Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 102 (1999), p. 125-140

<http://www.numdam.org/item?id=RSMUP_1999__102__125_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 102 (1999)

Existence of Minimizers and Lower Semicontinuity of Integral Functionals in the Vectorial Case.

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ABSTRACT - We study the existence of minimizers and the lower semicontinuity of functionals of the type $I(u) = \int_\Omega f(x, \det Du(x))\, dx$ and $F(u) = \int_\Omega f(u(x), \det Du(x))\, dx$ respectively, where f is a convex integrand satisfying some assumptions which are usual in the setting of nonlinear elasticity.

1. Introduction.

In this paper we study the existence of minimizers for functionals of the type

$$I(u) = \int_\Omega f(x, \det Du(x))\, dx \tag{1.1}$$

and the lower semicontinuity of the functional

$$F(u) = \int_\Omega f(u(x), \det Du(x))\, dx \tag{1.2}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and f is a convex integrand such that $f(s, t) = +\infty$ if and only if $t \leqslant 0$. These assumptions are usually verified by stored energy functions that are considered in nonlinear elasticity (see [D], [FT]).

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The main difficulty in the study of the existence of the minimizers for energy functionals (1.1) or (1.2) lies in the inapplicability of direct methods by reason of lack of coerciveness of the functional respect any $W^{1,p}$ space.

The existing minimum results base itself on the existence of solutions of the following Dirichlet problem

$$(\mathcal{P}) \qquad \begin{cases} \det Du(x) = g(x), & \mathrm{x} \in \Omega \\ u(x) = x, & x \in \partial\Omega \end{cases}$$

which has been solved by Dacorogna and Moser ([DMo]) in the case of the datum g Hölder continuous.

These results are related to functionals with stored energy functions which depend only on *detDu* ([D]) and to the particular case of the displacement problem for elastic crystal ([FT]).

The problem of the existence of minimizers for (1.1) under general assumptions is still open and it might be solved by proving the existence of solutions of $(\mathcal{P})$ in the case of $g \in L^{\beta}(\Omega)$ for a suitable $\beta > 1$ depending on the growth condition of the integrand.

In section 3 we give an existence result for $(\mathcal{P})$ in the case of $g \in C(\overline{\Omega})$. This is a partial answer (we don't know if the solution belong to $C^1(\overline{\Omega}, \mathbb{R}^n)$) to a conjecture in [DMo].

We obtain the result as a direct consequence of the existence of solutions for the following problem related to the Monge-Ampere equation

$$(\mathcal{P}_1) \qquad \begin{cases} \det D^2 u(x) = g(x), & x \in \Omega \\ \nabla u(x) = x, & x \in \partial\Omega . \end{cases}$$

We refer to [GT] for classical results. Recently Brenier ([B]) has proved the existence of weak solutions of the Monge-Ampere equation satisfying the range condition $\nabla u(\overline{\Omega}) = \overline{\Omega}$ under mild assumptions on g. Afterwards Caffarelli ($[C_1], [C_2]$) provided some regularity theorems for this problem. In this paper we prove the existence of solutions of the boundary problem $(\mathcal{P}_1)$ if $g \in C(\overline{\Omega})$. The proof base itself on the techniques used by Gangbo ([G]) in the setting of the polar factorization of vector valued functions.

The section ends with an existence results for the minimizers of (1.1) which is an easy consequence of the existence theorem related to problem $(\mathcal{P})$.

Finally we recall that a different Dirichlet problem related to the minimum problem for (1.1) has been recently studied in [DMa] and [Z].

In the last section we prove the weak lower semicontinuity of (1.2) in a suitable affine subspace of $W^{1,p}(\Omega; \mathbb{R}^n)$. We observe that the convex integrand f is merely measurable respect the first variable. In this setting the lower semicontinuity of (1.2) has been proved in the scalar case ([DGBDM], [A]) and in the vectorial case ([V]). In these results, the assumption $f(s, 0) < +\infty$ for every s is crucial.

