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Rendiconti del Seminario Matematico della Università di Padova, tome 99 (1998), p. 247-254

<http://www.numdam.org/item?id=RSMUP_1998__99__247_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 99 (1998)

Relations Between Localizations and I-Adic Completions in Commutative Domains.

PAOLO ZANARDO (*)

Introduction.

Let R be a commutative domain, endowed with a Hausdorff I-adic topology, where I is an ideal of R; let $\mathfrak{P}$ be a prime ideal of R containing I. If R is complete in its I-adic topology, it is natural to ask when the localization $R_{\mathfrak{P}}$ of R at $\mathfrak{P}$ is again complete in its $I_{\mathfrak{P}}$-adic topology, where $I_{\mathfrak{P}}$ denotes the extended ideal $IR_{\mathfrak{P}}$. Of course, the question makes sense only if $R_{\mathfrak{P}} \neq R$.

When R is a valuation domain, the answer is affirmative; in this case, every Hausdorff I-adic topology is equivalent to the natural topology (that is, every nonzero ideal is open). A valuation domain R endowed with the natural topology is complete if and only if every localization $R_{\mathfrak{P}}$ of R is complete in its own natural topology. This fact follows, for instance, from Theorems 1 and 15 of Matlis' book [6], since $R_{\mathfrak{P}}/R$ is a bounded R-module.

On the other hand, we shall give an example of a domain R with a non maximal prime ideal $\mathfrak{P}$ such that the $\mathfrak{P}$-adic topology is not equivalent to the natural topology, R is complete in the $\mathfrak{P}$-adic topology, and $R_{\mathfrak{P}}$ is complete in its $\mathfrak{P}_{\mathfrak{P}}$-adic topology.

However, the purpose of the present paper is to show that this situation cannot occur if the domain considered has enough good properties, for instance when it is noetherian. We shall always deal with the following situation: R is a commutative domain endowed with a Hausdorff I-adic topology, and $\mathfrak{P}$ is a prime ideal of R, containing I.

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Lavoro eseguito con il contributo del MURST.

We have a first result which actually follows from an old theorem by F. K. Schmidt [9]: if there exists a prime element $y \in R$ not in $\mathfrak{P}$ such that $\bigcap_n y^n R = \{0\}$, then $R_{\mathfrak{P}}$ is not complete in the $I_{\mathfrak{P}}$-adic topology (Theorem 1); note that we do not assume that R is complete in the I-adic topology. With a proof similar to that of Theorem 1, we get the remarkable fact that if R is a noetherian domain or a Krull domain (in particular a unique factorization domain) endowed with a Hausdorff I-adic topology, and $\mathfrak{P} \supseteq I$ is any prime ideal of R such that $R_{\mathfrak{P}} \neq R$, then $R_{\mathfrak{P}}$ is never complete in the $I_{\mathfrak{P}}$-adic topology (Proposition 2). We remark that the point of both results is to show that $R_{\mathfrak{P}}$ cannot be henselian with respect to the ideal $I_{\mathfrak{P}}$.

We shall say that a domain R satisfies the «Krull intersection property» (shorty: KIP) if for every proper ideal J of R, we have $\bigcap_n J^n = \{0\}$. Recall that Krull's theorem implies that every noetherian domain has the KIP (see e.g. Th.10.17. and Cor. 10.18. of [2]); D. D. Anderson in [1] proved that the KIP is satisfied by polynomial rings in any set of indeterminates over a noetherian domain and by the integral closure of noetherian domains; it is also important to observe that, in general, factorial domains do not satisfy the Krull intersection property (see the remark after Proposition 2).

Actually, we shall deal with those more general domains which satisfy the KIP on two-generated ideals. Let us then assume that R satisfies the KIP on two-generated ideals. We show in Theorem 3 that if I contains a prime element x of R and $R/xR \neq (R/xR)_{\mathfrak{P}/xR}$, then $R_{\mathfrak{P}}$ is never complete in its $I_{\mathfrak{P}}$-adic topology. Note that $R/xR \neq (R/xR)_{\mathfrak{P}/xR}$ holds when $R \neq R_{\mathfrak{P}}$ and either $\mathfrak{P}$ is not maximal or R is complete in the I-adic topology.

However, we remark that when R is not complete and $\mathfrak{P}$ is maximal, it can happen that $R_{\mathfrak{P}}$ is complete, as a consequence of the results in [10].

