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Some New Partially Symmetric Designs and their Resolution.

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ABSTRACT - In this note we study a resolution (a generalisation of a parallelism) in the (new) partially symmetric designs of the type $\mathcal{S} = \mathcal{D} \setminus \mathcal{D}'$ where \mathcal{D}' is a tight (Baer) subdesign in the symmetric $2 - (v, k, \lambda)$ design \mathcal{D} (with $\lambda > 1$).

1. - Introduction.

Throughout this note let \mathcal{D} be a symmetric $2 - (v, k, \lambda)$ design (with $\lambda > 1$) and let \mathcal{D}' be a symmetric $2 - (v', k', \lambda')$ subdesign of \mathcal{D} . By Jungnickel [4] \mathcal{D}' is a *tight* subdesign of \mathcal{D} iff each block of $\mathcal{D} \setminus \mathcal{D}'$ meets \mathcal{D}' in a constant number x of points. Furthermore if $\lambda = \lambda'$ (and then $x = 1$) we say \mathcal{D}' is Baer subdesign of \mathcal{D} .

By Hughes [2] a square 1-design \mathcal{S} is a *partial symmetric design* (a PSD) if there exist integers $\lambda_1, \lambda_2 \geq 0$ such that two points are on λ_1 or λ_2 common blocks; two blocks of \mathcal{S} contains λ_1 or λ_2 common points and all such that \mathcal{S} is connected. We say then \mathcal{S} is PSD for $(v_1, k_1, \lambda_1, \lambda_2)$ (where v_1 is the number of points (blocks) in \mathcal{S} and k_1 block (point)-size of \mathcal{S}).

The concept of a divisibility and resolution (a generalisation of the parallelism) we take as in [3] (pp. 206, 154) and [1] (pp. 45, 39).

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2. – Results.

2.1 PROPOSITION. *Let \mathcal{O}' be a Baer subdesign of \mathcal{O} . Then $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ is a PSD for $(v_1 = v - v', k_1 = k - 1, \lambda_1 = \lambda - 1, \lambda_2 = \lambda)$ and the relation \parallel (for arbitrary blocks b and c in \mathcal{S})*

$b \parallel c$ iff b and c lie on the same point in \mathcal{O}'

is an equivalence relation and, in this sense, \mathcal{S} is divisible.

PROOF. It is clear that \mathcal{S} is a square 1-design with $v_1 = v - v'$ points (blocks) and with point (block)-size $k - 1$. Each point in $\mathcal{O} \setminus \mathcal{O}'$ is exactly on one block of \mathcal{O}' and thus two points in $\mathcal{O} \setminus \mathcal{O}'$ lie exactly on $\lambda - 1$ or λ blocks of $\mathcal{O} \setminus \mathcal{O}'$. Further any two blocks of $\mathcal{O} \setminus \mathcal{O}'$ lie exactly on 0 or 1 points in \mathcal{O}' and their integers in $\mathcal{O} \setminus \mathcal{O}'$ are $\lambda - 1$ or λ .

The partition of the set of all blocks in \mathcal{S} onto subsets of blocks passing through some point from \mathcal{O}' is disjoint and therefore the \parallel is an equivalence relation. ■

2.2 REMARK. By 2.1, $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ (where \mathcal{O}' is a Baer subdesign of \mathcal{O}) is a PSD with a divisibility. But, in general, \mathcal{S} cannot have a resolution. For instance $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$, where \mathcal{O} is a symmetric $2 - (16, 6, 2)$ design with a symmetric $2 - (4, 3, 2)$ subdesign \mathcal{O}' .

But we have

2.3 PROPOSITION. *Let $\mathcal{O} = PG_2(3, q)$ (q a prime power). Then \mathcal{O} has Baer $2 - (q + 1, q + 1, q + 1)$ subdesign \mathcal{O}' and the $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ has a strong resolution.*

PROOF. By [4] \mathcal{O}' exists. Any class of blocks are all blocks in $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ through any point in \mathcal{O}' . Two different classes are disjoint.

Each of these classes have exactly $m = q^2 + q + 1 - (q - 1) = q^2$ blocks and any two blocks in the same class have exactly $\lambda - 1$ points in common. Two blocks in the different classes have exactly λ points in common. Finally, each point in $\mathcal{O} \setminus \mathcal{O}'$ lies exactly on $\lambda - 1$ blocks of any one class. Thus, \mathcal{S} have a strong resolution. ■

In general, let $\mathcal{O} = PG_{2d}(2d + 1, q)$ ($d \geq 2$) be the design of points and hyperplanes of the $(2d + 1)$ -dimensional projective space over $GF(q)$. Then, by [4], \mathcal{O} has a tight (c, c, c) -subdesign \mathcal{O}' with $c = q^d + \dots + q + 1$.

In general, we cannot say anything of the relation \parallel (as in 2.1 and 2.3) in $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$. Namely, the partition of \mathcal{S} , corresponding to \parallel , is not

disjoint. Further, we cannot say anything of a inner (outer) constant (for \parallel). But we have

2.4 PROPOSITION. *Let $\mathcal{O} = PG_{2d}(2d+1, q)$ ($d \geq 2$) and let \mathcal{O}' be a tight (c, c, c) -subdesign. Then $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ is a PSD having a strong resolution.*

