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## Some New Partially Symmetric Designs and their Resolution.

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**ABSTRACT** - In this note we study a resolution (a generalisation of a parallelism) in the (new) partially symmetric designs of the type  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$  where  $\mathcal{O}'$  is a tight (Baer) subdesign in the symmetric  $2 - (v, k, \lambda)$  design  $\mathcal{O}$  (with  $\lambda > 1$ ).

### 1. – Introduction.

Throughout this note let  $\mathcal{O}$  be a symmetric  $2 - (v, k, \lambda)$  design (with  $\lambda > 1$ ) and let  $\mathcal{O}'$  be a symmetric  $2 - (v', k', \lambda')$  subdesign of  $\mathcal{O}$ . By Jungnickel [4]  $\mathcal{O}'$  is a *tight* subdesign of  $\mathcal{O}$  iff each block of  $\mathcal{O} \setminus \mathcal{O}'$  meets  $\mathcal{O}'$  in a constant number  $x$  of points. Furthermore if  $\lambda = \lambda'$  (and then  $x = 1$ ) we say  $\mathcal{O}'$  is Baer subdesign of  $\mathcal{O}$ .

By Hughes [2] a square 1-design  $\mathcal{S}$  is a *partial symmetric design* (a PSD) if there exist integers  $\lambda_1, \lambda_2 \geq 0$  such that two points are on  $\lambda_1$  or  $\lambda_2$  common blocks; two blocks of  $\mathcal{S}$  contains  $\lambda_1$  or  $\lambda_2$  common points and all such that  $\mathcal{S}$  is connected. We say then  $\mathcal{S}$  is PSD for  $(v_1, k_1, \lambda_1, \lambda_2)$  (where  $v_1$  is the number of points (blocks) in  $\mathcal{S}$  and  $k_1$  block (point)-size of  $\mathcal{S}$ ).

The concept of a divisibility and resolution (a generalisation of the parallelism) we take as in [3] (pp. 206, 154) and [1] (pp. 45, 39).

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## 2. – Results.

**2.1 PROPOSITION.** Let  $\mathcal{O}'$  be a Baer subdesign of  $\mathcal{O}$ . Then  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$  is a PSD for  $(v_1 = v - v', k_1 = k - 1, \lambda_1 = \lambda - 1, \lambda_2 = \lambda)$  and the relation  $\parallel$  (for arbitrary blocks  $b$  and  $c$  in  $\mathcal{S}$ )

$$b \parallel c \text{ iff } b \text{ and } c \text{ lie on the same point in } \mathcal{O}'$$

is an equivalence relation and, in this sense,  $\mathcal{S}$  is divisible.

**PROOF.** It is clear that  $\mathcal{S}$  is a square 1-design with  $v_1 = v - v'$  points (blocks) and with point (block)-size  $k - 1$ . Each point in  $\mathcal{O} \setminus \mathcal{O}'$  is exactly on one block of  $\mathcal{O}'$  and thus two points in  $\mathcal{O} \setminus \mathcal{O}'$  lie exactly on  $\lambda - 1$  or  $\lambda$  blocks of  $\mathcal{O} \setminus \mathcal{O}'$ . Further any two blocks of  $\mathcal{O} \setminus \mathcal{O}'$  lie exactly on 0 or 1 points in  $\mathcal{O}'$  and their integers in  $\mathcal{O} \setminus \mathcal{O}'$  are  $\lambda - 1$  or  $\lambda$ .

The partition of the set of all blocks in  $\mathcal{S}$  onto subsets of blocks passing through some point from  $\mathcal{O}'$  is disjoint and therefore the  $\parallel$  is an equivalence relation. ■

**2.2 REMARK.** By 2.1,  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$  (where  $\mathcal{O}'$  is a Baer subdesign of  $\mathcal{O}$ ) is a PSD with a divisibility. But, in general,  $\mathcal{S}$  cannot have a resolution. For instance  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ , where  $\mathcal{O}$  is a symmetric  $2 - (16, 6, 2)$  design with a symmetric  $2 - (4, 3, 2)$  subdesign  $\mathcal{O}'$ .

