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New linking theorems

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New Linking Theorems.

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SUMMARY - We prove new linking theorems related to those of Schechter-Tintarev which allow us to obtain linking for sets which did not link under the older theories. This allows us to prove new theorems for nonlinear problems.

1. – Introduction.

Let G be a C^1 functional on a Banach space E, and assume that $E = M \oplus N$, where M, N are closed subspaces, one of which is finite dimensional. Assume that

$$a_0 := \sup_{M \cap \partial B_\delta} G \leqslant b_0 := \inf_N G \tag{1.1}$$

for some $\delta > 0$, where

$$B_r = \{u \in E : \|u\| < r\} . \tag{1.2}$$

One of the results of the present paper is

THEOREM 1.1. *Under the above hypotheses, there is a sequence* $\{u_k\} \subset E$ *such that*

$$G(u_k) \to c , \qquad b_0 \leqslant c < \infty , \qquad (1 + \|u_k\|)\, G'(u_k) \to 0 . \tag{1.3}$$

Interest in such a theorem stems from the fact that for many applica-

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tions, (1.3) implies the existence of a solution of

$$G(u) = c\,, \qquad G'(u) = 0\,. \tag{1.4}$$

We shall present some of these applications here. When $\dim M < \infty$, Theorem 1.1 is well known (cf. [Ra, Theorem 4.6]). However, the proof rests completely on the fact that M is finite dimensional. This is so much so, that no one seems to have suspected that the theorem is true even when $\dim M = \infty$. We shall show that this indeed is the case. As a result we can solve problems which could not be considered before.

We apply Theorem 1.1 to semilinear boundary value problems. Let Ω be a bounded domain in $\boldsymbol{R}^n$, and let A be a selfadjoint operator on $L^2(\Omega)$ with compact resolvent and eigenvalues

$$0 < \lambda_0 < \lambda_1 < \ldots < \lambda_j < \ldots .$$

We assume that the eigenfunctions of A are bounded. Let $f(x, t)$ be a Carathéodory function on $\Omega \times \boldsymbol{R}$ satisfying

$$|f(x, t)| \leqslant C|t| + V(x)\,, \qquad x \in \Omega\,, \quad t \in \boldsymbol{R} \tag{1.4}$$

and

$$f(x, t)/t \to \alpha_{\pm}(x) \qquad \text{a.e. as } t \to \pm\infty \tag{1.5}$$

where $V(x) \in L^2(\Omega)$ and the only solution of

$$Au = \alpha_+ u^+ - \alpha_- u^-\,, \qquad u^{\pm} = \max\{\pm u, 0\} \tag{1.6}$$

is $u \equiv 0$. We let

$$F(x, t) = \int_0^t f(x, s)\, ds\,. \tag{1.7}$$

We have

THEOREM 1.2. *Assume that for some* $l > 0$ *there are constants* $\nu > \lambda_{l-1}$ *and* $\delta > 0$ *such that*

$$\nu t^2 \leqslant 2F(x, t)\,, \qquad x \in \Omega\,, \quad t \in \boldsymbol{R}\,, \tag{1.8}$$

$$\lambda_l t^2 \leqslant 2F(x, t)\,, \qquad x \in \Omega\,, \ |t| < \delta\,, \tag{1.9}$$

$$\alpha_{\pm}(x) \leqslant \lambda_l, \quad x \in \Omega\,. \tag{1.10}$$

Then the equation

$$Au = f(x, u) \tag{1.11}$$

has at least one nontrivial solution.