1. – For general notions and results about I-adic topologies we refer to [2] and [4].

We start showing an example of a domain R complete in its $\mathfrak{P}$-adic topology, $\mathfrak{P}$ non maximal, where $R_{\mathfrak{P}}$ is complete in the $\mathfrak{P}_{\mathfrak{P}}$-adic topology, and the $\mathfrak{P}$-adic topology is not equivalent to the natural one.

EXAMPLE. Let $R = \mathbb{Z} + (X, Y)\mathbb{Q}[[X, Y]]$ be the ring of formal power series over the indeterminates X, Y, with coefficients in $\mathbb{Q}$ and constant term in $\mathbb{Z}$; let $\mathfrak{P} = (X, Y)$; note that $\mathfrak{P}$ is not a maximal ideal: $R \supsetneq (a, X, Y) \supsetneq (X, Y)$ for all $a \in \mathbb{Z}$ nonzero and not a unit. It is plain that R is complete in the $\mathfrak{P}$-adic topology: the limit of a Cauchy sequence of formal power series with constant term in $\mathbb{Z}$ has constant term in $\mathbb{Z}$. It is also immediate to check that $R_{\mathfrak{P}} = \mathbb{Q}[[X, Y]]$, therefore $R_{\mathfrak{P}}$ is complete in its $\mathfrak{P}_{\mathfrak{P}}$-adic topology. Finally, the $\mathfrak{P}$-adic topology of R is not equivalent to the natural one: the ideal (X) of R is not open, since it does not contain any power of $\mathfrak{P}$. Of course, R is neither noetherian nor a UFD: if $0 \neq a \in \mathbb{Z}$ is not a unit in $\mathbb{Z}$, then a is not a unit in R, too; but $a^{-n}X \in R$ for all $n \in \mathbb{N}$ implies that $X \in \bigcap_n a^n R$, which cannot happen in a factorial or in a noetherian domain.

In the sequel we shall not assume that R is complete in the I-adic topology.

I owe the following observations, concerned with the next Theorem 1, to the referee of a previous version of the present paper, where Theorem 1 was proved in a different, and much longer, way.

We recall a result by F. K. Schmidt ([9]; see also [3], Satz 2.3.11. p. 60): if a local domain D with maximal ideal $\mathfrak{M}$ is henselian, then any discrete valuation ring of the field of fractions of D must contain D. Our Theorem 1 follows by this result, with slight modifications of the argument, to take care of the fact that the ideal $I_{\mathfrak{P}}$ is not necessarily maximal in $R_{\mathfrak{P}}$; we have thought it appropriate to give a direct proof of it, based on the argument in [3].

Recall that a local ring R is henselian with respect to its ideal I if the following property is satisfied: let f, g_0, h_0 be monic polynomials in $R[X]$, and let $\overline{f}$, $\overline{g}_0$, $\overline{h}_0$ denote their reductions in $(R/I)[X]$; if $\overline{f} = \overline{g}_0\overline{h}_0$, and $\overline{g}_0, \overline{h}_0$ are coprime in $(R/I)[X]$ (which means $\overline{g}_0(R/I)[X] + \overline{h}_0(R/I)[X] = (R/I)[X]$), then there exist $g, h \in R[X]$ such that $f = gh$ and $\overline{g} = \overline{g}_0$, $\overline{h} = \overline{h}_0$. If R is complete in the I-adic topology, then it is henselian with respect to I (see Th. 3.7 of [4]).

THEOREM 1. *Let R be a commutative domain endowed with a Hausdorff I-adic topology, and let $\mathfrak{P}$ be a prime ideal of R which contains I. If there exists a prime element $y \in R$ not in $\mathfrak{P}$, such that $\bigcap_n y^n R = \{0\}$, then $R_{\mathfrak{P}}$ is not henselian with respect to $I_{\mathfrak{P}}$. In particular $R_{\mathfrak{P}}$ is not complete in the $I_{\mathfrak{P}}$-adic topology.*