But we have

**2.3 PROPOSITION.** Let  $\mathcal{O} = PG_2(3, q)$  ( $q$  a prime power). Then  $\mathcal{O}$  has Baer  $2 - (q + 1, q + 1, q + 1)$  subdesign  $\mathcal{O}'$  and the  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$  has a strong resolution.

**PROOF.** By [4]  $\mathcal{O}'$  exists. Any class of blocks are all blocks in  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$  through any point in  $\mathcal{O}'$ . Two different classes are disjoint.

Each of these classes have exactly  $m = q^2 + q + 1 - (q - 1) = q^2$  blocks and any two blocks in the same class have exactly  $\lambda - 1$  points in common. Two blocks in the different classes have exactly  $\lambda$  points in common. Finally, each point in  $\mathcal{O} \setminus \mathcal{O}'$  lies exactly on  $\lambda - 1$  blocks of any one class. Thus,  $\mathcal{S}$  have a strong resolution. ■

In general, let  $\mathcal{O} = PG_{2d}(2d + 1, q)$  ( $d \geq 2$ ) be the design of points and hyperplanes of the  $(2d + 1)$ -dimensional projective space over  $GF(q)$ . Then, by [4],  $\mathcal{O}$  has a tight  $(c, c, c)$ -subdesign  $\mathcal{O}'$  with  $c = q^d + \dots + q + 1$ .

In general, we cannot say anything of the relation  $\parallel$  (as in 2.1 and 2.3) in  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ . Namely, the partition of  $\mathcal{S}$ , corresponding to  $\parallel$ , is not

disjoint. Further, we cannot say anything of a inner (outer) constant (for  $\parallel$ ). But we have

**2.4 PROPOSITION.** *Let  $\mathcal{O} = PG_{2d}(2d+1, q)$  ( $d \geq 2$ ) and let  $\mathcal{O}'$  be a tight  $(c, c, c)$ -subdesign. Then  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$  is a PSD having a strong resolution.*

**PROOF.** The parameters of  $\mathcal{O}$  and  $\mathcal{O}'$  are  $v = q^{2d+1} + \dots + q + 1$ ,  $k = q^{2d} + \dots + q + 1$ ,  $\lambda = q^{2d-1} + \dots + q + 1$  and  $c = q^d + \dots + q + 1$ . It is not difficult to check that any point in  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$  is exactly on  $x$  blocks of  $\mathcal{O}'$  where  $x = q^{d-1} + \dots + q + 1$ . Thus  $\mathcal{S}$  is a square  $1 - (v - c, k - x, k - x)$  design. Further we have:

$$\text{from } 1 + (x(x-1))/\bar{\lambda}_1 = c \quad \text{we get } \bar{\lambda}_1 = q^{d-2} + \dots + q + 1;$$

$$k - c = \dots = q^{d+1} \cdot (q^{d-1} + \dots + q + 1) = q^{d+1} \cdot x$$

and

$$\lambda - c = \dots = q^{d+1} (q^{d-2} + \dots + q + 1) = q^{d+1} \bar{\lambda}_1.$$

Here there is an automorphism  $y \mapsto y + u$  exchanging the points resp. the blocks (pointwise) in  $\mathcal{O}'$  (Singer cycle in  $\mathcal{O}'$ , generated additively with  $u$  in  $Z_v$ ). Thus we conclude that the blocks in  $\mathcal{O}'$  ( $\cong PG_{d-1}(d, q)$ ) have  $x$  sets, each of these sets has exactly  $q^{d+1}$  points (in  $\mathcal{O} \setminus \mathcal{O}'$ ) and any two blocks in  $\mathcal{O}'$  have exactly  $\bar{\lambda}_1$  sets in common. We are calling these sets «points». So any two points of  $\mathcal{O} \setminus \mathcal{O}'$  are in one or in two «points». Therefore through two points in  $\mathcal{O} \setminus \mathcal{O}'$  pass, according with this,  $x$  or  $\bar{\lambda}_1$  blocks from  $\mathcal{O}'$ . Thus, on two points of  $\mathcal{S}$  lie  $\lambda_1 = \lambda - x$  or  $\lambda_2 = \lambda - \bar{\lambda}_1$  common blocks. By [4] (2.1), we get this for the intersections of the blocks in  $\mathcal{S}$ . Thus,  $\mathcal{S}$  is a PSD for

