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New Linking Theorems.

MARTIN SCHECHTER (*)

SUMMARY - We prove new linking theorems related to those of Schechter-Tintarev which allow us to obtain linking for sets which did not link under the older theories. This allows us to prove new theorems for nonlinear problems.

1. - Introduction.

Let G be a C^1 functional on a Banach space E , and assume that $E = M \oplus N$, where M, N are closed subspaces, one of which is finite dimensional. Assume that

$$(1.1) \quad a_0 := \sup_{M \cap \partial B_\delta} G \leq b_0 := \inf_N G$$

for some $\delta > 0$, where

$$(1.2) \quad B_r = \{u \in E : \|u\| < r\}.$$

One of the results of the present paper is

THEOREM 1.1. *Under the above hypotheses, there is a sequence $\{u_k\} \subset E$ such that*

$$(1.3) \quad G(u_k) \rightarrow c, \quad b_0 \leq c < \infty, \quad (1 + \|u_k\|) G'(u_k) \rightarrow 0.$$

Interest in such a theorem stems from the fact that for many applica-

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tions, (1.3) implies the existence of a solution of

$$(1.4) \quad G(u) = c, \quad G'(u) = 0.$$

We shall present some of these applications here. When $\dim M < \infty$, Theorem 1.1 is well known (cf. [Ra, Theorem 4.6]). However, the proof rests completely on the fact that M is finite dimensional. This is so much so, that no one seems to have suspected that the theorem is true even when $\dim M = \infty$. We shall show that this indeed is the case. As a result we can solve problems which could not be considered before.

We apply Theorem 1.1 to semilinear boundary value problems. Let Ω be a bounded domain in \mathbf{R}^n , and let A be a selfadjoint operator on $L^2(\Omega)$ with compact resolvent and eigenvalues

$$0 < \lambda_0 < \lambda_1 < \dots < \lambda_j < \dots$$

We assume that the eigenfunctions of A are bounded. Let $f(x, t)$ be a Carathéodory function on $\Omega \times \mathbf{R}$ satisfying

$$(1.4) \quad |f(x, t)| \leq C|t| + V(x), \quad x \in \Omega, \quad t \in \mathbf{R}$$

and

$$(1.5) \quad f(x, t)/t \rightarrow \alpha_{\pm}(x) \quad \text{a.e. as } t \rightarrow \pm \infty$$

where $V(x) \in L^2(\Omega)$ and the only solution of

$$(1.6) \quad Au = \alpha_+ u^+ - \alpha_- u^-, \quad u^{\pm} = \max\{\pm u, 0\}$$

is $u \equiv 0$. We let

$$(1.7) \quad F(x, t) = \int_0^t f(x, s) ds.$$

We have

THEOREM 1.2. *Assume that for some $l > 0$ there are constants $\nu > \lambda_{l-1}$ and $\delta > 0$ such that*

$$(1.8) \quad \nu t^2 \leq 2F(x, t), \quad x \in \Omega, \quad t \in \mathbf{R},$$

$$(1.9) \quad \lambda_l t^2 \leq 2F(x, t), \quad x \in \Omega, \quad |t| < \delta,$$

$$(1.10) \quad \alpha_{\pm}(x) \leq \lambda_l, \quad x \in \Omega.$$

Then the equation

$$(1.11) \quad Au = f(x, u)$$

has at least one nontrivial solution.

We also have

THEOREM 1.3. *Assume that for some $l \geq 0$ there are constants $\nu < \lambda_{l+1}$ and $\delta > 0$ such that*

$$(1.12) \quad 2F(x, t) \leq \nu t^2, \quad x \in \Omega, \quad t \in \mathbf{R},$$

$$(1.13) \quad 2F(x, t) \leq \lambda_l t^2, \quad x \in \Omega, \quad |t| < \delta,$$

$$(1.14) \quad \lambda_l \leq \alpha_{\pm}(x), \quad x \in \Omega.$$

Then (1.11) has at least one nontrivial solution.

The equation (1.6) approximates (1.11) when $|u(x)|$ is large. Theorem 1.2 cannot be proved by using previous linking theorems. On the other hand, Theorem 1.3 does follow [Si, Theorem 1.15]. It is included here because of its similarity to Theorem 1.2.