PROOF. By contradiction, let us assume that $R_{\mathfrak{P}}$ is henselian with respect to $I_{\mathfrak{P}}$. Let L be the field of fractions of R; the conditions on y imply that $R_{(y)}$ is the rank one discrete valuation ring of the y-adic valuation defined on L, which we denote by w. Let us first verify that $I_{\mathfrak{P}} \subset R_{(y)}$. We fix $a \in I_{\mathfrak{P}}$. Let p_0 be the characteristic of $R_{\mathfrak{P}}$ and let us denote by $\overline{h}$ the class of the integer h modulo $p_0\mathbb{Z}$; let us observe that $\mathfrak{P}$ may contain at most one element of the form $\overline{p}$, with p a prime integer; hence, for almost all prime numbers q, the class $\overline{q}$ is a unit of the local ring $R_{\mathfrak{P}}$. Now, for every positive prime integer q different from the characteristic of R/I and such that $\overline{q}$ is a unit of $R_{\mathfrak{P}}$, the polynomial $X^q - 1$ in $(R_{\mathfrak{P}}/I_{\mathfrak{P}})[X]$ is the product of $X-1$ by $u(X) = X^{q-1} + \ldots + q + 1$; by the choice of q, the polynomials $X-1$ and $u(X)$ are coprime in $(R_{\mathfrak{P}}/I_{\mathfrak{P}})[X]$, since $\overline{q} + I_{\mathfrak{P}} \in \in (X-1, u(X))(R_{\mathfrak{P}}/I_{\mathfrak{P}})[X]$ and $\overline{q} + I_{\mathfrak{P}}$ is a unit of $(R_{\mathfrak{P}}/I_{\mathfrak{P}})$. The henselianity of $R_{\mathfrak{P}}$ now ensures the existence of $\eta_q \in R_{\mathfrak{P}}$ which is a root of the polynomial $X^q - (1+a) \in R_{\mathfrak{P}}[X]$. From $\eta_q^q = 1 + a$ we deduce that $w(1 + + a) = qw(\eta_q)$ is divisible by arbitrarily large integers. This is possible only if $w(1+a) = 0$, whence $1 + a \in R_{(y)}$ and so $a \in R_{(y)}$, too. Since a was arbitrary, we have $I_{\mathfrak{P}} \subset R_{(y)}$. We now show that $I_{\mathfrak{P}} \subset R_{(y)}$ implies $R_{\mathfrak{P}} \subset R_{(y)}$. In fact, if $b \in R_{\mathfrak{P}}$ is such that $w(b) < 0$, then, for any $a \in I_{\mathfrak{P}}$, there exists a positive integer m such that $w(b^m a) < 0$, whence $b^m a \notin R_{(y)}$; but $b^m a \in \in I_{\mathfrak{P}} \subset R_{(y)}$, impossible. On the other hand, $R_{\mathfrak{P}} \subset R_{(y)}$ gives a plain contradiction, since y is a unit of $R_{\mathfrak{P}}$ but not a unit in $R_{(y)}$. We conclude that $R_{\mathfrak{P}}$ cannot be henselian with respect to $I_{\mathfrak{P}}$, as desired. ∎

We observe that, in Theorem 1, it is essential to suppose that $\bigcap_n y^n R = \{0\}$, otherwise we could be in a situation like that of the preceding example. Our next proposition extends the analogous result proved in Theorem 62 of [6] for principal ideals domains. We are in the most general setting: we simply exclude that, when the domain R is local, $\mathfrak{P}$ is the unique maximal ideal of R; of course, this assumption is minimal, since otherwise $R = R_{\mathfrak{P}}$ could be complete (or just henselian). For the definition of *Krull domain* see, e.g., Nagata's book [8].

PROPOSITION 2. *Let R be an integral domain whose integral closure is a Krull domain. Let R be endowed with a Hausdorff I-adic topology, and let $\mathfrak{P}$ be a prime ideal of R which contains I and is such that $R \neq R_{\mathfrak{P}}$. Then $R_{\mathfrak{P}}$ is not henselian with respect to $I_{\mathfrak{P}}$, and hence is not complete in the $I_{\mathfrak{P}}$-adic topology. In particular, the assertion holds if R is a noetherian domain or a unique factorization domain.*

PROOF. Let S denote the integral closure of R. Let us choose a non-unit $y \in R \backslash \mathfrak{P}$. By the definition of Krull domain, y is contained in a prime ideal J of S such that S_J is a discrete valuation ring. Let us now assume, by contradiction, that $R_{\mathfrak{P}}$ is henselian with respect to $I_{\mathfrak{P}}$. Arguing as in the proof of Theorem 1, we can prove that $S_J \supset R_{\mathfrak{P}}$, impossible, since y is not a unit of S_J. Let us now observe that a UFD is a Krull domain, and that the integral closure of a noetherian domain is a Krull domain (see §33 of [8]). The desired conclusion follows. ∎

REMARK. To see that Proposition 2 and the subsequent Theorem 3 do not overlap, it is worth noting that not every factorial domain satisfies the Krull intersection property. This fact appears rather reasonable, but counterexamples seem not to be commonly known.