We also have

THEOREM 1.3. *Assume that for some $l \geq 0$ there are constants $\nu < \lambda_{l+1}$ and $\delta > 0$ such that*

$$2F(x, t) \leq \nu t^2, \qquad x \in \Omega, \quad t \in \boldsymbol{R}, \tag{1.12}$$

$$2F(x, t) \leq \lambda_l t^2, \qquad x \in \Omega, \quad |t| < \delta, \tag{1.13}$$

$$\lambda_l \leq \alpha_{\pm}(x), \qquad x \in \Omega. \tag{1.14}$$

Then (1.11) *has at least one nontrivial solution.*

The equation (1.6) approximates (1.11) when $|u(x)|$ is large. Theorem 1.2 cannot be proved by using previous linking theorems. On the other hand, Theorem 1.3 does follow [Si, Theorem 1.15]. It is included here because of its similarity to Theorem 1.2.

Theorem 1.1 is proved in Section 4 along with other theorems on linking stated in Section 2. Theorems 1.2 and 1.3 are proved in Section 3. They are based on a slight variation of Theorem 1.1. Other linking methods can be found in [MW, BN, Ra, Si].

2. – The method.

We present a refined version of the new linking concept introduced in [ST]. Let E be a Banach space and let Φ be the set of all continuous maps $\Gamma = \Gamma(t)$ from $E \times [0, 1]$ to E such that

1) $\Gamma(0) = I$, the identity map.

2) For each $t \in [0, 1)$, $\Gamma(t)$ is a homeomorphism of E onto E and $\Gamma^{-1}(t) \in C(E \times [0, 1), E)$.

3) $\Gamma(1)E$ is a single point in E and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$.

4) For each $t_0 \in [0, 1)$ and each bounded set $A \subset E$

$$\sup_{\substack{0 \leq t \leq t_0 \\ u \in A}} \{\|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\|\} < \infty . \tag{2.1}$$

DEFINITION. For $A, B \subset E$ we say that A links B if

a) $A \cap B = \emptyset$,

b) for each $\Gamma \in \Phi$ there is a $t \in (0, 1]$ such that

$$\Gamma(t) A \cap B \neq \emptyset . \tag{2.2}$$

We have

PROPOSITION 2.1. *If $A, B \subset E$ are closed and bounded, $E \backslash A$ is pathwise connected and A links B, then B links A.*

PROPOSITION 2.2. *If $F \in C(E, \boldsymbol{R}^n)$ and $Q \subset E$ is such that $F_0 = F|_Q$ is a homeomorphism of Q onto the closure of a bounded open subset Ω of $\boldsymbol{R}^n$, then $\partial Q \equiv F_0^{-1}(\partial\Omega)$ links $F^{-1}(p)$ for each $p \in \Omega$.*

THEOREM 2.3. *Let G be a C^1 functional on E, and let A, B be subsets of E such that A is bounded and links B. Assume*

$$a_0 := \sup_A G \leq b_0 := \inf_B G , \tag{2.3}$$

$$a := \inf_{\Gamma \in \Phi} \sup_{\substack{0 \leq s \leq 1 \\ u \in A}} G(\Gamma(s)\, u) < \infty . \tag{2.4}$$

Let $\psi(t)$ be a positive nonincreasing function on $(0, \infty)$ such that

$$\int_\infty^1 \psi(r)\, dr = \infty . \tag{2.5}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a , \qquad G'(u_k) = o(\psi(\|u_k\|)) . \tag{2.6}$$

COROLLARY 2.4. *Under the hypotheses of Theorem 2.3, there is a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \to a , \qquad (1 + \|u_k\|)\, G'(u_k) \to 0 . \tag{2.7}$$

PROPOSITION 2.5. *Let H be a homeomorphism of E onto itself such that H and H^{-1} map bounded sets into bounded sets. If $A, B \subset E$ and A links B, then HA links HB.*

PROPOSITION 2.6. *Let A, B_n, $n = 1, 2, \ldots$, be subsets of E such that A is bounded and links B_n for each n. Suppose*

$$B_n = B_n' \cup B_n'' \tag{2.8}$$

where

$$d(B_n'', 0) \to \infty \quad \text{as } n \to \infty \tag{2.9}$$

and there is a set $B \subset E$ such that

$$A \cap B = \emptyset, \quad B_n' \subset B, \quad n = 1, 2, \ldots . \tag{2.10}$$

Then A links B.