The semicontinuity theorem present in this paper (Theorem 4.5) is proved under the usual assumptions of the nonlinear elasticity. In particular $f(s, \cdot)$ has to satisfy a growth condition which permits to limit the study of the semicontinuity of (1.2) on a space of homeomorphisms. Then, by using the change variable formula, we overcome the difficulty related to the lack of regularity of the integrand.

2. Notations and definitions.

Throughout this work Ω will denote a nonempty, bounded, open subset of $\mathbb{R}^n$, where $n \geq 2$.

If $R \in \mathbb{R}$, $R > 0$, we denote by B_R the open ball with center the origin and radius R.

Besides, if $m \geq 1$, $\mathcal{B}(\mathbb{R}^m)$ and $\mathcal{L}(\mathbb{R}^m)$ will be the Borel and Lebesgue σ-algebras on $\mathbb{R}^m$ respectively.

Let $1 \leq p \leq +\infty$, $m \geq 1$ and $k \geq 1$; we say that $u \in W^{k,p}(\Omega; \mathbb{R}^m)$ if the coordinate functions of u belong to $L^p(\Omega)$ together with their distributional derivatives up to k^{th} order; we denote by Du the matrix of the first derivatives of u; besides, if $n = m$, $\det Du$ and $Adj\, Du$ will be the Jacobian determinant of Du and the transpose of the matrix of cofactors of Du. In the case $m = 1$, we denote by ∇u and by $D^2 u$ the gradient and the Hessian matrix of u rispectively. Finally, if $\{u_n\}_n$ is a sequence in $W^{1,p}(\Omega; \mathbb{R}^m)$ and $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, we denote by $u_n \rightharpoonup u$ the convergence of the sequence to u respect the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m)$.

If $m \geq 1$ and $f\colon \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is a mapping, we denote by $\partial^- f(x)$ the subdifferential of f at the point $x \in \mathbb{R}^m$ and by f^* the Fenchel conjugate of f; besides we denote $f^{**} = (f^*)^*$ (see [ET] for definitions).

Now we recall the definition of convex integrand.

Let $m \geq 1$, $B \in \mathcal{B}(\mathbb{R}^m)$ and $g\colon B \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a $\mathcal{L}(B) \times \mathcal{B}(\mathbb{R})$-

measurable function. We say that g is a convex integrand if $g(s,\cdot)$ is convex and lower semicontinuous on $\mathbb{R}$ for a.e. $s \in B$.

Besides, in the case of integrand, by $g^*(s,t)$ and $\partial_t^- g(s,p)$ we will mean $[g(s,\cdot)]^*(t)$ and $\partial^-[g(s,\cdot)](p)$ respectively.

In the section 3 we widely use the following definitions.

DEFINITION 2.1. *Let B be an open subset of $\mathbb{R}^n$ such that* $\text{meas}(\partial B) = 0$ *and $h\colon \overline{B} \to \mathbb{R}^n$ be a mapping. We say that h is a measure-preserving mapping on B if h satisfies the following equivalent properties*:

$$(a) \qquad h^{-1}(A) \in \mathcal{L}(\overline{B}) \ \textit{and} \ \text{meas}(h^{-1}(A)) = \text{meas}(A)$$

for every $A \in \mathcal{L}(\overline{B})$

$$(b) \qquad k \circ h \in L^1(B), \qquad \int_B k \circ h(x)\,dx = \int_B k(x)\,dx$$

for every $k \in L^1(B)$.

DEFINITION 2.2. *Let $B \subset \mathbb{R}^n$ be an open set and $h\colon B \to \mathbb{R}^n$ be a mapping.*

h is said to satisfy the N-property if $\text{meas}(h(A)) = 0$ *for each* $A \in \mathcal{L}(B)$ *such that* $\text{meas}(A) = 0$.

h is said to satisfy the N^{-1}-property if $\text{meas}(h^{-1}(A)) = 0$ *for each* $A \in \mathcal{L}(\mathbb{R}^n)$ *such that* $meas(A) = 0$.