Theorem 1.1 is proved in Section 4 along with other theorems on linking stated in Section 2. Theorems 1.2 and 1.3 are proved in Section 3. They are based on a slight variation of Theorem 1.1. Other linking methods can be found in [MW, BN, Ra, Si].

2. - The method.

We present a refined version of the new linking concept introduced in [ST]. Let E be a Banach space and let Φ be the set of all continuous maps $\Gamma = \Gamma(t)$ from $E \times [0, 1]$ to E such that

- 1) $\Gamma(0) = I$, the identity map.
- 2) For each $t \in [0, 1)$, $\Gamma(t)$ is a homeomorphism of E onto E and $\Gamma^{-1}(t) \in C(E \times [0, 1), E)$.
- 3) $\Gamma(1)E$ is a single point in E and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \rightarrow 1$ for each bounded set $A \subset E$.

4) For each $t_0 \in [0, 1)$ and each bounded set $A \subset E$

$$(2.1) \quad \sup_{\substack{0 \leq t \leq t_0 \\ u \in A}} \{ \|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\| \} < \infty .$$

DEFINITION. For $A, B \subset E$ we say that A links B if

a) $A \cap B = \emptyset$,

b) for each $\Gamma \in \Phi$ there is a $t \in (0, 1]$ such that

$$(2.2) \quad \Gamma(t) A \cap B \neq \emptyset .$$

We have

PROPOSITION 2.1. *If $A, B \subset E$ are closed and bounded, $E \setminus A$ is pathwise connected and A links B , then B links A .*

PROPOSITION 2.2. *If $F \in C(E, \mathbf{R}^n)$ and $Q \subset E$ is such that $F_0 = F|_Q$ is a homeomorphism of Q onto the closure of a bounded open subset Ω of \mathbf{R}^n , then $\partial Q \equiv F_0^{-1}(\partial\Omega)$ links $F^{-1}(p)$ for each $p \in \Omega$.*

THEOREM 2.3. *Let G be a C^1 functional on E , and let A, B be subsets of E such that A is bounded and links B . Assume*

$$(2.3) \quad a_0 := \sup_A G \leq b_0 := \inf_B G ,$$

$$(2.4) \quad a := \inf_{\Gamma \in \Phi} \sup_{\substack{0 \leq s \leq 1 \\ u \in A}} G(\Gamma(s)u) < \infty .$$

Let $\psi(t)$ be a positive nonincreasing function on $(0, \infty)$ such that

$$(2.5) \quad \int_{\infty}^1 \psi(r) dr = \infty .$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$(2.6) \quad G(u_k) \rightarrow a , \quad G'(u_k) = o(\psi(\|u_k\|)) .$$

COROLLARY 2.4. *Under the hypotheses of Theorem 2.3, there is a sequence $\{u_k\} \subset E$ such that*

$$(2.7) \quad G(u_k) \rightarrow a , \quad (1 + \|u_k\|) G'(u_k) \rightarrow 0 .$$

PROPOSITION 2.5. *Let H be a homeomorphism of E onto itself such that H and H^{-1} map bounded sets into bounded sets. If $A, B \subset E$ and A links B , then HA links HB .*

PROPOSITION 2.6. *Let $A, B_n, n = 1, 2, \dots$, be subsets of E such that A is bounded and links B_n for each n . Suppose*

$$(2.8) \quad B_n = B'_n \cup B''_n$$

where

$$(2.9) \quad d(B''_n, 0) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

and there is a set $B \subset E$ such that

$$(2.10) \quad A \cap B = \emptyset, \quad B'_n \subset B, \quad n = 1, 2, \dots$$

Then A links B .

COROLLARY 2.7. *Let M, N be closed subspaces of E , one of which is finite dimensional and such that*

$$(2.11) \quad E = M \oplus N.$$

If

$$(2.12) \quad B_R := \{u \in E: \|u\| < R\}$$

then $M \cap \partial B_R$ links N for each $R > 0$.

COROLLARY 2.8. *Let M, N be closed subspaces of E such that (2.11) holds with one of them being finite dimensional. Let w_0 be an element of $M \setminus \{0\}$, and let $0 < r < R$,*

$$A = \{w \in M: \|w\| = R\},$$

$$B = \{v \in N: \|v\| \geq r\} \cup \{u = v + sw_0: v \in N, s \geq 0, \|u\| = r\}.$$

Then A links B .

3. – The application.

We now give the proof of Theorems 1.2, 1.3.