D. F. Anderson suggested examination of Example 5.7. in the paper [5], which exhibits a Krull domain T containing a two-generated ideal J such that $\bigcap_n J^n$ is nonzero. We observe that this example is due to P. Eakin and that the authors of [5] did not need that T be a UFD, for their purposes.

As a matter of fact, T turns out to be a UFD, which does not satisfy the KIP on two-generated ideals. We omit the description of T and the verification that it is actually a UFD, since this matter is extraneous to the topic of the present paper. I thank R. Gilmer and D. F. Anderson for the help they gave to me in finding this example.

The above recalled Schmidt's theorem is not useful to prove the next result. Typical examples of domains which satisfy the KIP only on finitely generated ideals, not in general, are valuations domains V of Krull dimension one; for such a V, it is also easily proved that $V[\![X]\!]$ satisfies the KIP on two-generated ideals.

THEOREM 3. *Let R be a domain satisfying the Krull intersection property on two-generated ideals, endowed with a I-adic topology, where the ideal I contains a prime element x of R; let $\mathfrak{P} \supseteq I$ be a prime ideal of R, such that $R/xR \neq (R/xR)_{\mathfrak{P}/xR}$. Then $R_{\mathfrak{P}}$ is not complete in its $I_{\mathfrak{P}}$-adic topology.*

PROOF. Again by contradiction, we assume that $R_{\mathfrak{P}}$ is complete. Let us first note that the condition $R/xR \neq (R/xR)_{\mathfrak{P}/xR}$ is equivalent to say that there exists $y \in R \backslash \mathfrak{P}$ such that (x, y) is a proper ideal of R. In fact $\mathfrak{P}/Rx$ is not the unique maximal ideal of R/xR if and only if $y + xR$ is not a unit of R/xR for some $y \in R \backslash \mathfrak{P}$, if and only if $1 \notin (x, y)$. Let us then choose $y \in R \backslash \mathfrak{P}$ such that the ideal (x, y) of R is proper. Now, $y \notin \mathfrak{P}$ implies that $x^n y^{-k} \in I_{\mathfrak{P}}^n$ for all positive integers n, k. Thus we deduce that, for every sequence $\{a_n\}$ of elements of R and every map $\phi\colon n \mapsto \phi n$, the elements $\sum_{n=0}^{k} a_n x^n / y^{\phi n} \in R_{\mathfrak{P}}$ form a Cauchy sequence with respect to the $I_{\mathfrak{P}}$-adic topology; since $R_{\mathfrak{P}}$ is complete, the sequence has a limit in $R_{\mathfrak{P}}$, which we shall denote, as usual, by $\sum_{n=0}^{\infty} a_n x^n / y^{\phi n}$. In particular, we have that $\sum_{n=0}^{\infty} (x/y)^{n!} \in R_{\mathfrak{P}}$. Then we can write

$$(1) \qquad \sum_{n=0}^{\infty} (x/y)^{n!} = f/g \qquad \text{where} \quad f, \ g \in R, \ \ g \notin \mathfrak{P} .$$

Let us now fix a positive integer k. Multiplying both members of (1) by $gy^{k!}$ we get:

$$g(y^{k!}+x^{1!}y^{k!-1!}+\ldots+x^{(k-1)!}y^{k!-(k-1)!}+x^{k!}+\sum_{n>k}x^{n!}/y^{n!-k!})=y^{k!}f; \tag{2}$$

from (2) we readily obtain

$$gx^{k!}\left(1+\sum_{n>k}(x/y)^{n!-k!}\right)=y^k h_k \tag{3}$$

for a suitable $h_k\in R$. Now, since $n!-k!\geqslant k$ and $R_{\mathfrak{P}}$ is complete, we can write

$$\sum_{n>k}(x/y)^{n!-k!}=x^k\sum_{n>k}x^{n!-k!-k}/y^{n!-k!}=x^k u_k/v_k\ , \tag{4}$$

