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Generalized solutions of time dependent
impulsive control systems

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ABSTRACT - This paper is concerned with the impulsive Cauchy problem
\[ \dot{x}(t) = f(t, x) + g(t, x) u(t), \quad t \in [0, T], \quad x(0) = \bar{x} \]
where \( u \) is a possibly discontinuous control function and the vector fields \( f, g : R \times R^n \rightarrow R^n \) are measurable in \( t \) and Lipschitz continuous in \( x \). If \( g \) is smooth w.r.t. the variable \( x \) and satisfies
\[ \| g(t, \cdot) - g(s, \cdot) \|_2 \leq \phi(t) - \phi(s), \]
for some increasing function \( \phi \) and every \( s < t \), we show that the above Cauchy problem is well posed as \( u \) ranges in the space \( L^1(d\phi) \).

1. Introduction.

Consider the Cauchy problem for an impulsive control system of the form
\[ \dot{x}(t) = f(t, x) + g(t, x) u(t), \quad t \in [0, T], \quad x(0) = \bar{x} \in R^n, \]
where \( u \) is a scalar control function and the dot denotes a derivative w.r.t. time. We assume that the vector fields \( f, g : R \times R^n \rightarrow R^n \) are bounded, measurable in \( t \) and Lipschitz continuous in \( x \), so that
\begin{align*}
(1.2) \quad |f(t, x)| & \leq M, \quad |g(t, x)| \leq M, \\
(1.3) \quad |f(t, x) - f(t, y)| & \leq L|x - y|, \quad |g(t, x) - g(t, y)| \leq L|x - y|,
\end{align*}

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for some constants $M, L$. Under these conditions, for any continuously differentiable scalar function $u$, the right hand side of (1.1) is measurable in $t$ and Lipschitz in $x$. Therefore, a well known theorem of Carathéodory [1] provides the existence and uniqueness of the corresponding solution $t \mapsto x(t, u)$. Aim of this paper is to show that, under suitable assumptions on $g$, the map $u \mapsto x(T, u)$ can be continuously extended to a much larger space of (possibly discontinuous) control functions. Besides (1.2)-(1.3), let $g$ be twice continuously differentiable w.r.t. $x$, say

$$
(1.4) \quad \|g(t, \cdot)\|_{C^2} = \\
\sup_x \left\{ \left| g(t, x) \right| + \sum_{i=1}^{n} \left| \frac{\partial g(t, x)}{\partial x_i} \right| + \sum_{i,j=1}^{n} \left| \frac{\partial^2 g(t, x)}{\partial x_i \partial x_j} \right| \right\} \leq M.
$$

Moreover, we shall assume that the total variation of $g$ w.r.t. time is bounded:

$$
(1.5) \quad \|g(t, \cdot) - g(s, \cdot)\|_{C^1} \leq \phi(t) - \phi(s), \quad 0 \leq s < t \leq T
$$

for some non-decreasing function $\phi$. Observe that, if $u$ is a $C^1$ function, the solution of (1.1) is not affected by changing $g$ on a set of times of measure zero. For simplicity, we shall thus assume that both $g$ and $\phi$ are right continuous functions of time. By possibly replacing $\phi$ with

$$
\tilde{\phi}(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 + t + \phi(t) & \text{if } 0 \leq t < T, \\ 2 + T + \phi(T) & \text{if } t \geq T, 
\end{cases}
$$

it is not restrictive to assume that

$$
(1.6) \quad \phi(0^+) - \phi(0^-) \geq 1, \quad \phi(T^+) - \phi(T^-) \geq 1, \quad \dot{\phi}(t) \geq 1 \text{ a.e.}.
$$

By (1.6), the positive Radon measure $d\phi$ contains an atom at $t = 0$ and at $t = T$, and satisfies $d\phi \geq dx$, where $dx$ denotes the standard Lebesgue measure. We can now state the main result of this paper.

**Theorem 1.1.** Consider a set of bounded, measurable control functions of the form $u \in \{ u : [0, T] \mapsto [-M_1, M_1] \mid u \in C^1 \}$. For $u \in U$, call $x(t, u)$ the corresponding solution of the Cauchy problem (1.1). Then,
under the assumptions (1.2)-(1.6), the map \( u \mapsto x(T, u) \) satisfies

\[
|x(T, u) - x(T, v)| \leq C\int_0^T |u(t) - v(t)| \, d\phi(t),
\]

for some constant \( C \) and all \( u, v \in \mathcal{U} \).

As a consequence, the map \( x(T, u) \) can be uniquely extended by continuity to the closure of \( \mathcal{U} \) in the space \( L^1(d\phi) \). This provides a natural definition of solution of (1.1) also for a discontinuous control \( u \),

\[
x(T, u) = \lim_{n \to \infty} x(T, v_n),
\]

where \( \{v_n\}_{n \geq 1} \) is any bounded sequence of \( C^1 \) functions, tending to \( u \) in the space \( L^1(d\phi) \).

**Remark 1.2.** In the case where \( g \) is a piecewise smooth function of \( t, x \), with finitely many jumps at times \( 0 = t_0 < t_1 < \ldots < t_n = T \), one can always construct a function \( \phi \) such that (1.5) holds. Indeed, for suitable constants \( C_1, C_2 \), one can take

\[
\phi(t) = C_1 t + C_2 \cdot \sup \{k; \ t_k \leq t\}.
\]

**Remark 1.3.** Our results can be extended to systems of the form

\[
\dot{x} = f(t, x, u) + g(t, x, u) \dot{u}.
\]

Indeed, the dependence on \( u \) is easily removed by introducing an additional coordinate \( x_0 = u \), with \( \dot{x}_0 = \dot{u} \).