PROOF OF THEOREM 1.2. Let

$$(3.1) \quad G(u) = \|u\|_D^2 - 2 \int_{\Omega} F(x, u) dx, \quad u \in D$$

where $D = D(A^{1/2})$ and

$$(3.2) \quad \|u\|_D = \|A^{1/2}u\|.$$

With this norm D becomes a Hilbert space. Under hypothesis (1.4) it is easily shown that G is a C^1 functional on D and

$$(3.3) \quad (G'(u), v)/2 = (u, v)_D - (f(u), v).$$

From this it follows that u is a solution of (1.10) if

$$(3.4) \quad G'(u) = 0.$$

Let N be the subspace of $L^2(\Omega)$ spanned by the eigenfunctions of A corresponding to the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_l$, and let $E(\lambda_l)$ be the eigenspace of λ_l . Let $M = N^\perp \cap D$. By (1.8) we have

$$(3.5) \quad G(v) \leq \|v\|_D^2 - \nu \|v\|^2 \leq \left(1 - \frac{\nu}{\lambda_{l-1}}\right) \|v\|_D^2, \quad v \in N_{l-1}.$$

Moreover, I claim that for each $\varrho > 0$ sufficiently small either (a) there is a $y \in E(\lambda_l) \setminus \{0\}$ satisfying

$$(3.6) \quad Ay = f(x, y) = \lambda_l y, \quad \|y\|_D = \varrho$$

or (b) there is an $\varepsilon > 0$ such that

$$(3.7) \quad G(v + y) \leq -\varepsilon, \quad v \in N_{l-1}, \quad y \in E(\lambda_l), \quad \|v + y\|_D = \varrho.$$

Assume this for the moment. Since (3.6) exhibits a nontrivial solution of (1.11), we need only address option (b). Let ϱ, ε be such that (3.7) holds. Let $y_0 \in E(\lambda_l) \setminus \{0\}$ and take

$$(3.8) \quad B = \{v \in N_{l-1}: \|v\|_D \geq \varrho\} \cup \\ \cup \{u = v + sy_0: v \in N_{l-1}, s \geq 0, \|u\|_D = \varrho\}.$$

Then (3.5) and (3.7) imply

$$(3.9) \quad G(v) \leq -\varepsilon_0, \quad v \in B$$

for some $\varepsilon_0 > 0$. On the other hand, (1.10) implies

$$(3.10) \quad G(w) \rightarrow \infty \quad \text{as } \|w\|_D \rightarrow \infty, \quad w \in M_{l-1}.$$

To see this, let $\{w_k\}$ be any sequence in $M = M_{l-1}$ such that $\varrho_k =$

$= \|w_k\|_D \rightarrow \infty$. Let $\tilde{w}_k = w_k/\varrho_k$. Then

$$(3.11) \quad G(w_k)/\varrho_k^2 = 1 - 2 \int_{\Omega} F(x, w_k)/\varrho_k^2 dx .$$

Now $\|\tilde{w}_k\|_D = 1$. Hence there is a renamed subsequence such that $\tilde{w}_k \rightarrow \tilde{w}$ weakly in D , strongly in $L^2(\Omega)$ and a.e. in Ω . Moreover, (1.4) implies

$$(3.12) \quad |F(x, t)| \leq Ct^2 + V(x) |t|$$

and (1.5) implies

$$(3.14) \quad 2F(x, t)/t^2 \rightarrow \alpha_{\pm}(x) \quad \text{a.e. as } t \rightarrow \pm \infty .$$

Thus

$$(3.15) \quad 2 \int_{\Omega} F(x, w_k) dx/\varrho_k^2 \rightarrow \int_{\Omega} \{ \alpha_+(\tilde{w}^+)^2 + \alpha_-(\tilde{w}^-)^2 \} dx , \quad k \rightarrow \infty$$

and

$$(3.16) \quad G(w_k)/\varrho_k^2 \rightarrow 1 - \int_{\Omega} \{ \alpha_+(\tilde{w}^+)^2 + \alpha_-(\tilde{w}^-)^2 \} dx , \quad k \rightarrow \infty .$$

Since $\|\tilde{w}\|_D \leq 1$, (1.10) implies that the right hand side of (3.16) is ≥ 0 . The only way it could vanish is if $\tilde{w} \in E(\lambda_l)$ and

$$(3.17) \quad \int_{\Omega} \{ (\lambda_l - \alpha_+)(\tilde{w}^+)^2 + (\lambda_l - \alpha_-)(\tilde{w}^-)^2 \} dx = 0 .$$