PROOF. The parameters of \mathcal{O} and \mathcal{O}' are $v = q^{2d+1} + \dots + q + 1$, $k = q^{2d} + \dots + q + 1$, $\lambda = q^{2d-1} + \dots + q + 1$ and $c = q^d + \dots + q + 1$. It is not difficult to check that any point in $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$ is exactly on x blocks of \mathcal{O}' where $x = q^{d-1} + \dots + q + 1$. Thus \mathcal{S} is a square $1 - (v - c, k - x, k - x)$ design. Further we have:

$$\text{from } 1 + (x(x-1))/\bar{\lambda}_1 = c \quad \text{we get } \bar{\lambda}_1 = q^{d-2} + \dots + q + 1;$$

$$k - c = \dots = q^{d+1} \cdot (q^{d-1} + \dots + q + 1) = q^{d+1} \cdot x$$

and

$$\bar{\lambda} - c = \dots = q^{d+1}(q^{d-2} + \dots + q + 1) = q^{d+1}\bar{\lambda}_1.$$

Here there is an automorphism $y \mapsto y + u$ exchanging the points resp. the blocks (pointwise) in \mathcal{O}' (Singer cycle in \mathcal{O}' , generated additively with u in Z_v). Thus we conclude that the blocks in \mathcal{O}' ($\cong PG_{d-1}(d, q)$) have x sets, each of these sets has exactly q^{d+1} points (in $\mathcal{O} \setminus \mathcal{O}'$) and any two blocks in \mathcal{O}' have exactly $\bar{\lambda}_1$ sets in common. We are calling these sets «points». So any two points of $\mathcal{O} \setminus \mathcal{O}'$ are in one or in two «points». Therefore through two points in $\mathcal{O} \setminus \mathcal{O}'$ pass, according with this, x or $\bar{\lambda}_1$ blocks from \mathcal{O}' . Thus, on two points of \mathcal{S} lie $\lambda_1 = \lambda - x$ or $\lambda_2 = \lambda - \bar{\lambda}_1$ common blocks. By [4] (2.1), we get this for the intersections of the blocks in \mathcal{S} . Thus, \mathcal{S} is a PSD for

$$(v - c, k - x, \lambda_1 = \lambda - x, \lambda_2 = \lambda - \bar{\lambda}_1) =$$

$$= (q^{2d+1} + \dots + q^{d+1}, q^{2d} + \dots + q^d, q^{2d-1} + \dots + q^d, q^{2d-1} + \dots + q^{d-1}).$$

Any resolution-class in \mathcal{S} is formed from all blocks in \mathcal{S} passing through any block in $2 - (c, x, \bar{\lambda}_1)$ design \mathcal{O}' . One has exactly

$$\frac{k - c}{x} = \frac{q^{d+1}x}{x} = q^{d+1}$$

blocks in each resolution-class and exactly

$$\frac{v - c}{q^{d+1}} = \frac{q^{2d+1} + \dots + q + 1 - (q^d + \dots + q + 1)}{q^{d+1}} = \dots = c$$

(disjoint!) classes. There are exactly $(\lambda - x)/x = \dots = q^d$ blocks from each resolution-class passing through any point in $\mathcal{S} = \mathcal{O} \setminus \mathcal{O}'$.

Finally, our resolution is strong with inner and outer constant $\lambda - x$ and $\lambda - \bar{\lambda}_1$ respectively. ■

AN ILLUSTRATION. $\mathcal{O} = PG_4(5, 2)$ (with a tight subdesign for $(7, 7, 7)$). The initial block in \mathcal{O} (in the form of a difference set) is

$1_0 = 0, 1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 16, 18, 19, 24, 26, 27, 28,$

$32, 33, 35, 36, 38, 41, 45, 48, 49, 52, 54, 56.$

(All blocks 1_i are formed by (mod. 63) addition of the $1, 2, \dots, 62$ respectively.) The points in \mathcal{O}' are $0, 9, 18, 27, 36, 45, 54$.

The resolution-classes are:

$$\begin{aligned} (0) \wedge (9) &= (0) \wedge (45) = (9) \wedge (45) = \\ &= \{1_7, 1_{31}, 1_{37}, 1_{39}, 1_{44}, 1_{56}, 1_{59}, 1_{60}\}, \end{aligned}$$

$$\begin{aligned} (0) \wedge (18) &= (0) \wedge (27) = (18) \wedge (27) = \\ &= \{1_{11}, 1_{14}, 1_{15}, 1_{25}, 1_{49}, 1_{55}, 1_{57}, 1_{62}\}, \end{aligned}$$

$$\begin{aligned} (0) \wedge (36) &= (0) \wedge (54) = (36) \wedge (54) = \\ &= \{1_{22}, 1_{28}, 1_{30}, 1_{35}, 1_{47}, 1_{50}, 1_{51}, 1_{61}\}, \end{aligned}$$

$$\begin{aligned} (9) \wedge (18) &= (9) \wedge (54) = (18) \wedge (54) = \\ &= \{1_2, 1_5, 1_6, 1_{16}, 1_{40}, 1_{46}, 1_{48}, 1_{53}\}, \end{aligned}$$

$$\begin{aligned} (9) \wedge (27) &= (9) \wedge (36) = (27) \wedge (36) = \\ &= \{1_1, 1_3, 1_8, 1_{20}, 1_{23}, 1_{24}, 1_{34}, 1_{58}\}, \end{aligned}$$

$$\begin{aligned} (18) \wedge (36) &= (18) \wedge (45) = (36) \wedge (45) = \\ &= \{1_4, 1_{10}, 1_{12}, 1_{17}, 1_{29}, 1_{32}, 1_{33}, 1_{43}\}, \end{aligned}$$

$$\begin{aligned} (27) \wedge (45) &= (27) \wedge (54) = (45) \wedge (54) = \\ &= \{1_{13}, 1_{19}, 1_{21}, 1_{26}, 1_{38}, 1_{41}, 1_{42}, 1_{52}\}, \end{aligned}$$

This is the 4-resolution with the constants 12 and 14.

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