$$(v - c, k - x, \lambda_1 = \lambda - x, \lambda_2 = \lambda - \bar{\lambda}_1) = \\ = (q^{2d+1} + \dots + q^{d+1}, q^{2d} + \dots + q^d, q^{2d-1} + \dots + q^d, q^{2d-1} + \dots + q^{d-1}).$$

Any resolution-class in  $\mathcal{S}$  is formed from all blocks in  $\mathcal{S}$  passing through any block in  $2 - (c, x, \bar{\lambda}_1)$  design  $\mathcal{O}'$ . One has exactly

$$\frac{k - c}{x} = \frac{q^{d+1}x}{x} = q^{d+1}$$

blocks in each resolution-class and exactly

$$\frac{v - c}{q^{d+1}} = \frac{q^{2d+1} + \dots + q + 1 - (q^d + \dots + q + 1)}{q^{d+1}} = \dots = c$$

(disjoint!) classes. There are exactly  $(\lambda - x)/x = \dots = q^d$  blocks from each resolution-class passing through any point in  $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ .

Finally, our resolution is strong with inner and outer constant  $\lambda - x$  and  $\lambda - \lambda_1$  respectively. ■

AN ILLUSTRATION.  $\mathcal{O} = PG_4(5, 2)$  (with a tight subdesign for  $(7, 7, 7)$ ). The initial block in  $\mathcal{O}$  (in the form of a difference set) is

$$1_0 = 0, 1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 16, 18, 19, 24, 26, 27, 28,$$

$$32, 33, 35, 36, 38, 41, 45, 48, 49, 52, 54, 56.$$

(All blocks  $1_i$  are formed by (mod. 63) addition of the  $1, 2, \dots, 62$  respectively.) The points in  $\mathcal{O}'$  are  $0, 9, 18, 27, 36, 45, 54$ .

The resolution-classes are:

$$(0) \wedge (9) = (0) \wedge (45) = (9) \wedge (45) =$$

$$= \{1_7, 1_{31}, 1_{37}, 1_{39}, 1_{44}, 1_{56}, 1_{59}, 1_{60}\},$$

$$(0) \wedge (18) = (0) \wedge (27) = (18) \wedge (27) =$$

$$= \{1_{11}, 1_{14}, 1_{15}, 1_{25}, 1_{49}, 1_{55}, 1_{57}, 1_{62}\},$$

$$(0) \wedge (36) = (0) \wedge (54) = (36) \wedge (54) =$$

$$= \{1_{22}, 1_{28}, 1_{30}, 1_{35}, 1_{47}, 1_{50}, 1_{51}, 1_{61}\},$$

$$(9) \wedge (18) = (9) \wedge (54) = (18) \wedge (54) =$$

$$= \{1_2, 1_5, 1_6, 1_{16}, 1_{40}, 1_{46}, 1_{48}, 1_{53}\},$$

$$(9) \wedge (27) = (9) \wedge (36) = (27) \wedge (36) =$$

$$= \{1_1, 1_3, 1_8, 1_{20}, 1_{23}, 1_{24}, 1_{34}, 1_{58}\},$$

$$(18) \wedge (36) = (18) \wedge (45) = (36) \wedge (45) =$$

$$= \{1_4, 1_{10}, 1_{12}, 1_{17}, 1_{29}, 1_{32}, 1_{33}, 1_{43}\},$$

$$(27) \wedge (45) = (27) \wedge (54) = (45) \wedge (54) =$$

$$= \{1_{13}, 1_{19}, 1_{21}, 1_{26}, 1_{38}, 1_{41}, 1_{42}, 1_{52}\},$$

This is the 4-resolution with the constants 12 and 14.

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