In the case where the vector fields \( f, g \) do not depend on time, solutions of the impulsive Cauchy problem (1.1) were studied in [2]. For a special class of Lagrangean systems with piecewise continuous dependence on a time-like variable, the impulsive control problem was recently considered in [6]. The present approach is simpler than [6], since it does not require any smoothing approximation of the vector field \( g \).

The proof of Theorem 1.1 is given in the next two sections. We first introduce a suitable definition of solution of (1.1), valid when \( u \) lies in the set

\[
\mathcal{U}' = \{u: [0, T] \to [-M_1, M_1] \mid u \text{ is piecewise constant and all of its jumps occur at times } t \neq 0, T \text{ where } \phi \text{ is continuous}\}.
\]
For $u \in \mathcal{U}'$, we show that the inequality (1.7) holds, hence the map $\nu \mapsto x(T, \nu)$ can be continuously extended to the closure of $\mathcal{U}'$ in the space $L^1(d\phi)$. When $u \in C^1$, this continuous extension coincides with the usual Carathéodory definition. Since the closures of $\mathcal{U}$ and $\mathcal{U}'$ coincide, the result will be proved.

2. Definition of generalized solutions and preliminary lemmas.

Let $k:\[0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a (time dependent) vector field, and fix a time $\tau \in [0, T]$. Denote by $t \mapsto \exp\{tk(\tau)\} \tilde{x}$ the solution of the Cauchy problem

\begin{equation}
\dot{x}(t) = k(\tau, x(t)), \quad x(0) = \tilde{x}.
\end{equation}

We assume that for every $x \in \mathbb{R}^n$, the map $t \mapsto k(t, x)$ is measurable and for every $t \in [0, T]$, the map $x \mapsto k(t, x)$ is continuously differentiable. Moreover, denote by $t \mapsto \Phi(t, k(\tau), \tilde{x})$ the fundamental matrix solution of the linear differential equation

\begin{equation}
\dot{v}(t) = D_x k(\tau, \exp\{tk(\tau)\} \tilde{x}) \cdot v(t),
\end{equation}

with $\Phi(0, k(\tau), \tilde{x})$ the identity matrix. Here $D_x k(\tau, \cdot)$ represents the Jacobian matrix of first order partial derivatives of $k(\tau, \cdot)$ with respect to $x$.

The matrix $\Phi(t, k(\tau), \tilde{x})$ has the following properties.

**Lemma 2.1.** Let $M_2$ be a constant such that

$$|D_x k(\tau, \exp\{tk(\tau)\} \tilde{x}) \cdot w| \leq M_2 |w|$$

for every $\tilde{x}, w \in \mathbb{R}^n$, $\tau \in [0, T]$ and $|t| \leq M_1$. Then $|\Phi(t, k(\tau), \tilde{x}) \cdot w| \leq \leq |w| e^{M_2 |t|}.$

**Proof.** Since $d/dt |\Phi(t, k(\tau), \tilde{x}) \cdot w| \leq M_2 |\Phi(t, k(\tau), \tilde{x}) \cdot w|$, by Gronwall's inequality $|\Phi(t, k(\tau), \tilde{x}) \cdot w| \leq |w| e^{M_2 |t|}.$

**Lemma 2.2.** Let $k$ be twice continuously differentiable w.r.t. $x$ and let $\tau \in [0, T]$. Suppose that for any $x, y \in \mathbb{R}^n$

$$|k(\tau, x) - k(\tau, y)| \leq L|x - y|$$
and \( \|k(\tau, \cdot)\|_\infty \leq M \). Then for any \( 0 \leq t \leq M_1 \) and \( x_1, x_2, w \in \mathbb{R}^n \),

\[
|\Phi(t, k(\tau), x_1) \cdot w - \Phi(t, k(\tau), x_2) \cdot w| \leq n^3 M M_1 |x_1 - x_2| |w| e^{2LM_1 + n^2 MM_1}.
\]

**Proof.** Let \( \tau \in [0, T] \) and \( x_1, x_2, w \in \mathbb{R}^n \). We put \( v_1(t) = \Phi(t, k(\tau), x_1) \cdot w \) and \( v_2(t) = \Phi(t, k(\tau), x_2) \cdot w \). For \( i = 1 \) and \( 2 \), \( v_i(t) \) is the value at time \( t \) of the solution of the linear differential equation

\[
\dot{v}_i(t) = D_x k(\tau, \exp \{k(\tau)\} x_i) \cdot v_i(t), \quad v_i(0) = w.
\]

Observing that for any \( v, x, y \in \mathbb{R}^n \),

\[
|D_x k(\tau, x) \cdot v| \leq n^2 M |v|
\]

and

\[
|D_x k(\tau, x) \cdot v - D_x k(\tau, y) \cdot v| \leq n^3 M |x - y| |v|,
\]
due to Lemma 2.1

\[
\frac{d}{dt} |v_1(t) - v_2(t)| \leq
\]

\[
\leq |D_x k(\tau, \exp \{k(\tau)\} x_1) \cdot v_1(t) - D_x k(\tau, \exp \{k(\tau)\} x_2) \cdot v_2(t)| \leq
\]

\[
\leq |D_x k(\tau, \exp \{k(\tau)\} x_1) \cdot v_1(t) - D_x k(\tau, \exp \{k(\tau)\} x_1) \cdot v_2(t)| +
\]

\[
+ |D_x k(\tau, \exp \{k(\tau)\} x_1) \cdot v_2(t) - D_x k(\tau, \exp \{k(\tau)\} x_2) \cdot v_2(t)| \leq
\]

\[
\leq n^2 M |v_1(t) - v_2(t)| + n^3 M |x_1 - x_2| |w| e^{LM_1 + n^2 MM_1}.
\]