COROLLARY 2.7. *Let M, N be closed subspaces of E, one of which is finite dimensional and such that*

$$E = M \oplus N . \tag{2.11}$$

If

$$B_R := \{u \in E: \|u\| < R\} \tag{2.12}$$

then $M \cap \partial B_R$ links N for each $R > 0$.

COROLLARY 2.8. *Let M, N be closed subspaces of E such that* (2.11) *holds with one of them being finite dimensional. Let w_0 be an element of $M \backslash \{0\}$, and let $0 < r < R$,*

$$A = \{w \in M: \|w\| = R\},$$

$$B = \{v \in N: \|v\| \geq r\} \cup \{u = v + sw_0: v \in N, s \geq 0, \|u\| = r\}.$$

Then A links B.

3. – The application.

We now give the proof of Theorems 1.2, 1.3.

PROOF OF THEOREM 1.2. Let

$$G(u) = \|u\|_D^2 - 2\int_\Omega F(x, u)\, dx, \qquad u \in D \tag{3.1}$$

where $D = D(A^{1/2})$ and

$$\|u\|_D = \|A^{1/2} u\| . \tag{3.2}$$

With this norm D becomes a Hilbert space. Under hypothesis (1.4) it is easily shown that G is a C^1 functional on D and

$$(G'(u), v)/2 = (u, v)_D - (f(u), v) . \tag{3.3}$$

From this it follows that u is a solution of (1.10) if

$$G'(u) = 0 . \tag{3.4}$$

Let N be the subspace of $L^2(\Omega)$ spanned by the eigenfunctions of A corresponding to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_l$, and let $E(\lambda_l)$ be the eigenspace of λ_l. Let $M = N^{\perp} \cap D$. By (1.8) we have

$$G(v) \leqslant \|v\|_D^2 - \nu\|v\|^2 \leqslant \left(1 - \frac{\nu}{\lambda_{l-1}}\right)\|v\|_D^2, \qquad v \in N_{l-1}. \tag{3.5}$$

Moreover, I claim that for each $\varrho > 0$ sufficiently small either (a) there is a $y \in E(\lambda_l)\backslash\{0\}$ satisfying

$$Ay = f(x, y) = \lambda_l y , \qquad \|y\|_D = \varrho \tag{3.6}$$

or (b) there is an $\varepsilon > 0$ such that

$$G(v + y) \leqslant -\varepsilon , \qquad v \in N_{l-1} , \qquad y \in E(\lambda_l), \qquad \|v + y\|_D = \varrho . \tag{3.7}$$

Assume this for the moment. Since (3.6) exhibits a nontrivial solution of (1.11), we need only address option (b). Let ϱ, ε be such that (3.7) holds. Let $y_0 \in E(\lambda_l)\backslash\{0\}$ and take

$$\begin{aligned} B = \{v \in N_{l-1}\colon \|v\|_D \geqslant \varrho\} \cup \\ \cup \{u = v + sy_0\colon v \in N_{l-1}, s \geqslant 0, \ \|u\|_D = \varrho\} . \end{aligned} \tag{3.8}$$

Then (3.5) and (3.7) imply

$$G(v) \leqslant -\varepsilon_0 , \qquad v \in B \tag{3.9}$$

for some $\varepsilon_0 > 0$. On the other hand, (1.10) implies

$$G(w) \to \infty \quad \text{as } \|w\|_D \to \infty , \ w \in M_{l-1}. \tag{3.10}$$

To see this, let $\{w_k\}$ be any sequence in $M = M_{l-1}$ such that $\varrho_k =$

$= \|w_k\|_D \to \infty$. Let $\tilde{w}_k = w_k/\varrho_k$. Then

(3.11) $$G(w_k)/\varrho_k^2 = 1 - 2\int_\Omega F(x, w_k)/\varrho_k^2\,dx\,.$$

Now $\|\tilde{w}_k\|_D = 1$. Hence there is a renamed subsequence such that $\tilde{w}_k \to \tilde{w}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. Moreover, (1.4) implies