Since the integrand in (3.17) is nonnegative, we must have

$$\begin{aligned} \alpha_+(x) &\equiv \lambda_l & \text{when } \tilde{w}(x) > 0 , \\ \alpha_-(x) &\equiv \lambda_l & \text{when } \tilde{w}(x) < 0 . \end{aligned}$$

From this it follows that \tilde{w} is a solution of (1.6). By hypothesis, this implies that $\tilde{w} \equiv 0$, showing that the right hand side of (3.16) does not vanish. Hence the left hand side of (3.16) converges to a positive limit for every such sequence, showing that (3.10) holds. Once we know this, we take R such that

$$(3.18) \quad G(w) \geq 0 , \quad w \in M_{l-1} \cap \partial B_R \equiv A .$$

Let $G_1(u) = -G(u)$. Then

$$(3.19) \quad \sup_A G_1 \leq 0 < \varepsilon_0 \leq \inf_B G_1.$$

By Corollary 2.8, A links B . Hence there is a sequence $\{u_k\} \subset D$ such that

$$(3.20) \quad G_1(u_k) \rightarrow c_1, \quad \varepsilon_0 \leq c_1 < \infty, \quad G_1'(u_k) \rightarrow 0.$$

Thus

$$(3.21) \quad \|u_k\|_D^2 - 2 \int_{\Omega} F(x, u_k) dx \rightarrow -c_1$$

and

$$(3.22) \quad (u_k, v)_D - (f(u_k), v) \rightarrow 0, \quad v \in D$$

where we write $f(u)$ in place of $f(x, u)$. If $\varrho_k = \|u_k\|_D \rightarrow \infty$, we let $\tilde{u}_k = u_k/\varrho_k$. Then $\|\tilde{u}_k\|_D = 1$ and there is a renamed subsequence such that $\tilde{u}_k \rightarrow \tilde{u}$ weakly in D , strongly in $L^2(\Omega)$ and a.e. in Ω . We then obtain

$$(3.23) \quad \int_{\Omega} \{\alpha_+ (\tilde{u}^+)^2 + \alpha_- (\tilde{u}^-)^2\} dx = 1$$

from (3.21) and

$$(3.24) \quad (\tilde{u}, v)_D = \int_{\Omega} (\alpha_+ \tilde{u}^+ - \alpha_- \tilde{u}^-) v dx, \quad v \in D$$

from (3.22). This implies that $\|\tilde{u}\|_D^2$ equals the left hand side of (3.23) and that \tilde{u} is a solution of (1.6). Thus by hypothesis we must have $\tilde{u} \equiv 0$. But this contradicts (3.23). Hence the ϱ_k are bounded. We can now apply well known techniques to show that there is a solution of (1.11) satisfying $G_1(u) = c_1 \geq \varepsilon_0 > 0$. Since $G_1(0) = 0$, we see that u is a nontrivial solution of (1.11).

It remains to prove (3.7). First we have

LEMMA 3.1. *If (1.9) holds, then for each $\varrho > 0$ sufficiently small there is a positive ε such that*

$$(3.25) \quad G(v + y) \leq -\varepsilon \|v\|_D^2, \quad v \in N_{l-1}, \quad y \in E(\lambda_l), \quad \|v + y\|_D \leq \varrho.$$

PROOF. Since $E(\lambda_l) \subset L^\infty(\Omega)$, there is a $\varrho > 0$ such that

$$(3.26) \quad \|y\|_D \leq \varrho \text{ implies } \|y\|_\infty \leq \delta/2, \quad y \in E(\lambda_l)$$

where δ is the constant in (1.9). Let $w = v + y$, where $v \in N_{l-1}$, $y \in E(\lambda_l)$.
If

$$(3.27) \quad \|w\|_D \leq \rho \quad \text{and} \quad |w(x)| \geq \delta$$

then

$$(3.28) \quad \delta \leq |w(x)| \leq |v(x)| + |y(x)| \leq |v(x)| + \delta/2.$$

Consequently, (3.27) implies

$$(3.30) \quad |w(x)| \leq 2|v(x)|.$$

By (3.12)