Gronwall's inequality implies that

\[
|v_1(t) - v_2(t)| \leq n^3 M M_1 |x_1 - x_2| |w| e^{LM_1 + 2n^2 MM_1}.
\]

When \( u \in \mathcal{U} \), the corresponding generalized solution \( x(t, u) \) of (1.1) can be defined in a straightforward manner. Indeed, let \( u \) have jumps at points \( t_i \), with \( 0 < t_1 < \ldots < t_n < T \). In this case, \( x(t, u) \) is the function which solves the differential equation

\[
(2.3) \quad \dot{x}(t) = f(t, x(t))
\]
on each subinterval \( ]t_{i-1}, t_i[ \), together with the boundary conditions

\[
(2.4) \quad x(0) = \overline{x}, \quad x(t_i +) = \exp \{u(t_i +) - u(t_i -)) g(t_i)\} x(t_i -),
\]

\[
i = 1, \ldots, n.
\]
To study the continuous dependence of these solutions on the control $u \in U'$, it is convenient to introduce an alternative representation, in terms of a new variable $\xi$, which will remove the discontinuities due to the jumps in $u$.

Choose points $c_i$ with $c_1 = 0$, $c_{n+1} = T$, such that $c_i < t_i < c_{i+1}$ for each $i = 1, \ldots, n$. Since $u$ is constant outside the points $t_i$, on each subinterval $I_i = [c_i, c_{i+1}]$, the function $x(t, u)$ provides a solution to

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t)) \dot{u}(t), \quad c_i \leq t \leq c_{i+1}.$$  

(2.5)

Defining the auxiliary variable $\xi(t) = \exp \{-u(t) g(t_i)\} x(t)$, it is known [2,3] that $\xi$ is an absolutely continuous function which satisfies

$$\dot{\xi}(t) = F^*(t, t_i, \xi(t), u(t)), \quad \text{a.e. on } [c_i, c_{i+1}],$$  

(2.6)

where $F^* : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is defined by

(2.7)

$$F^*(t, \tau, \xi, u) = \Phi(-u, g(\tau), \exp \{u g(\tau)\} \xi) \cdot f(t, \exp \{u g(\tau)\} \xi).$$

For $u \in U'$, the corresponding solution $t \mapsto x(t, u)$ can thus be obtained by setting

$$x(t, u) = \exp \{u(t_1) g(t_i)\} \xi(t, u), \quad t \in I_i,$$

(2.8)

where $t \mapsto \xi(t, u)$ is the piecewise continuous function such that

$$\dot{\xi}(t) = F^*(t, t_i, \xi(t), u(t)), \quad t \in I_i,$$

(2.9)

$$\begin{cases}
\xi(0) = \exp \{-u(0) g(t_1)\} \overline{x}, \\
\xi(c_i^+) = \exp \{-u(c_i) g(t_i)\} (\exp \{u(c_i) g(t_{i-1})\} \xi(c_i^-)).
\end{cases}$$

(2.10)

The main advantage of the representation (2.9)-(2.10) is the following. The total variation of $u$, and hence of $x$, can be arbitrarily large. On the other hand, the total variation of $\xi$ is related to the total variation of $g$, which by (1.5) is bounded in terms of $\phi$. For this reason, it is convenient to study the solution of (1.1) in terms of the variable $\xi$, which is much better behaved than $u$ or $x$.

From now on, we assume that $f$ and $g$ satisfy all the hypotheses in Theorem 1.1. The following lemma shows that the map $F^*$ defined in (2.7) is Lipschitz continuous w.r.t. both variables $\xi, u$. 

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LEMMA 2.3. There exists $L_1 > 0$ such that for any $t, \tau \in [0, T], \xi_1, \xi_2 \in \mathbb{R}^n$ and $|u_1|, |u_2| \leq M_1$,

\begin{equation}
|F^*(t, \tau, \xi_1, u_1) - F^*(t, \tau, \xi_2, u_2)| \leq L_1(|\xi_1 - \xi_2| + |u_1 - u_2|).
\end{equation}

PROOF. By (1.4), we can easily see that for any $\xi, w \in \mathbb{R}^n$ and $|t| \leq M_1$, $|D_2 g(\tau, \exp \{tg(\tau)\} \xi) \cdot w| \leq n^2 M |w|$. By Lemma 2.1 and Lemma 2.2,

\begin{align*}
|F^*(t, \tau, \xi_1, u_1) - F^*(t, \tau, \xi_2, u_1)| &\leq |\Phi(-u_1, g(\tau), \exp \{u_1g(\tau)\} \xi_1) \cdot f(t, \exp \{u_1g(\tau)\} \xi_1) - \\
&- \Phi(-u_1, g(\tau), \exp \{u_1g(\tau)\} \xi_2) \cdot f(t, \exp \{u_1g(\tau)\} \xi_2)| + \\
&+ |\Phi(-u_1, g(\tau), \exp \{u_1g(\tau)\} \xi_2) \cdot f(t, \exp \{u_1g(\tau)\} \xi_1) - \\
&- \Phi(-u_1, g(\tau), \exp \{u_1g(\tau)\} \xi_2) \cdot f(t, \exp \{u_1g(\tau)\} \xi_2)| \\
&\leq n^3 M^2 M_1 e^{2LM_1 + 2n^2MM_1} |\xi_1 - \xi_2| + Le^{LM_1 + n^2MM_1} |\xi_1 - \xi_2| = C_1 |\xi_1 - \xi_2|
\end{align*}