Finally we recall the definition of topological degree.

Now let $f\colon \Omega \to \mathbb{R}^n$ be a continuous mapping, A a domain such that $A \Subset \Omega$ and $y \in \mathbb{R}^n$. Suppose that $y \notin f(\partial A)$. Then there exists $r \in \mathbb{R}$, $0 < r < 1$, small enough, such that f induces a homomorphism of cohomology groups

$$f^*\colon H_{n+1}(\overline{B(y, r^{-1})}; \overline{B(y, r^{-1})} \setminus B(y,r)) \to H_{n-1}(\overline{A}; \partial A).$$

If g_1, g_2 are convenient generators of the cohomology groups, there exists an integer, denote it $\mu(y,f,A)$, such that $f^*(g_1) = \mu(y,f,A)\, g_2$; $\mu(y,f,A)$ is called the topological degree of y with respect to the pair (f, A).

3. The existence results.

The main result of this section is related to a boundary value problem for the Monge-Ampère equation.

Let f, g be two real, bounded functions defined on Ω. We say that u, a Lipschitz real function, is a weak solution of the Monge-Ampère equation

$$g(\nabla u)\det D^2 u = f$$

if

$$\int_\Omega k(y)\, g(y)\, dy = \int_\Omega k(\nabla u(x))\, f(x)\, dx$$

for any $k \in L^\infty_{loc}(\mathbb{R}^n)$.

Besides we define the operator $\mathfrak{J}$: $C(B_R) \cap L^\infty(B_R) \to \mathrm{Lip}(\mathbb{R}^n)$ by

$$\mathfrak{J}(v)(y) = \sup_{z \in B_R} \{yz - v(z)\} \qquad \forall y \in \mathbb{R}^n.$$

By Proposition 4.5 in [G], the operator $\mathfrak{J}$ is well defined.

In the proof of Theorem 3.1 we will consider the space

$$V_0 = \{k \in C^1_c(B_R) \cap C(\mathbb{R}^n)\colon \nabla k(y) = 0 \ \ \forall y \in \partial\Omega\}.$$

If $\mathrm{meas}(\partial\Omega) = 0$, it is straightforward to prove, by using cut-off functions, that V_0 is dense in $L^1(B_R)$.

THEOREM 3.1. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with* $\mathrm{meas}(\partial\Omega) = 0$. *Let $f \in C(\overline{\Omega})$ such that $f > 0$ in $\overline{\Omega}$ and $\int_\Omega f(x)\, dx =$ $= \mathrm{meas}(\Omega)$. Then there exists $u \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$ for every $p < +\infty$ such that*

$$(\mathcal{P}_2) \qquad \begin{cases} \det D^2 u(x) = f(x), & a.e. \ in \ \Omega \\ \nabla u(x) = x, & x \in \partial\Omega. \end{cases}$$

PROOF. Let $R > 0$ such that $\overline{\Omega} \subset B_R$. Then by using Tietze extension theorem and Uryshon Lemma we can construct a mapping g that is a continuous extension of f on $\mathbb{R}^n$, such that $g > 0$ in $\mathbb{R}^n$ and $\int_{B_R} g(x)\, dx =$ $= \mathrm{meas}(B_R)$. Then, by Theorem 5 in [DMo] there exists a homeomorphism $\phi\colon \overline{B}_R \to \overline{B}_R$ satisfying $\int_A g(x)\, dx = \mathrm{meas}(\phi(A))$ for every open set $A \subset B_R$.