$$\begin{aligned} G(w) &\leq \|w\|_D^2 - \lambda_l \int_{|w| < \delta} w^2 dx + C \int_{|w| > \delta} (w^2 + |w|) dx \leq \\ &\leq \|w\|_D^2 - \lambda_l \|w\|^2 + C' \int_{|w| > \delta} (w^2 + |w|) dx \leq \\ &\leq \|v\|_D^2 - \lambda_l \|v\|^2 + C'' \int_{2|v| > \delta} |v|^\sigma dx \leq \left(1 - \frac{\lambda_l}{\lambda_{l-1}} - C''' \|v\|_D^{\sigma-2}\right) \|v\|_D^2 \end{aligned}$$

where $\sigma > 2$. If we take ρ sufficiently small, this implies (3.25). ■

Once we have inequality (3.25), we prove (3.7) by assuming, on the contrary, that there is a sequence $w_k = v_k + y_k$, $v_k \in N_{l-1}$, $y_k \in E(\lambda_l)$ such that $\|w_k\|_D = \rho$ and

$$(3.32) \quad G(w_k) \rightarrow 0.$$

By (3.25) we see that $v_k \rightarrow 0$. Thus $\|y_k\|_D \rightarrow \rho$. Since $E(\lambda_l)$ is finite dimensional, there is a renamed subsequence such that $y_k \rightarrow y$ in $E(\lambda_l)$ and $\|y\|_D = \rho$. By (3.32)

$$(3.33) \quad G(y) = 0.$$

If ρ is such that (3.27) holds, then (1.9) implies

$$(3.34) \quad \lambda_l y(x)^2 \leq 2F(x, y(x)), \quad x \in \Omega.$$

But (3.33) says

$$(3.35) \quad \int_{\Omega} \{\lambda_l y(x)^2 - 2F(x, y(x))\} dx = 0.$$

From (3.34) and (3.35) we see that

$$\lambda_l y(x)^2 \equiv F(x, y(x)), \quad x \in \Omega.$$

Let $\zeta(x)$ be any function in $C_0^\infty(\Omega)$. Then for $t > 0$ sufficiently small

$$\lambda_l [(y + t\zeta)^2 - y^2]/t \leq 2[F(x, y + t\zeta) - F(x, y)]/t.$$

If we take the limit as $t \rightarrow 0$, we have

$$\lambda_l y(x)^2 \zeta(x) \leq f(x, y(x)) \zeta(x), \quad x \in \Omega.$$

From this we conclude that

$$\lambda_l y(x)^2 \equiv f(x, y(x)), \quad x \in \Omega.$$

Since $y \in E(\lambda_l)$, this implies that y is a solution of (3.6), the option which we discarded. This completes the proof of the theorem. ■

PROOF OF THEOREM 1.3. We only sketch the proof because of its similarity to that of Theorem 1.2. We use the notation of the proof of Theorem 1.2. By (1.12)

$$(3.36) \quad G(w) \geq \|w\|_D^2 - \nu \|w\|^2 \geq \left(1 - \frac{\nu}{\lambda_{l+1}}\right) \|w\|^2 D, \quad w \in M.$$

Moreover, (1.13) implies that for each $\varrho > 0$ sufficiently small, either (a) there is a solution $y \in E(\lambda_l) \setminus \{0\}$ of (3.6) or (b) there is an $\varepsilon > 0$ such that

$$(3.37) \quad G(w + y) \geq \varepsilon, \quad w \in M, y \in E(\lambda_l), \quad \|w + y\| = \varrho.$$

This is proved by the same method used in the proof of (3.7). Since the existence of a solution of (3.6) implies the conclusion of the theorem, we may assume that (3.37) holds. Let ϱ, ε be such that (3.37) holds and let $y_0 \in E(\lambda_l) \setminus \{0\}$ be fixed. Take

$$(3.38) \quad B = \{w \in M: \|w\|_D \geq \varrho\} \cup \\ \cup \{u = w + sy_0: w \in M, s \geq 0, \|u\|_D = \varrho\}.$$

Then (3.36) and (3.37) imply that

$$(3.39) \quad G(w) \geq \varepsilon_0, \quad w \in B$$

for some $\varepsilon_0 > 0$. Moreover, (1.14) implies

$$(3.40) \quad G(v) \rightarrow -\infty \quad \text{as } \|v\|_D \rightarrow \infty, v \in N$$

Again, this is proved in the same way that we proved (3.10). Next we take R so large that

$$(3.41) \quad G(v) \leq 0, \quad v \in N \cap \partial B_R \equiv A.$$

Then

$$(3.42) \quad \sup_A G \leq 0 < \varepsilon_0 \leq \inf_B G.$$

We know that A links B (this follows from Corollary 2.8, but it was known previously [Si, Lemma 1.14]). Thus by Theorem 2.3 there is a sequence $\{u_k\} \subset E$ such that (3.20) holds with G_1 replaced by G . The rest of the proof proceeds as before. ■

4. - The linking theorems.