and

\begin{align*}
|F^*(t, \tau, \xi_2, u_1) - F^*(t, \tau, \xi_2, u_2)| &\leq |\Phi(-u_1, g(\tau), \exp \{u_1g(\tau)\} \xi_2) \cdot f(t, \exp \{u_1g(\tau)\} \xi_2) - \\
&- \Phi(-u_1, g(\tau), \exp \{u_2g(\tau)\} \xi_2) \cdot f(t, \exp \{u_2g(\tau)\} \xi_2)| + \\
&+ |\Phi(-u_1, g(\tau), \exp \{u_2g(\tau)\} \xi_2) \cdot f(t, \exp \{u_2g(\tau)\} \xi_2) - \\
&- \Phi(-u_2, g(\tau), \exp \{u_2g(\tau)\} \xi_2) \cdot f(t, \exp \{u_2g(\tau)\} \xi_2)| \\
&\leq LMe^{n^2MM_1} |u_1 - u_2| + n^3 M^3 M_1 e^{LM_1 + 2n^2MM_1} |u_1 - u_2| + \\
&+ n^2 M^2 e^{n^2MM_1} |u_1 - u_2| = C_2 |u_1 - u_2|
\end{align*}

where $C_1 = n^3 M^2 M_1 e^{2LM_1 + 2n^2MM_1} + Le^{LM_1 + n^2MM_1}$ and $C_2 = e^{n^2MM_1}(n^3 M^3 M_1 e^{LM_1 + n^2MM_1} + LM + n^2 M^2)$. Thus (2.11) holds for $L_1 = C_1 + C_2$. $\blacksquare$
Lemma 2.4. Let $\bar{x} \in \mathbb{R}^n$ and $0 \leq \tau_1 < \tau_2 \leq T$. Then for any $t \in \mathbb{R}$,

\begin{equation}
|\exp \{tg(\tau_1)\} \bar{x} - \exp \{tg(\tau_2)\} \bar{x}| \leq e^{L|t|} (\phi(\tau_2) - \phi(\tau_1))|t|.
\end{equation}

Proof. Replacing $t$ with $-t$, it is not restrictive to assume $t > 0$. Observing that

\[
\frac{d}{dt} |\exp \{tg(\tau_1)\} \bar{x} - \exp \{tg(\tau_2)\} \bar{x}| \leq \\
\leq |g(\tau_1, \exp \{tg(\tau_1)\} \bar{x}) - g(\tau_2, \exp \{tg(\tau_2)\} \bar{x})| \leq \\
\leq |g(\tau_1, \exp \{tg(\tau_1)\} \bar{x}) - g(\tau_1, \exp \{tg(\tau_2)\} \bar{x})| + \\
+ |g(\tau_1, \exp \{tg(\tau_2)\} \bar{x}) - g(\tau_2, \exp \{tg(\tau_2)\} \bar{x})| \leq \\
\leq L |\exp \{tg(\tau_1)\} \bar{x} - \exp \{tg(\tau_2)\} \bar{x}| + (\phi(\tau_2) - \phi(\tau_1)),
\]

Gronwall's inequality implies

\[
|\exp \{tg(\tau_1)\} \bar{x} - \exp \{tg(\tau_2)\} \bar{x}| \leq e^{L|t|} (\phi(\tau_2) - \phi(\tau_1))|t|.
\]

Let $t_1, t_2 \in [0, T]$ and $p, q, v \in \mathbb{R}^n$. From (1.4) and (1.5), we can easily see that

\begin{align}
(2.13) & \quad |(D_z g(t_1, p) - D_z g(t_2, p)) \cdot v| \leq n^2 |\phi(t_2) - \phi(t_1)| |v|, \\
(2.14) & \quad |g(t_1, p) - g(t_1, q)| \leq n^2 M |p - q|, \\
(2.15) & \quad |D_z g(t_1, p) \cdot v| \leq n^2 M |v|
\end{align}

and

\begin{align}
(2.16) & \quad |(D_z g(t_1, p) - D_z g(t_1, q)) \cdot v| \leq n^3 M |p - q| |v|.
\end{align}

We define a map

\[k(t_i, \tau) = g(t_i, (1 - \tau) q + \tau p), \quad \tau \in [0, 1], \quad i = 1, 2.\]

Then $k$ is differentiable w.r.t. $\tau$ and we have

\begin{align}
(2.17) & \quad |g(t_2, p) - g(t_2, q) - g(t_1, p) + g(t_1, q)| = \\
\quad = |k(t_2, 1) - k(t_2, 0) - k(t_1, 1) + k(t_1, 0)| = \left| \int_0^1 \frac{d}{d\tau} (k(t_2, \tau) - k(t_1, \tau)) d\tau \right| = \\
\end{align}
In the similar way, we have that

\[
\text{PROPOSITION 2.5. Let } x_0, y_0 \in \mathbb{R}^n \text{ and let } t_1 \text{ and } t_2 \text{ be points on } [0, T].
\]

Define a map \( K: [-M_1, M_1] \to \mathbb{R}^n \) by

\[
(2.19) \quad K(s) = \exp \left\{ -s g(t_2) \right\} x_0 - \exp \left\{ -s g(t_2) \right\} \exp \left\{ s g(t_1) \right\} y_0.
\]

Then there exists \( B_1 > 0 \) such that for any \( s \in [-M_1, M_1] \),

\[
(2.20) \quad |K(s)| \leq |x_0 - y_0| e^{B_1|\phi(t_2) - \phi(t_1)|}.
\]

**PROOF.** Let

\[
\begin{align*}
p_1 &= \exp \{ s g(t_1) \} x_0, \\
p_2 &= \exp \{ -s g(t_2) \} p_1, \\
q_1 &= \exp \{ s g(t_1) \} y_0 \quad \text{and} \quad q_2 = \exp \{ -s g(t_2) \} q_1.
\end{align*}
\]