Now we consider the set

$$W=\left\{(u,v): u,\ v\in C(B_R)\cap L^{\infty}(B_R)\ \text{and}\ \begin{cases} u(y)+v(z)\geqslant yz, & \forall y,\ z\in B_R \\ u(y)+v(y)=\|y\|^2, & \forall y\in\partial\Omega\,. \end{cases}\right\}.$$

Finally we set $\psi=\phi^{-1}$ and define the functional

$$J(u,\ v)=\int_{B_R}[u(\psi(x))+v(x)]\,dx$$

for every $(u,\ v)\in W$. Now we consider the problem Min $\{J(u,\ v):(u,\ v)\in\in W\}$ and we prove the existence of a minimizer. Following [G], let $\{(u_m,\ v_m)\}_m$ be a minimizing sequence in W such that $\inf_{y\in B_R} u_m(y)=0$. Now we regularize the sequence in this way: we set

$$\tilde{u}_m=\mathfrak{J}(u_m)\,,\qquad \tilde{v}_m=\mathfrak{J}(\tilde{u}_m)\,. \tag{3.1}$$

By Proposition 4.5 in [G] the sequence $\{(\tilde{v}_m,\ \tilde{u}_m)\}_m$ is compact for the uniform topology on the compact subset of $\mathbb{R}^n$. Besides we observe that $\{(\tilde{v}_m,\ \tilde{u}_m)\}_m$ is a minimizing sequence. Indeed, since $(u_m,\ v_m)\in W$ and by (3.1), we deduce that

$$\tilde{u}_m(z)+\tilde{v}_m(y)\geqslant yz\qquad \forall(z,\ y)\in B_R\times B_R$$

and

$$\tilde{u}_m(z)\leqslant v_m(z)\qquad \forall z\in B_R,$$

$$\tilde{v}_m(y)\leqslant u_m(y)\qquad \forall y\in B_R.$$

From these inequalities, it follows that $\tilde{u}_m(y)+\tilde{v}_m(y)=\|y\|^2$ for every $y\in\partial\Omega$. Therefore $(\tilde{v}_m,\ \tilde{u}_m)\in W$ and $\{(\tilde{v}_m,\ \tilde{u}_m)\}_m$ is still a minimizing sequence. Then there exist two locally Lipschitz functions $p,\ q:\mathbb{R}^n\to\mathbb{R}$ such that $\tilde{v}_m$ and $\tilde{u}_m$ converges uniformly on compact subsets of $\mathbb{R}^n$ to p and q respectively. It is easy to check that p and q are convex functions and $(p,\ q)\in W$. Therefore we have that $J(p,\ q)=\text{Min}\{J(u,\ v):(u,\ v)\in\in W\}$. Finally we consider

$$w=\mathfrak{J}(q)\,. \tag{3.2}$$

From the definition of Fenchel conjugate and (3.2) we easily get that

$$w(y)\leqslant p(y)\qquad \forall y\in B_R, \tag{3.3}$$

$$w^*(z)\leqslant q(z)\qquad \forall z\in B_R. \tag{3.4}$$

and now it is easy to check that $(w, w^*) \in W$ and $J(w, w^*) = \operatorname{Min}\{J(u, v): (u, v) \in W\}$. Since ψ satisfies *N-property*, (3.3) and (3.4) imply that $p = w$ and $q = w^*$ on B_R.

For each $h \in C(B_R) \cap L^\infty(B_R)$, we define

$$G(h) = J(\mathfrak{J}(h), h)\,.$$

Now, we have that $\mathfrak{J}(w^*) = \mathfrak{J}(q) = w = w^{**}$ and then, since ψ satisfies N^{-1}-*property* and by Lemma 2.4 in [G], we obtain that

$$\langle G'(w^*), k\rangle = \int_{B_R} k(x) - k(\nabla w(\psi(x)))\,dx \qquad \forall k \in C(B_R) \cap L^\infty(B_R) \tag{3.5}$$

where G' denotes the Gâteaux derivative of G. Now, for each $k \in V_0$ and $r \in \mathbb{R}$, we define

$$\tilde{w}_r(z) = w^*(z) + rk(z)\,, \qquad \forall z \in \mathbb{R}^n,$$

$$w_r = \mathfrak{J}(\tilde{w}_r)\,. \tag{3.6}$$

Now we check that

$$\tilde{w}_r(y) + w_r(y) = \|y\|^2 \qquad \forall y \in \partial\Omega\,.$$

Let $y \in \partial\Omega$. Since $k \in V_0$, we have that

$$\partial^- \tilde{w}_r(y) = \partial^-[w^* + rk](y) = \{\alpha + r\nabla k(y) \colon \alpha \in \partial^- w^*(y)\} = \partial^- w^*(y)\,.$$