(3.12) $$|F(x, t)| \leqslant Ct^2 + V(x)\,|t|$$

and (1.5) implies

(3.14) $$2F(x, t)/t^2 \to \alpha_\pm(x) \quad \text{a.e. as } t \to \pm\infty\,.$$

Thus

(3.15) $$2\int_\Omega F(x, w_k)\,dx/\varrho_k^2 \to \int_\Omega \{\alpha_+(\tilde{w}^+)^2 + \alpha_-(\tilde{w}^-)^2\}\,dx\,, \qquad k \to \infty$$

and

(3.16) $$G(w_k)/\varrho_k^2 \to 1 - \int_\Omega \{\alpha_+(\tilde{w}^+)^2 + \alpha_-(\tilde{w}^-)^2\}\,dx\,, \qquad k \to \infty\,.$$

Since $\|\tilde{w}\|_D \leqslant 1$, (1.10) implies that the right hand side of (3.16) is $\geqslant 0$. The only way it could vanish is if $\tilde{w} \in E(\lambda_l)$ and

(3.17) $$\int_\Omega \{(\lambda_l - \alpha_+)(\tilde{w}^+)^2 + (\lambda_l - \alpha_-)(\tilde{w}^-)^2\}\,dx = 0\,.$$

Since the integrand in (3.17) is nonnegative, we must have

$$\alpha_+(x) \equiv \lambda_l \quad \text{when } \tilde{w}(x) > 0\,,$$

$$\alpha_-(x) \equiv \lambda_l \quad \text{when } \tilde{w}(x) < 0\,.$$

From this it follows that $\tilde{w}$ is a solution of (1.6). By hypothesis, this implies that $\tilde{w} \equiv 0$, showing that the right hand side of (3.16) does not vanish. Hence the left hand side of (3.16) converges to a positive limit for every such sequence, showing that (3.10) holds. Once we know this, we take R such that

(3.18) $$G(w) \geqslant 0\,, \quad w \in M_{l-1} \cap \partial B_R \equiv A\,.$$

Let $G_1(u) = -G(u)$. Then

$$\sup_A G_1 \leqslant 0 < \varepsilon_0 \leqslant \inf_B G_1 . \tag{3.19}$$

By Corollary 2.8, A links B. Hence there is a sequence $\{u_k\} \subset D$ such that

$$G_1(u_k) \to c_1, \qquad \varepsilon_0 \leqslant c_1 < \infty , \qquad G_1'(u_k) \to 0 . \tag{3.20}$$

Thus

$$\|u_k\|_D^2 - 2 \int_\Omega F(x, u_k)\, dx \to -c_1 \tag{3.21}$$

and

$$(u_k, v)_D - (f(u_k), v) \to 0 , \qquad v \in D \tag{3.22}$$

where we write $f(u)$ in place of $f(x, u)$. If $\varrho_k = \|u_k\|_D \to \infty$, we let $\tilde{u}_k = u_k/\varrho_k$. Then $\|\tilde{u}_k\|_D = 1$ and there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. We then obtain

$$\int_\Omega \{\alpha_+ (\tilde{u}^+)^2 + \alpha_- (\tilde{u}^-)^2\}\, dx = 1 \tag{3.23}$$

from (3.21) and

$$(\tilde{u}, v)_D = \int_\Omega (\alpha_+ \tilde{u}^+ - \alpha_- \tilde{u}^-)\, v\, dx , \qquad v \in D \tag{3.24}$$

from (3.22). This implies that $\|\tilde{u}\|_D^2$ equals the left hand side of (3.23) and that $\tilde{u}$ is a solution of (1.6). Thus by hypothesis we must have $\tilde{u} \equiv 0$. But this contradicts (3.23). Hence the ϱ_k are bounded. We can now apply well known techniques to show that there is a solution of (1.11) satisfying $G_1(u) = c_1 \geqslant \varepsilon_0 > 0$. Since $G_1(0) = 0$, we see that u is a nontrivial solution of (1.11).