Then

\[
K(s) = p_2 - q_2
\]

and

\[
K'(s) = -g(t_2, p_2) + \Phi(-s, g(t_2), p_1) \cdot g(t_1, p_1) + g(t_2, q_2) -
\]

\[
- \Phi(-s, g(t_2), q_1) \cdot g(t_1, q_1) = -g(t_2, p_2) + g(t_2, q_2) +
\]

\[
+ \Phi(-s, g(t_2), \exp \{ s g(t_2) \} p_2) \cdot g(t_1, \exp \{ s g(t_2) \} p_2) -
\]

\[
- \Phi(-s, g(t_2), \exp \{ s g(t_2) \} q_2) \cdot g(t_1, \exp \{ s g(t_2) \} q_2).
\]

Define a map \( H: [-M_1, M_1] \to \mathbb{R}^n \) by

\[
(2.21) \quad H(s) = \Phi(-s, g(t_2), \exp \{ s g(t_2) \} p_2) \cdot g(t_1, \exp \{ s g(t_2) \} p_2) -
\]

\[
- \Phi(-s, g(t_2), \exp \{ s g(t_2) \} q_2) \cdot g(t_1, \exp \{ s g(t_2) \} q_2).
\]
Observing that for any \( p \in \mathbb{R}^n \),

\[
\frac{d}{ds} \Phi(-s, g(t_2), \exp \{ s g(t_2) \} \ p) \cdot g(t_1, \exp \{ s g(t_2) \} \ p) =
\]

\[
= -\Phi(-s, g(t_2), \exp \{ s g(t_2) \} \ p) \cdot D_x g(t_2, \exp \{ s g(t_2) \} \ p) \cdot g(t_1, \exp \{ s g(t_2) \} \ p) + \Phi(-s, g(t_2), \exp \{ s g(t_2) \} \ p) \cdot D_x g(t_1, \exp \{ s g(t_2) \} \ p) \cdot g(t_2, \exp \{ s g(t_2) \} \ p),
\]

we have

\[
(2.22) \quad H'(s) = \Phi(-s, g(t_2), \exp \{ s g(t_2) \} \ p_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp \{ s g(t_2) \} \ p_2) - \Phi(-s, g(t_2), \exp \{ s g(t_2) \} q_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp \{ s g(t_2) \} q_2),
\]

where \([g(t_2, \cdot), g(t_1, \cdot)](p) = D_x g(t_1, p) \cdot g(t_2, p) - D_x g(t_2, p) \cdot g(t_1, p)\). If there exists \( B_2 > 0 \) such that

\[
(2.23) \quad |K'(s)| \leq B_2 |\phi(t_2) - \phi(t_1)| \cdot |p_2 - q_2|,
\]

then by Gronwall's inequality

\[
(2.24) \quad |K(s)| \leq |x_0 - y_0| e^{B_1 |\phi(t_2) - \phi(t_1)|},
\]

where \( B_1 = M_1 B_2 \). We thus only have to show that inequality (2.23) holds for some \( B_2 > 0 \). Since

\[
(2.25) \quad K'(s) = H(s) - g(t_2, p_2) + g(t_2, q_2) =
\]

\[
= H(s) - (g(t_1, p_2) - g(t_1, q_2)) + (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)) =
\]

\[
= H(s) - H(0) + (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)) =
\]

\[
= \int_0^s H'(\tau)d\tau + (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2))
\]

and by (2.17)

\[
|g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)| \leq n^2 |\phi(t_2) - \phi(t_1)| \cdot |p_2 - q_2|,
\]

to claim inequality (2.23) we shall show that there exists \( B_3 > 0 \) such that
for any \( \tau \in [-M_1, M_1] \)

\[ |H'(\tau)| \leq B_3 |\phi(t_2) - \phi(t_1)| |p_2 - q_2| . \]

We fix \( \tau \in [-M_1, M_1] \) and define maps \( v_1, v_2: [-M_1, M_1] \to \mathbb{R}^n \) by

\[ v_1(\sigma) = \Phi(\sigma, g(t_2), \exp \{ \tau g(t_2) \} p_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp \{ \tau g(t_2) \} p_2) \]

and

\[ v_2(\sigma) = \Phi(\sigma, g(t_2), \exp \{ \tau g(t_2) \} q_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp \{ \tau g(t_2) \} q_2) . \]

Then

\[ H'(\tau) = v_1(-\tau) - v_2(-\tau) \]

and \( v_1, v_2 \) satisfy

\[ \frac{d}{d\sigma} v_1(\sigma) = D_x g(t_2, \exp \{ \sigma g(t_2) \} p_3) \cdot v_1(\sigma), \]

\[ v_1(0) = [g(t_2, \cdot), g(t_1, \cdot)](p_3), \]

(2.29)

\[ \frac{d}{d\sigma} v_2(\sigma) = D_x g(t_2, \exp \{ \sigma g(t_2) \} q_3) \cdot v_2(\sigma), \]

\[ v_2(0) = [g(t_2, \cdot), g(t_1, \cdot)](q_3), \]

where \( p_3 = \exp \{ \tau g(t_2) \} p_2 \) and \( q_3 = \exp \{ \tau g(t_2) \} q_2 \). We compute a bound for \( |v_1(0) - v_2(0)| \) to get

\[ |v_1(0) - v_2(0)| = |D_x g(t_1, p_3) \cdot g(t_2, p_3) - D_x g(t_2, p_3) \cdot g(t_1, p_3) - D_x g(t_1, q_3) \cdot g(t_2, q_3) + D_x g(t_2, q_3) \cdot g(t_1, q_3)| = \]