Now we have that

$$w(y) + w^*(y) = \|y\|^2;$$

therefore

$$y \in \partial^- w^*(y) = \partial^- \tilde{w}_r(y)$$

and so

$$\|y\|^2 = \tilde{w}_r(y) + \tilde{w}_r^*(y) \geq \tilde{w}_r(y) + w_r(y)\,.$$

The opposite inequality follows from (3.6).

Now it is straightforward to check that $(w_r, \tilde{w}_r) \in W\ \forall r \in \mathbb{R}$. Hence, since $J(w, w^*) = \operatorname{Min}\{J(u, v): (u, v) \in W\}$ and by (3.5), we obtain

$$\int_{B_R} k(x)\,dx = \int_{B_R} k(\nabla w(\psi(x)))\,dx \qquad \forall k \in V_0\,. \tag{3.7}$$

This implies that $\nabla w \circ \psi$ satisfies N^{-1}-*property* on B_R and, since V_0 is dense in $L^1(B_R)$, we have that (3.7) holds for every $k \in L^1(B_R)$. Therefore $\nabla w \circ \psi$ is a measure-preserving mapping on B_R and, consequently, ∇w and ∇w^* satisfy N^{-1}-*property*. Hence

$$\int_{B_R} k(\psi(x))\,dx = \int_{B_R} k\big(\nabla w^*(\nabla w(\psi(x)))\big)\,dx = \int_{B_R} k(\nabla w^*(x))\,dx$$

for every $k \in L^\infty_{\rm loc}(\mathbb{R}^n)$. Thus

$$(3.8) \int_{B_R} k(\nabla w^*(x))\,dx = \int_{B_R} k(\psi(x))\,dx = \int_{B_R} k(y)\,d\phi(y) = \int_{B_R} k(y)\,g(y)\,dy$$

for every $k \in L^\infty_{\rm loc}(\mathbb{R}^n)$. Then w^* is a weak solution of $g(\nabla u)\det D^2 u = 1$ on B_R and, by the regularity theorem in $[C_1]$, $w^* \in W^{2,p}_{\rm loc}(B_R) \cap C^1(B_R)$ for every $p < +\infty$; besides w^* is strictly convex. This last property implies that w is differentiable everywhere on B_R; therefore $\nabla w^* = (\nabla w)^{-1}$ on B_R and $\nabla w \in C(B_R)$. By (3.8) it follows that

$$\int_{B_R} \eta(x)\,dx = \int_{B_R} \eta(\nabla w(y))g(y)\,dy$$

for every $\eta \in L^\infty_{\rm loc}(\mathbb{R}^n)$.

This means that w is a weak solution of $\det D^2 u = g$ in B_R and $w \in W^{2,p}_{\rm loc}(B_R) \cap C^1(B_R)$ for every $p < +\infty$ ($[C_1]$). Let us now prove that w is a solution of $(\mathcal{P}_2)$. First we observe that, since $(w, w^*) \in W$, $w(y) + w^*(y) = \|y\|^2$ for every $y \in \partial\Omega$ and this means that $\nabla w(y) = y$ for every $y \in \partial\Omega$. Besides, for any $\eta \in L^\infty(\Omega)$, we have ([RR], § V.3.4, Thm.2)

$$\int_\Omega \eta(y)\,f(y)\,dy = \int_{\nabla w(\Omega)} \eta(\nabla w^*(x))\,dx = \int_\Omega \eta(y)\det D(\nabla w(y))\,dy$$

since $\mu(x, \nabla w, \Omega) = \mu(x, Id, \Omega) = 1$ for every $x \in \nabla w(\Omega)$, ([RR], § II.2.3 Thm. 6). Therefore we obtain $\det D^2(w(y)) = f(y)$ a.e. in Ω. ■