It remains to prove (3.7). First we have

LEMMA 3.1. *If* (1.9) *holds, then for each* $\varrho > 0$ *sufficiently small there is a positive* ε *such that*

$$G(v + y) \leqslant -\varepsilon \|v\|_D^2, \qquad v \in N_{l-1}, \qquad y \in E(\lambda_l), \qquad \|v + y\|_D \leqslant \varrho . \tag{3.25}$$

PROOF. Since $E(\lambda_l) \subset L^\infty(\Omega)$, there is a $\varrho > 0$ such that

$$\|y\|_D \leqslant \varrho \text{ implies } \|y\|_\infty \leqslant \delta/2 , \qquad y \in E(\lambda_l) \tag{3.26}$$

where δ is the constant in (1.9). Let $w = v + y$, where $v \in N_{l-1}$, $y \in E(\lambda_l)$. If

$$\|w\|_D \leqslant \varrho \quad \text{and} \quad |w(x)| \geqslant \delta \tag{3.27}$$

then

$$\delta \leqslant |w(x)| \leqslant |v(x)| + |y(x)| \leqslant |v(x)| + \delta/2 \,. \tag{3.28}$$

Consequently, (3.27) implies

$$|w(x)| \leqslant 2\,|v(x)| \,. \tag{3.30}$$

By (3.12)

$$\begin{aligned} G(w) &\leqslant \|w\|_D^2 - \lambda_l \int_{|w| < \delta} w^2\,dx + C \int_{|w| > \delta} (w^2 + |w|)\,dx \leqslant \\ &\leqslant \|w\|_D^2 - \lambda_l \|w\|^2 + C' \int_{|w| > \delta} (w^2 + |w|)\,dx \leqslant \\ &\leqslant \|v\|_D^2 - \lambda_l \|v\|^2 + C'' \int_{2|v| > \delta} |v|^\sigma\,dx \leqslant \left(1 - \frac{\lambda_l}{\lambda_{l-1}} - C''' \|v\|_D^{\sigma - 2}\right) \|v\|_D^2 \end{aligned}$$

where $\sigma > 2$. If we take ϱ sufficiently small, this implies (3.25). ■

Once we have inequality (3.25), we prove (3.7) by assuming, on the contrary, that there is a sequence $w_k = v_k + y_k$, $v_k \in N_{l-1}$, $y_k \in E(\lambda_l)$ such that $\|w_k\|_D = \varrho$ and

$$G(w_k) \to 0 \,. \tag{3.32}$$

By (3.25) we see that $v_k \to 0$. Thus $\|y_k\|_D \to \varrho$. Since $E(\lambda_l)$ is finite dimensional, there is a renamed subsequence such that $y_k \to y$ in $E(\lambda_l)$ and $\|y\|_D = \varrho$. By (3.32)

$$G(y) = 0 \,. \tag{3.33}$$

If ϱ is such that (3.27) holds, then (1.9) implies

$$\lambda_l y(x)^2 \leqslant 2F(x, y(x)) \,, \qquad x \in \Omega \,. \tag{3.34}$$

But (3.33) says

$$\int_\Omega \{\lambda_l y(x)^2 - 2F(x, y(x))\}\,dx = 0 \,. \tag{3.35}$$

From (3.34) and (3.35) we see that

$$\lambda_l y(x)^2 \equiv F(x, y(x)), \qquad x \in \Omega .$$

Let $\zeta(x)$ be any function in $C_0^\infty(\Omega)$. Then for $t > 0$ sufficiently small

$$\lambda_l[(y + t\zeta)^2 - y^2]/t \leqslant 2[F(x, y + t\zeta) - F(x, y)]/t .$$

