoral thesis (1994).

Manoscritto pervenuto in redazione il 22 novembre 1996.