In this section we give the proof of the theorems of Section 2. Proofs of Proposition 2.1 and 2.2 were given in [ST] (the definition of the set Φ given there was slightly different from that given in Section 2, but the proofs are not affected.)

PROOF OF THEOREM 2.3. If the theorem were false, there would be a $\delta > 0$ and a ψ satisfying (2.5) such that

$$(4.1) \quad \psi(\|u\|) \leq \|G'(u)\|$$

when

$$(4.2) \quad u \in Q := \{u \in E : |G(u) - a| \leq 3\delta\}.$$

Assume first that $b_0 < a$, and reduce δ so that $3\delta < a - b_0$. Since $G \in C^1(E, \mathbf{R})$, there is a locally Lipschitz continuous mapping $Y(u)$ of $\widehat{E} = \{u \in E : G'(u) = 0\}$ into E such that

$$(4.3) \quad \|Y(u)\| \leq 1, \theta \|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \widehat{E}$$

holds for some $\theta > 0$. Let

$$Q_0 = \{u \in E : |G(u) - a| < 2\delta\},$$

$$Q_1 = \{u \in E : |G(u) - a| < \delta\},$$

$$Q_2 = E \setminus Q_0, \quad \eta(u) = d(u, Q_2) / [d(u, Q_1) + d(u, Q_2)],$$

and let $\sigma(t)$ be the flow generated by

$$(4.4) \quad W(u) = -\eta(u) Y(u).$$

The mapping $W(u)$ is locally Lipschitz continuous on the whole of E and is bounded in norm by 1. We have

$$(4.5) \quad \begin{aligned} dG(\sigma(t))/dt = \eta(\sigma(t) u)(G'(\sigma(t) u), Y(\sigma(t) u)) &\leq \\ &\leq -\theta\eta(\sigma)\|G'(\sigma)\| \leq -\theta\eta(\sigma)\psi(\|\sigma\|) \leq -\theta\eta(\sigma)\psi(\|u\| + t) \end{aligned}$$

since

$$(4.6) \quad \|\sigma(t) u - u\| \leq t, \quad t > 0$$

and ψ is nonincreasing. By the definition (2.4) of a , there a $\Gamma \in \Phi$ such that

$$(4.7) \quad G(\Gamma(s)u) < a + \delta, \quad s \in [0, 1], \quad u \in A.$$

Let

$$(4.8) \quad M = \sup \{ \|\Gamma(s) u\| : s \in [0, 1], u \in A \}.$$

Since A is bounded, M is finite by the definition of Φ . Let T be such that

$$(4.9) \quad 2\delta < \theta \int_M^{T+M} \psi(t) dt.$$

This can be accomplished because ψ satisfies (2.5). Let $v = \Gamma(s)u$, where $s \in [0, 1]$ and $u \in A$. If there is a $t_1 \leq T$ such that $\sigma(t_1)v \in Q_1$, then

$$(4.10) \quad G(\sigma(T) v) < a - \delta$$

by (4.5) and (4.7). Otherwise, $\sigma(t)v \in Q_1$ for all $t \in [0, T]$ and

$$G(\sigma(T) v) \leq a + \delta - \theta \int_0^T \psi(M + t) dt < a - \delta.$$

Hence

$$(4.11) \quad G(\sigma(T) \Gamma(s) u) < a - \delta, \quad s \in [0, 1], \quad u \in A.$$

Let

$$(4.12) \quad \Gamma_1(s) = \begin{cases} \sigma(2sT), & 0 \leq s \leq \frac{1}{2}, \\ \sigma(T) \Gamma(2s - 1), & \frac{1}{2} < s \leq 1. \end{cases}$$