THEOREM 3.2. *Let Ω a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary. Let $f \in C(\overline{\Omega})$ and $\phi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that $f > 0$ and $\det D\phi > 0$ on $\overline{\Omega}$ and $\int_\Omega f(x)\,dx = \int_\Omega \det D\phi(x)\,dx$.*

Then there exists $u \in W^{1,p}(\Omega; \mathbb{R}^n) \cap C(\overline{\Omega})$ *for every* $p < +\infty$ *such that*

$$\begin{cases} \det Du(x) = f(x), & a.e.\ in\ \Omega \\ u(x) = \phi(x), & x \in \partial\Omega . \end{cases}$$

PROOF. Let us consider the following problems

$$\begin{cases} \det Dv(x) = f(x) \dfrac{\text{meas}(\Omega)}{\int_\Omega f(x)\, dx}, & \text{a.e. in } \Omega \\ v(x) = x , & x \in \partial\Omega . \end{cases}$$

and

$$\begin{cases} \det Dw(x) = \det D\phi(x) \dfrac{\text{meas}(\Omega)}{\int_\Omega \det D\phi(x)\, dx}, & \text{a.e. in } \Omega \\ w(x) = x , & x \in \partial\Omega . \end{cases}$$

By Theorem 3.1, these problems have solutions $v, w \in W^{1,p}(\Omega; \mathbb{R}^n) \cap \cap C(\overline{\Omega})$ for every $p < +\infty$. Besides w is invertible in Ω and $w^{-1} \in \in W^{1,p}(\Omega; \mathbb{R}^n) \cap C(\overline{\Omega})$. Then we set $g = w^{-1} \circ v$ and $u = \phi \circ g$. We have that $g, u \in W^{1,p} \cap C(\overline{\Omega})$ for every $p < +\infty$ and the chain rule holds. This follows from Theorem 2.9 in [R] and by a density argument since v satisfy N^{-1}-property. Therefore we obtain

$$\det Du(x) = \det D\phi(g(x)) \det Dg(x) =$$

$$= \det Dw(g(x)) \frac{\int_\Omega \det D\phi(x)\, dx}{\text{meas}(\Omega)} \det Dw^{-1}(v(x)) \det Dv(x) = f(x)$$

a.e. in Ω, since w satisfies N-property and v satisfies N^{-1}-property. Besides we have that $u(x) = \phi(x)$ if $x \in \partial\Omega$. ∎

Now we are able to prove the existence result for the functional (1.1).

Given $f: \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ a convex integrand, we define the multifunction $\Gamma: \overline{\Omega} \to \mathcal{P}(\mathbb{R})$ by

$$\Gamma(x) = \partial^- f_s^*(x, 0).$$

THEOREM 3.3. *Let Ω a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary. Let $\phi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that $\det D\phi > 0$ in $\overline{\Omega}$. Besides let $f: \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a convex integrand satisfying the following condition:*

(a) *for every $x \in \overline{\Omega}$, $f(x, t) = +\infty$ if and only $t \leq 0$,*

(b) *there exists $p: \overline{\Omega} \to \mathbb{R}$ a continuous selection of Γ,*

(c) $\int_\Omega p(x)\, dx = \int_\Omega \det D\phi(x)\, dx.$

Then there exists $u \in W^{1,p}(\Omega; \mathbb{R}^n) \cap C(\overline{\Omega})$ for every $p < +\infty$ solution of the problem

$$(\mathcal{P}_3) \qquad \inf\left\{ I(u) = \int_\Omega f(x, \det Du(x))\, dx : u \in W^{1,1}(\Omega; \mathbb{R}^n),\ u = \phi \text{ on } \partial\Omega \right\}$$

REMARK. Assumption (b) is satisfied by any continuous integrand f, strictly convex in the last variable such that

$$f(x, t) \geq ah(t) + b$$

for every $(x, t) \in \overline{\Omega} \times \mathbb{R}$, where a, b are positive costants and $h: \mathbb{R} \to \mathbb{R}$ is a lower semicontinuous function satysfing $\lim_{t \to +\infty} h(t) = +\infty$.