If we take the limit as $t \to 0$, we have

$$\lambda_l y(x)^2 \zeta(x) \leqslant f(x, y(x))\zeta(x), \qquad x \in \Omega .$$

From this we conclude that

$$\lambda_l y(x)^2 \equiv f(x, y(x)), \qquad x \in \Omega .$$

Since $y \in E(\lambda_l)$, this implies that y is a solution of (3.6), the option which we discarded. This completes the proof of the theorem. ■

PROOF OF THEOREM 1.3. We only sketch the proof because of its similarity to that of Theorem 1.2. We use the notation of the proof of Theorem 1.2. By (1.12)

$$G(w) \geqslant \|w\|_D^2 - \nu\|w\|^2 \geqslant \left(1 - \frac{\nu}{\lambda_{l+1}}\right)\|w\|^2 D, \qquad w \in M . \tag{3.36}$$

Moreover, (1.13) implies that for each $\varrho > 0$ sufficiently small, either (a) there is a solution $y \in E(\lambda_l)\setminus\{0\}$ of (3.6) or (b) there is an $\varepsilon > 0$ such that

$$G(w + y) \geqslant \varepsilon , \quad w \in M, \ y \in E(\lambda_l), \quad \|w + y\| = \varrho . \tag{3.37}$$

This is proved by the same method used in the proof of (3.7). Since the existence of a solution of (3.6) implies the conclusion of the theorem, we may assume that (3.37) holds. Let ϱ, ε be such that (3.37) holds and let $y_0 \in E(\lambda_l)\setminus\{0\}$ be fixed. Take

$$B = \{w \in M\colon \|w\|_D \geqslant \varrho\} \cup \\ \cup \{u = w + sy_0\colon w \in M, \ s \geqslant 0, \ \|u\|_D = \varrho\} . \tag{3.38}$$

Then (3.36) and (3.37) imply that

$$G(w) \geqslant \varepsilon_0 , \qquad w \in B \tag{3.39}$$

for some $\varepsilon_0 > 0$. Moreover, (1.14) implies

$$G(v) \to -\infty \qquad \text{as } \|v\|_D \to \infty , \ v \in N \tag{3.40}$$

Again, this is proved in the same way that we proved (3.10). Next we take R so large that

$$G(v) \leqslant 0\,, \qquad v \in N \cap \partial B_R \equiv A\,. \tag{3.41}$$

Then

$$\sup_A G \leqslant 0 < \varepsilon_0 \leqslant \inf_B G\,. \tag{3.42}$$

We know that A links B (this follows from Corollary 2.8, but it was known previously [Si, Lemma 1.14]). Thus by Theorem 2.3 there is a sequence $\{u_k\} \subset E$ such that (3.20) holds with G_1 replaced by G. The rest of the proof proceeds as before. ■

4. – The linking theorems.

In this section we give the proof of the theorems of Section 2. Proofs of Proposition 2.1 and 2.2 were given in [ST] (the definition of the set Φ given there was slightly different from that given in Section 2, but the proofs are not affected.)

PROOF OF THEOREM 2.3. If the theorem were false, there would be a $\delta > 0$ and a ψ satisfying (2.5) such that

$$\psi(\|u\|) \leqslant \|G'(u)\| \tag{4.1}$$

when

$$u \in Q := \{u \in E\colon\ |G(u) - a| \leqslant 3\delta\}\,. \tag{4.2}$$

Assume first that $b_0 < a$, and reduce δ so that $3\delta < a - b_0$. Since $G \in C^1(E, \boldsymbol{R})$, there is a locally Lipschitz continuous mapping $Y(u)$ of $\widehat{E} = \{u \in E\colon\ G'(u) = 0\}$ into E such that

$$\|Y(u)\| \leqslant 1\,,\ \theta\|G'(u)\| \leqslant (G'(u),\, Y(u))\,, \qquad u \in \widehat{E} \tag{4.3}$$