Then $\Gamma_1 \in \Phi$. Since

$$(4.13) \quad G(\sigma(t) u) \leq a_0, \quad t \geq 0, \quad u \in A$$

we see by (4.11) that

$$(4.14) \quad G(\Gamma_1(s) u) < a - \delta, \quad s \in [0, 1], \quad u \in A.$$

But this contradicts the definition (2.4) of a . Hence (4.1) cannot hold for u satisfying (4.2). If $b_0 = a$, we proceed as before, but we cannot use (4.13) to imply (4.14). However, we note that (4.5) implies

$$(4.15) \quad G(\sigma(t) u) \leq b_0 - \theta \int_0^t \eta(\sigma(\tau) u) \psi(\|\sigma(\tau) u\|) dt$$

for $u \in A$. This shows that

$$(4.16) \quad \sigma(t) A \cap B = \emptyset, \quad t \geq 0.$$

For the only way we can have $\sigma(t) u \in B$ is if

$$\eta(\sigma(\tau) u) \equiv 0, \quad 0 \leq \tau \leq t.$$

But this implies $\sigma(\tau) u \in \bar{Q}_2$. Consequently,

$$G(\sigma(\tau) u) < a - \delta, \quad 0 \leq \tau \leq t$$

which cannot happen if $\sigma(t) u \in B$. Thus (4.16) holds. Similarly, (4.11) shows that

$$(4.17) \quad \sigma(T) \Gamma(t) A \cap B = \emptyset, \quad t \in [0, 1].$$

Combining (4.16) and (4.17), we see that

$$\Gamma_1(s) A \cap B = \emptyset, \quad 0 \leq s \leq 1$$

contradicting the fact that A links B . This completes the proof of the theorem. ■

We prove Corollary 2.4 by taking $\psi(r) = 1/(1+r)$.

PROOF OF PROPOSITION 2.5. Let Γ be an arbitrary map in Φ . Then $H^{-1}\Gamma(s)H$ is in Φ . If A links B , then there is an $s_1 \in [0, 1]$ such that

$$H^{-1}\Gamma(s_1)HA \cap B \neq \emptyset.$$

Thus

$$\Gamma(s_1) HA \cap HB \neq \emptyset.$$

Since Γ was arbitrary, HA links HB . ■

PROOF OF PROPOSITION 2.6. Let Γ be any map in Φ . Then

$$(4.18) \quad K = \sup \{ \|\Gamma(t) u\| : t \in [0, 1], u \in A \} < \infty.$$

For n sufficiently large

$$(4.19) \quad d(B_n'', 0) > K.$$

Now

$$(4.20) \quad \Gamma(t_1) A \cap B_n \neq \emptyset$$

for some $t_1 \in [0, 1]$. But

$$\Gamma(t_1) A \cap B_n'' = \emptyset$$

by (4.18) and (4.19). Hence

$$\Gamma(t_1) A \cap B_n' \neq \emptyset.$$

Consequently

$$\Gamma(t_1) A \cap B \neq \emptyset.$$

Thus A links B . ■

PROOF OF COROLLARY 2.7. If $\dim M < \infty$, the result follows from Proposition 2.2 if we take $Q = M \cap B_R$ and let F be the projection of E onto Q . If $\dim N < \infty$, let $A = M \cap \partial B_R$ and $B_n = \{v \in N : \|v\| \leq n\} \cup \{v + sw_0 : v \in N, s \geq 0, \|v + sw_0\| = n\}$, $Q = \{v + sw_0 : v \in N, s \geq 0, \|v + sw_0\| \leq n\}$, and

$$(4.21) \quad F(v + w) = v + \|w\|w_0, \quad v \in N, \quad w \in M$$

where $w_0 \in M$ and $\|w_0\| = 1$. It follows from Proposition 2.2 that B_n links A when $R < n$. By Proposition 2.1 we also have that A links B_n . If we now take $B = N$, we can apply Propositions 2.6 to conclude that A links B . ■

PROOF OF COROLLARY 2.8. Let

$$B_n = \{v \in N: r \leq \|v\| \leq n\} \cup \{u = v + sw_0: v \in N, s \geq 0, \|u\| = r\} \cup \\ \cup \{u = v + sw_0: v \in N, s \geq 0, \|u\| = n\}.$$

If $r < R < n$, it follows from Propositions 2.1 and 2.2 that A and B_n link each other (no matter which subspace is finite dimensional). We now apply Proposition 2.6 to obtain the desired conclusion. ■

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