\[ = |(D_x g(t_1, p_3) - D_x g(t_2, p_3)) \cdot (g(t_2, p_3) - g(t_2, q_3)) + (D_x g(t_2, q_3) - D_x g(t_2, q_3)) \cdot (g(t_1, p_3) - g(t_2, p_3)) + (D_x g(t_1, p_3) - D_x g(t_2, p_3) - D_x g(t_1, q_3) + D_x g(t_2, q_3)) \cdot g(t_2, q_3) + D_x g(t_2, q_3)(g(t_2, p_3) - g(t_1, p_3) - g(t_2, q_3) + g(t_1, q_3))| \leq \]

\[ \leq 4n^4 M |\phi(t_2) - \phi(t_1)| |p_3 - q_3| \leq 4n^4 M e^{LM_1} |\phi(t_2) - \phi(t_1)| |p_2 - q_2| . \]
By considering \(-a\) instead of \(a\), we assume that \(a \approx 0\). Observing that we have a bound for \(|v(\sigma) - v(\sigma)|\) as

\[
|v(\sigma) - v(\sigma)| =
\]

\[
= \left| \int_0^\sigma (D_x g(t_2, \exp \{\eta g(t_2)\} p_3 \cdot v_1(\eta) - D_x g(t_2, \exp \{\eta g(t_2)\} q_3 \cdot v_1(\eta) + D_x g(t_2, \exp \{\eta g(t_2)\} q_3 \cdot v_1(\eta) -
\]

\[
- D_x g(t_2, \exp \{\eta g(t_2)\} q_3 \cdot v_2(\eta)) \, d\eta + (v(0) - v(0)) \right| \leq
\]

\[
\leq \int_0^\sigma n^3 M |\exp \{\eta g(t_2)\} p_3 - \exp \{\eta g(t_2)\} q_3| \, |v_1(\eta)| \, d\eta +
\]

\[
+ \int_0^\sigma n^2 M |v_1(\eta) - v_2(\eta)| \, d\eta + |v_1(0) - v_2(0)| \leq
\]

\[
\leq 2n^5 M^2 M_1 e^{2LM_1 + n^2MM_1} |\phi(t_2) - \phi(t_1)| \, |p_2 - q_2| +
\]

\[
+ 4n^4 M e^{LM_1} |\phi(t_2) - \phi(t_1)| \, |p_2 - q_2| + \int_0^\sigma n^2 M |v_1(\eta) - v_2(\eta)| \, d\eta.
\]

By Gronwall’s inequality,

\[
|v(\sigma) - v(\sigma)| \leq B_3 |p_2 - q_2| \, |\phi(t_2) - \phi(t_1)| \text{ for any } 0 \leq \sigma \leq M_1,
\]

where \(B_3 = 2n^5 M^2 M_1 e^{2LM_1 + n^2MM_1} + 4n^4 M e^{LM_1} e^{n^2MM_1}\). By (2.27),

\[
|H'(\tau)| \leq B_3 |p_2 - q_2| \, |\phi(t_2) - \phi(t_1)|.
\]
We thus have that for any $s \in [-M_1, M_1]$

$$
(2.34) \quad \left| \int_{0}^{s} H' (r) \, dr \right| \leq B_4 \left| p_2 - q_2 \right| \left| \phi (t_2) - \phi (t_1) \right| ,
$$

where $B_4 = B_2 M_1$. By (2.25)

$$
|K' (s)| \leq B_2 \left| p_2 - q_2 \right| \left| \phi (t_2) - \phi (t_1) \right| ,
$$

where $B_2 = B_4 + n^2$. As a consequence, the proposition is proved. □

3. Proof of the theorem.

Before proving that (1.7) holds for $u, v \in \mathcal{U}$, we show that it holds for $u, v \in \mathcal{U}'$. Let $u, v \in \mathcal{U}'$. Recall that the generalized solutions $x(t, u)$ and $x(t, v)$ can be defined in terms of (2.8)-(2.10). Assume that either $u$ or $v$ jumps at $t_i$ where

$$
0 < t_1 < t_2 < \ldots < t_n < T ,
$$

moreover, we may assume that $u$ and $v$ are left continuous since $\phi$ is continuous at each $t_i$ and the integral $\int_{0}^{T} \left| u(t) - v(t) \right| \, d\phi (t)$ is not affected by changing the value $\left| u(t_i) - v(t_i) \right|$. Let $c_1 = 0$ and $d_n = T$. We choose $c_i, d_{i-1} \in (t_{i-1}, t_i]$ with $c_i < d_{i-1}$ for $i = 2, \ldots, n$. Define the time intervals $I_i = [c_i, d_i]$, $i = 1, \ldots, n$. Since $u$ and $v$ are left continuous, it is not restrictive to assume that $d_{i-1} = t_i$. Define

$$
X(t) = \exp \left\{ -u(t) \, g(t) \right\} x(t, u) ,
$$

$$
Y(t) = \exp \left\{ -v(t) \, g(t) \right\} x(t, v) .
$$

Since

$$
| x(T, u) - x(T, v) | = | \exp \left\{ u(T) \, g(T) \right\} X(T) - \exp \left\{ v(T) \, g(T) \right\} Y(T) | ,
$$

we need to estimate the increase of $| X(t_i) - Y(t_i) |$ as $i$ increases. On each interval $I_i$, we define

$$
X_i(t) = \exp \left\{ -u(t) \, g(t_i) \right\} x(t, u) , \quad Y_i(t) = \exp \left\{ -v(t) \, g(t_i) \right\} x(t, v) .
$$
By (2.6), on the interval $I_i$, $X_i$ and $Y_i$ satisfy the differential equations

$$\dot{X}_i(t) = F^*(t, t_i, X_i(t), u(t)),$$ $$\dot{Y}_i(t) = F^*(t, t_i, Y_i(t), v(t)),$$

respectively. Due to Lemma 2.3, on the interval $[t_i, t_{i+1}]$ we have

$$\frac{d}{dt} |X_i(t) - Y_i(t)| \leq L_1 (|X_i(t) - Y_i(t)| + |u(t) - v(t)|).$$

We thus have an estimate by Gronwall's inequality

$$|X_i(t_{i+1}) - Y_i(t_{i+1})| \leq$$

$$\leq |X_i(t_i) - Y_i(t_i)| e^{L_1(t_{i+1} - t_i)} + L_1 e^{L_1 T} \int_{t_i}^{t_{i+1}} |u(s) - v(s)| \, ds.$$