holds for some $\theta > 0$. Let

$$Q_0 = \{u \in E\colon\ |G(u) - a| < 2\delta\}\,,$$

$$Q_1 = \{u \in E\colon\ |G(u) - a| < \delta\}\,,$$

$$Q_2 = E \backslash Q_0\,, \qquad \eta(u) = d(u,\, Q_2)/[d(u,\, Q_1) + d(u,\, Q_2)]\,,$$

and let $\sigma(t)$ be the flow generated by

$$W(u) = -\eta(u)\, Y(u)\,. \tag{4.4}$$

The mapping $W(u)$ is locally Lipschitz continuous on the whole of E and is bounded in norm by 1. We have

$$(4.5)\quad dG(\sigma(t))/dt = \eta(\sigma(t)\, u)(G'(\sigma(t)\, u),\, Y(\sigma(t)\, u)) \leqslant$$
$$\leqslant -\theta\eta(\sigma)\|G'(\sigma)\| \leqslant -\theta\eta(\sigma)\,\psi(\|\sigma\|) \leqslant -\theta\eta(\sigma)\,\psi(\|u\|+t)$$

since

$$(4.6)\qquad \|\sigma(t)\, u - u\| \leqslant t\,, \qquad t > 0$$

and ψ is nonincreasing. By the definition (2.4) of a, there a $\Gamma \in \Phi$ such that

$$(4.7)\qquad G(\Gamma(s)u) < a+\delta,\ s \in [0,1], \qquad u \in A\,.$$

Let

$$(4.8)\qquad M = \sup\{\|\Gamma(s)\, u\| \colon\ s \in [0,1],\ u \in A\}\,.$$

Since A is bounded, M is finite by the definition of Φ. Let T be such that

$$(4.9)\qquad 2\delta < \theta \int_M^{T+M} \psi(t)\, dt\,.$$

This can be accomplished because ψ satisfies (2.5). Let $v = \Gamma(s)u$, where $s \in [0,1]$ and $u \in A$. If there is a $t_1 \leqslant T$ such that $\sigma(t_1)v \notin Q_1$, then

$$(4.10)\qquad G(\sigma(T)\, v) < a - \delta$$

by (4.5) and (4.7). Otherwise, $\sigma(t)v \in Q_1$ for all $t \in [0, T]$ and

$$G(\sigma(T)\, v) \leqslant a + \delta - \theta \int_0^T \psi(M+t)\, dt < a - \delta\,.$$

Hence

$$(4.11)\qquad G(\sigma(T)\, \Gamma(s)\, u) < a - \delta\,, \qquad s \in [0,1],\ \ u \in A\,.$$

Let

$$(4.12)\qquad \Gamma_1(s) = \begin{cases} \sigma(2sT)\,, & 0 \leqslant s \leqslant \dfrac{1}{2}\,, \\[2ex] \sigma(T)\, \Gamma(2s-1)\,, & \dfrac{1}{2} < s \leqslant 1\,. \end{cases}$$

Then $\Gamma_1 \in \Phi$. Since

$$G(\sigma(t)\,u) \leqslant a_0, \qquad t \geqslant 0\,, \quad u \in A \tag{4.13}$$

we see by (4.11) that

$$G(\Gamma_1(s)\,u) < a - \delta\,, \qquad s \in [0,\,1]\,, \quad u \in A\,. \tag{4.14}$$