PROOF. By hypothesis (b) we have that, for every $x \in \overline{\Omega}$, $0 \in \partial^- f_s(x, p(x))$ and then $f(x, p(x)) \leq f(x, t)$ for every $(x, t) \in \overline{\Omega} \times \mathbb{R}$. This implies that $p(x) > 0$ for every $x \in \overline{\Omega}$ and $f(x, p(x)) \leq f(x, \det Dv(x))$ for almost every $x \in \overline{\Omega}$ and $v \in W^{1,1}(\Omega; \mathbb{R}^n)$.

Now, by Theorem 3.2, there exists $u \in W^{1,p}(\Omega; \mathbb{R}^n) \cap C(\overline{\Omega})$ for every $p < +\infty$ such that $\det Du(x) = p(x)$ a.e. in Ω and $u(x) = \phi(x)$ for every $x \in \partial\Omega$. Hence u is solution of $(\mathcal{P}_3)$. ■

Now we produce an example of integrand satysfing the assumptions of Theorem 3.3.

EXAMPLE. Let be Ω, ϕ as in the Theorem 3.3 and let be $f\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x, t) = \begin{cases} \alpha(x)\left(t + \dfrac{(p(x))^2}{t}\right) & \text{if } t > 0 \\ +\infty & \text{if } t \leq 0 \end{cases}$$

where α is a positive, measurable fuction on $\overline{\Omega}$, $p \in C(\overline{\Omega})$ is positive and such that

$$\int_\Omega p(x)\, dx = \int_\Omega \det D\phi(x)\, dx\,.$$

Then it easy to verify that f is a convex integrand and

$$f^*(x, s) = \begin{cases} -2p(x)\sqrt{\alpha(x)}\sqrt{\alpha(x) - s} & \text{if } s \leq \alpha(x) \\ +\infty & \text{if } s > \alpha(x) \end{cases}$$

so that $\Gamma(x) = p(x)$.

4. The semicontinuity result.

In this section we study the lower semicontinuity of integral functionals of the type

$$F(u) = \int_\Omega f(u(x), \det Du(x))\, dx$$

defined on the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$.

Throughout the section Ω will be a nonempty, bounded, connected and strongly Lipschitz open subset of $\mathbb{R}^n$ with $n \geq 2$. Besides, if $p > n$ and $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ we shall assume that the Hölder continuous representative of u has been choosen. Now, given $u_0 \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that u_0 be one-to-one in Ω, we consider the closed affine subspace $W^{1,p}_{u_0}(\Omega, \mathbb{R}^n)$ of $W^{1,p}(\Omega, \mathbb{R}^n)$ defined by

$$W^{1,p}_{u_0}(\Omega, \mathbb{R}^n) = \{u \in W^{1,p}(\Omega, \mathbb{R}^n) \colon u - u_0 \in W^{1,p}_0(\Omega, \mathbb{R}^n)\}$$

and the set

$$A_p = \{u \in W^{1,p}_{u_0}(\Omega, \mathbb{R}^n) \colon \det Du(x) > 0 \;\; a.e \;\; in \;\; \Omega\}\,.$$

The principal tools in our analysis will be the following two theorems due to Ball [B, Theorem 1 and 2].

THEOREM 4.1. *Let $p > n$ and $u \in A_p$. Then:*

(a) $u(\overline{\Omega}) = u_0(\overline{\Omega})$

(b) *The change of variables formula*

$$\int_A h(u(x)) \det Du(x)\, dx = \int_{u(A)} h(v)\, dv \tag{4.1}$$

holds for all measurable set $A \subset \overline{\Omega}$ and any measurable function $h: \mathbb{R}^n \to \mathbb{R}$, provided only that one of the integrals in (4.1) exists.