Next, we estimate the difference between

$$|X(t_{i+1}) - Y(t_{i+1})| \quad \text{and} \quad |X_i(t_{i+1}) - Y_i(t_{i+1})|.$$

If we put $x_0 = X_i(t_{i+1})$ and $y_0 = Y_i(t_{i+1})$, then

$$|X(t_{i+1}) - Y(t_{i+1})| =$$

$$= |\exp \{ -u(t_{i+1}) g(t_{i+1}) \} \exp \{ u(t_{i+1}) g(t_i) \} x_0 -$$

$$- \exp \{ -v(t_{i+1}) g(t_{i+1}) \} \exp \{ v(t_{i+1}) g(t_i) \} y_0| \leq$$

$$\leq |\exp \{ -u(t_{i+1}) g(t_{i+1}) \} \exp \{ u(t_{i+1}) g(t_i) \} x_0 -$$

$$- \exp \{ -u(t_{i+1}) g(t_{i+1}) \} \exp \{ u(t_{i+1}) g(t_i) \} y_0| +$$

$$+ |\exp \{ -u(t_{i+1}) g(t_{i+1}) \} \exp \{ u(t_{i+1}) g(t_i) \} y_0 +$$

$$- \exp \{ -v(t_{i+1}) g(t_{i+1}) \} \exp \{ v(t_{i+1}) g(t_i) \} y_0| = E_1 + E_2.$$

If in (3.3) $E_2 \leq C_3 (\phi(t_{i+1}) - \phi(t_i)) |u(t_{i+1}) - v(t_{i+1})|$ for some $C_3 > 0$, then by Proposition 2.5

$$|X(t_{i+1}) - Y(t_{i+1})| \leq$$

$$\leq |X_i(t_{i+1}) - Y_i(t_{i+1})| e^{B_1 (\phi(t_{i+1}) - \phi(t_i))} + C_3 \int_{t_i}^{t_{i+1}} |u(s) - v(s)| \, d\phi(s).$$
By (3.2) and (3.4),

\begin{equation}
|X(t_{i+1}) - Y(t_{i+1})| \leq \nonumber
\end{equation}

\begin{align*}
&\leq e^{B_1(\phi(t_{i+1}) - \phi(t_i))} \left| X(t_i) - Y(t_i) \right| e^{L_1(t_{i+1} - t_i)} + L_1 e^{L_1T} \int_{t_i}^{t_{i+1}} |u(s) - v(s)| \, ds + \\
&+ C_3 \int_{t_i}^{t_{i+1}} |u(s) - v(s)| \, d\phi(s).
\end{align*}

Observing that on the interval \([0, t_1]\) equation (1.1) is \(\dot{x} = f(t, x)\), \(x(t_1, u) = x(t_1, v)\) and

\begin{equation}
|X(t_1) - Y(t_1)| \leq M|u(0) - v(0)|.
\end{equation}

Due to (3.5) and (3.6), we can use the induction to obtain

\begin{align*}
|X(T) - Y(T)| &\leq e^{B_1(\phi(T) - \phi(0)) + L_1T} |X(t_1) - Y(t_1)| + \\
&+ L_1 e^{2L_1T + B_1(\phi(T) - \phi(0))} \int_0^T |u(s) - v(s)| \, d\phi(s) + \\
&+ C_3 e^{B_1(\phi(T) - \phi(0)) + L_1T} \int_0^T |u(s) - v(s)| \, ds \leq e^{B_1(\phi(T) - \phi(0)) + L_1T}.
\end{align*}

\begin{align*}
&\cdot \left( M|u(0) - v(0)| + (L_1 e^{L_1T} + C_3) \int_0^T |u(s) - v(s)| \, d\phi(s) \right) \leq \\
&\leq C_4 \int_0^T |u(s) - v(s)| \, d\phi(s),
\end{align*}

where \( C_4 = e^{B_1(\phi(T) - \phi(0)) + L_1T} (M + L_1 e^{L_1T} + C_3) \). We can estimate

\begin{align*}
|x(T, u) - x(T, v)| = |\exp \{ u(T) \, g(T) \} X(T) - \exp \{ v(T) \, g(T) \} Y(T)| \leq \\
\leq |\exp \{ u(T) \, g(T) \} X(T) - \exp \{ u(T) \, g(T) \} Y(T)| + \\
+ |\exp \{ u(T) \, g(T) \} Y(T) - \exp \{ v(T) \, g(T) \} Y(T)| \leq \\
\leq e^{LM_T} |X(T) - Y(T)| + M|u(T) - v(T)| \leq C_5 \int_0^T |u(s) - v(s)| \, d\phi(s),
\end{align*}

where \( C_5 = C_4 e^{LM_T} + M \). Hence (1.7) holds for \( u, v \in U' \).
Now we need to show that, in (3.3), $E_2 \leq C_3 (\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|$ for some constant $C_3 > 0$. By Lemma 2.4,

$$|\exp \{u(t_{i+1}) g(t_i)\} y_0 - \exp \{(u_{i+1}) g(t_{i+1})\} \exp \{\nu(t_{i+1}) g(t_i)\} y_0| \leq e^{2LM_1} (\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|$$

and

$$|\exp \{-u(t_{i+1}) g(t_{i+1})\} \exp \{u(t_{i+1}) g(t_i)\} y_0 - \exp \{-\nu(t_{i+1}) g(t_{i+1})\} \exp \{\nu(t_{i+1}) g(t_i)\} y_0| =$$

$$= |\exp \{-u(t_{i+1}) g(t_{i+1})\} \exp \{u(t_{i+1}) g(t_i)\} y_0 - \exp \{-\nu(t_{i+1}) g(t_{i+1})\} \exp \{\nu(t_{i+1}) g(t_i)\} y_0| \leq e^{3LM_1} (\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|.$$

Therefore $E_2 \leq C_3 (\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|$ for $C_3 = e^{3LM_1}$.