But this contradicts the definition (2.4) of a. Hence (4.1) cannot hold for u satisfying (4.2). If $b_0 = a$, we proceed as before, but we cannot use (4.13) to imply (4.14). However, we note that (4.5) implies

$$G(\sigma(t)\,u) \leqslant b_0 - \theta \int_0^t \eta(\sigma(\tau)\,u)\,\psi(\|\sigma(\tau)\,u\|)\,dt \tag{4.15}$$

for $u \in A$. This shows that

$$\sigma(t)\,A \cap B = \phi\,, \qquad t \geqslant 0\,. \tag{4.16}$$

For the only way we can have $\sigma(t)\,u \in B$ is if

$$\eta(\sigma(\tau)\,u) \equiv 0\,, \qquad 0 \leqslant \tau \leqslant t\,.$$

But this implies $\sigma(\tau)u \in \overline{Q}_2$. Consequently,

$$G(\sigma(\tau)\,u) < a - \delta\,, \qquad 0 \leqslant \tau \leqslant t$$

which cannot happen if $\sigma(t)\,u \in B$. Thus (4.16) holds. Similarly, (4.11) shows that

$$\sigma(T)\,\Gamma(t)\,A \cap B = \emptyset\,, \qquad t \in [0,\,1]\,. \tag{4.17}$$

Combing (4.16) and (4.17), we see that

$$\Gamma_1(s)\,A \cap B = \emptyset\,, \qquad 0 \leqslant s \leqslant 1$$

contradicting the fact that A links B. This completes the proof of the theorem. ■

We prove Corollary 2.4 be taking $\psi(r) = 1/(1+r)$.

PROOF OF PROPOSITION 2.5. Let Γ be an arbitrary map in Φ. Then $H^{-1}\Gamma(s)H$ is in Φ. If A links B, then there is an $s_1 \in [0,\,1]$ such that

$$H^{-1}\Gamma(s_1)\,HA \cap B \neq \emptyset\,.$$

Thus

$$\Gamma(s_1)\, HA \cap HB \neq \emptyset\,.$$

Since Γ was arbitrary, HA links HB. ■

PROOF OF PROPOSITION 2.6. Let Γ be any map in Φ. Then

$$K = \sup\{\|\Gamma(t)\, u\| \colon t \in [0, 1],\, u \in A\} < \infty\,. \tag{4.18}$$

For n sufficiently large

$$d(B_n'', 0) > K\,. \tag{4.19}$$

Now

$$\Gamma(t_1)\, A \cap B_n \neq \emptyset \tag{4.20}$$

for some $t_1 \in [0, 1]$. But

$$\Gamma(t_1)\, A \cap B_n'' = \emptyset$$

by (4.18) and (4.19). Hence

$$\Gamma(t_1)\, A \cap B_n' \neq \emptyset\,.$$

Consequently

$$\Gamma(t_1)\, A \cap B \neq \emptyset\,.$$

Thus A links B. ■

PROOF OF COROLLARY 2.7. If $\dim M < \infty$, the result follows from Proposition 2.2 if we take $Q = M \cap B_R$ and let F be the projection of E onto Q. If $\dim N < \infty$, let $A = M \cap \partial B_R$ and $B_n = \{v \in N \colon \|v\| \leqslant n\} \cup \cup \{v + sw_0 \colon v \in N, s \geqslant 0, \|v + sw_0\| = n\}$, $Q = \{v + sw_0 \colon v \in N, s \geqslant 0, \|v + + sw_0\| \leqslant n\}$, and

$$F(v + w) = v + \|w\| w_0\,, \qquad v \in N\,,\quad w \in M \tag{4.21}$$

where $w_0 \in M$ and $\|w_0\| = 1$. It follows from Proposition 2.2 that B_n links A when $R < n$. By Proposition 2.1 we also have that A links B_n. If we now take $B = N$, we can apply Propositions 2.6 to conclude that A links B. ■

PROOF OF COROLLARY 2.8. Let

$$B_n = \{v \in N:\ r \leqslant \|v\| \leqslant n\} \cup \{u = v + sw_0:\ v \in N,\ s \geqslant 0,\ \|u\| = r\} \cup \cup \{u = v + sw_0:\ v \in N,\ s \geqslant 0,\ \|u\| = n\}.$$

If $r < R < n$, it follows from Propositions 2.1 and 2.2 that A and B_n link each other (no matter which subspace is finite dimensional). We now apply Proposition 2.6 to obtain the desired conclusion. ■

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