with u_k, $v_k\in R$ and $v_k\notin\mathfrak{P}$, since $\sum_{n>k}x^{n!-k!-k}/y^{n!-k!}\in R_{\mathfrak{P}}$. Therefore from (3) and (4) we get

$$x^{k!}g(v_k+x^k u_k)=y^k h_k v_k; \tag{5}$$

since x is a prime element and does not divide $y^k v_k$, then $x^{k!}$ must divide h_k, $h_k=x^{k!}f_k$, say. Then (5) gives

$$v_k(g-f_k y^k)=-gu_k x^k; \tag{6}$$

again, since x is prime and $v_k\notin\mathfrak{P}$, we deduce that x^k divides $g-f_k y^k$ so that there exists $g_k\in R$ such that

$$g=f_k y^k+g_k x^k\in(x,\ y)^k\ . \tag{7}$$

Since k was arbitrary, $(x,\ y)$ is a proper ideal and R satisfies the KIP on two-generated ideals, from (7) we obtain $g=0$, impossible. The desired conclusion follows. ■

In the above notation, the condition $R/xR\neq(R/xR)_{\mathfrak{P}/xR}$ is trivially satisfied if $\mathfrak{P}$ is not a maximal ideal. It is worth noting that the condition also holds if $R\neq R_{\mathfrak{P}}$ and R is complete in the I-adic topology. In fact in this case I is contained in the Jacobson radical $J(R)$ of R. If now $y\notin\mathfrak{P}$ is not a unit of R, we have that $(x,\ y)$ is a proper ideal of R: from $1=ax+by$ $(a,\ b\in R)$ it follows that $by=1-ax$ is a unit, since $x\in J(R)$, whence y is a unit, impossible. From $(x,\ y)$ proper it follows $R/xR\neq(R/xR)_{\mathfrak{P}/xR}$, as shown in the above proof.

A case not covered by the hypothesis of Theorem 3 is when R is not complete and $\mathfrak{P}$ is a maximal ideal. In view of the results of [10], this case deserves some comments, which are the object of our final remark.

REMARK. In Theorem 7 of [10] it is proved that every domain T which is complete in a $\mathfrak{M}$-adic topology, $\mathfrak{M}$ a maximal ideal of T, is of the form $T = R_{\mathfrak{P}}$, where R is a suitable subring of T and $\mathfrak{P}$ is a maximal ideal of R such that R is not complete in its $\mathfrak{P}$-adic topology and $\mathfrak{M} = \mathfrak{P}R_{\mathfrak{P}}$. As a first consequence, it can happen that $R_{\mathfrak{P}}$ is complete in the $I_{\mathfrak{P}}$-adic topology while R is not complete in the I-adic topology. Moreover, the assumption in Theorem 3 that $R/xR \neq (R/xR)_{\mathfrak{P}/xR}$ cannot be deleted, as shown by the following example, based on the ideas of [10] (see also [7]), where R/xR will be a field. Let T be a complete DVR, with field of fractions Q; let V be a valuation domain of Q, not contained nor containing T and with Krull rank one. V may be constructed as follows (see also Prop. 3 of [7]). Let F be the prime subfield of Q and let B be a basis of transcendence of Q over F. Note that B is infinite, since T is a complete DVR. We may assume that B contains a *unit* t of T. Let us consider the field $L = F(B \setminus \{t\})$ and the discrete t-adic valuation v_t of the field $L(t)$. Since Q is an algebraic extension of $L(t) = F(B)$, we may extend v_t to a rank one valuation w of Q. Let V be the valuation domain associated to w. Note that t is a unit in T but not in V, whence the two rank one valuation domains T and V are incomparable by inclusion. Therefore V fullfils our requirements. Let us now set $R = T \cap V$; by Theorem 11.11 p. 38 of [8], R has exactly two maximal ideals, namely $xR = xT \cap R$, where xT is the maximal ideal of R (we can choose $x \in R$), and $\mathfrak{H} = \mathfrak{J} \cap R$, where $\mathfrak{J}$ is the maximal ideal of V. It follows that R satisfies the KIP on finitely generated ideals, since T and V are both of rank one. Setting $\mathfrak{P} = I = xR$, we have $R_{\mathfrak{P}} = T$ (Theorem 11.11 of [8]), so that $R_{\mathfrak{P}}$ is complete in the $I_{\mathfrak{P}}$-topology.

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Manoscritto pervenuto in redazione il 21 giugno 1996
e, in forma revisionata, il 22 novembre 1996.