Next, we claim that (1.7) holds for $u, v \in \mathcal{U}$. Suppose that for any $w \in \mathcal{U}$, there exists a sequence $\{w_n\}$ in $\mathcal{U}$ such that $w_n \rightarrow w$ in $L^1(d\phi)$ and $x(T, w_n) \rightarrow x(T, w)$ as $n \rightarrow \infty$. For $u$ and $v \in \mathcal{U}$, we have sequences $\{u_n\}$ and $\{v_n\}$ so that $u_n \rightarrow u$, $v_n \rightarrow v$ in $L^1(d\phi)$, $x(T, u_n) \rightarrow x(T, u)$ and $x(T, v_n) \rightarrow x(T, v)$ as $n \rightarrow \infty$. We thus have that for any $n \in \mathbb{N}$

$$|x(T, u) - x(T, v)| \leq$$

$$\leq |x(T, u) - x(T, u_n)| + |x(T, v) - x(T, v_n)| + |x(T, u_n) - x(T, v_n)| \leq$$

$$\leq |x(T, u) - x(T, u_n)| + |x(T, v) - x(T, v_n)| + C \int_0^T |u_n(s) - v_n(s)| d\phi(s)$$

and take $n \rightarrow \infty$ in (3.7) to get

$$|x(T, u) - x(T, v)| \leq C \int_0^T |u(s) - v(s)| d\phi(s).$$

Hence we only have to show that for any $w \in \mathcal{U}$, there exists a sequence $\{w_n\}$ in $\mathcal{U}$ such that $w_n \rightarrow w$ in $L^1(d\phi)$ and $x(T, w_n) \rightarrow x(T, w)$ as $n \rightarrow \infty$. Let $w \in \mathcal{U}$. We can construct a sequence $\{w_n\}$ in $\mathcal{U}$ such that $w_n \rightarrow w$ in $L^1(d\phi)$ and $w_n(t) = \sum_{i=1}^{\delta_n} \alpha_i^n \chi_{I_i}(t)$, where $I_i^n = [a_i^n, b_i^n]$, $I_i^n =$
For $n \in \mathbb{N}$ and $i = 2, \ldots, \delta(n)$, we choose $c_i^n \in (a_i^n, b_i^n)$. Put $c_i^n = 0$ and $c_{\delta(n)+1}^n = T$. Define the time intervals $J_i^n = [c_i^n, c_{i+1}^n]$ and $J_i^n = (c_i^n, c_{i+1}^n)$ for $i = 2, \ldots, \delta(n)$.

Before proving that $\lim_{n \to \infty} x(T, w_n) = x(T, w)$, we observe that for a control function $\tilde{w}$, if the solution or the generalized solution of the initial value problem

\begin{equation}
\dot{x}(t) = f(t, x) + g(b_i^n, x) \tilde{w}(t) \\
\text{for } t \in J_i^n, \quad i = 1, \ldots, \delta(n) \text{ and } x(0) = \bar{x}
\end{equation}

exists, then we denote by $y_n(t, \tilde{w})$ the solution or the generalized solution of equation (3.10) corresponding to a control function $\tilde{w}$. If $\tilde{w} \in \mathcal{U}$, then $y_n(t, \tilde{w})$ is the usual solution of (3.10). If $\xi(t, \tilde{w})$ is the solution of the differential equation such that on each interval $J_i^n$,

\begin{equation}
\begin{cases}
\dot{\xi}(t) = F^*(t, b_i^n, \xi(t), \tilde{w}(t)), \\
\xi(c_i^n) = \exp \{ -\tilde{w}(c_i^n) \cdot g(b_i^n) \} \cdot y_n(c_i^n, \tilde{w}),
\end{cases}
\end{equation}

then $y_n(t, \tilde{w})$ also satisfies that

\begin{equation}
y_n(t, \tilde{w}) = \exp \{ \tilde{w}(t) \cdot g(b_i^n) \} \cdot \xi(t, \tilde{w}), \quad t \in J_i^n.
\end{equation}

On the other hand, if $\tilde{w} \in \mathcal{U}'$, then $y_n(t, \tilde{w})$ is inductively defined by (3.11) and (3.12), in this case $y_n(t, \tilde{w}) = x(t, \tilde{w})$ is the generalized solution of (1.1) corresponding to $\tilde{w}$.

Simple computation yields that

\begin{equation}
\lim_{n \to \infty} y_n(T, w) = x(T, w),
\end{equation}

when we take $y_n(t, w)$ as a usual Carathéodory solution of (3.10) corresponding to $w$. By Theorem 5 in [2],

\begin{equation}
\lim_{n \to \infty} |y_n(T, w_n) - y_n(T, w)| = 0,
\end{equation}

Generalized solutions of time dependent etc.
when $y_n(t, w)$ is defined by (3.11) and (3.12) corresponding to $w$. As a consequence,

$$\lim_{n \to \infty} x(T, w_n) = x(T, w)$$

and (1.7) holds for $u, v \in U$. ■

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