The second theorem gives conditions under which a function $u \in A_p$ is a homeomorphism.

THEOREM 4.2. *Let $p > n$ and $u \in A_p$. Besides let $u_0(\Omega)$ satisfy the cone condition, and suppose that there exists $q > n$ such that*

$$\int_\Omega \|(Du)^{-1}(x)\|^q \det Du(x)\, dx < +\infty \tag{4.2}$$

then u is a homeomorphism of Ω onto $u_0(\Omega)$ and the inverse function $w \in W^{1,q}(u_0(\Omega), \mathbb{R}^n)$. Moreover we have:

$$\int_{u_0(\Omega)} \left| \frac{\partial w^i}{\partial y_j} \right|^q dy = \int_\Omega |(Adj\, Du)_{i,j}|^q (\det Du)^{1-q}\, dx\,. \tag{4.3}$$

Finally if $u_0(\Omega)$ is strongly Lipschitz, then u is a homeomorphism of $\overline{\Omega}$ onto $u_0(\overline{\Omega})$.

As a consequence of the previous theorem we have the following

COROLLARY 4.3. *Let $\beta > n-1$, $p > (\beta n(n-1))/(\beta + 1 - n)$ and $u_0(\Omega)$ be strongly Lipschitz; besides let $u \in A_p$ such that $(\det Du)^{-1} \in L^\beta(\Omega)$. Then u is a homeomorphism of $\overline{\Omega}$ onto $u_0(\overline{\Omega})$ and the inverse function $w \in W^{1,q}(u_0(\Omega), \mathbb{R}^n)$ for a suitable $q \in (n, \beta + 1)$.*

PROOF Let $u \in A_p$ such that $(\det Du)^{-1} \in L^\beta(\Omega)$. Since $p > (\beta n(n-1))/(\beta + 1 - n)$ and by continuity and monotonicity properties of the function $a(r) = (\beta + 1 - r)/(r(n-1))$, we have that there exists $q \in (n, \beta + 1)$ such that $\beta/p < (\beta + 1 - q)/(q(n-1))$. Now we consider $s \in$

$\in (\beta/(\beta+1-q), p/(q(n-1)))$ and we obtain $(q-1)((s/(s-1)) \leq \beta$ and $qs \leq p/(n-1)$. For such choice of s and q and by Hölder inequality, we obtain that there exists a costant $c = c(\Omega, n, q, s, \beta)$ s.t.

$$\int_\Omega \|(Du)^{-1}(x)\|^q \det Du(x)\, dx \leq c\left(\int_\Omega \|Adj\, Du(x)\|^{p/(n-1)}\, dx\right)^{((n-1)q)/p} \times$$

$$\times \left(\int_\Omega (\det Du(x))^{-\beta}\, dx\right)^{(q-1)/\beta}.$$

Therefore we have that condition (4.2) is satisfied. As a consequence of Theorem 4.3 u is a homeomorphism of $\overline{\Omega}$ onto $u_0(\overline{\Omega})$. ■

Now we state the hypothesis on the integrand of the functional F.

Let $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying the following assumptions:

(a) f is a convex integrand;

(b) for every $s \in \mathbb{R}^n$, $f(s, t) = +\infty$ if and only $t \leq 0$;

(c) there exist costants $c > 0, \beta > n-1$ such that $f(s, t) \geq ct^{-\beta}$, for every $s \in \mathbb{R}^n$ and $t > 0$. In the proof of the semicontinuity result we shall use the following Lemma.

LEMMA 4.4. *Let f as above, $p > n$ and let $\{u_n\}_{n\in N}$ be a sequence in $W^{1,p}(\Omega, \mathbb{R}^n)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^n)$ and $\liminf_n F(u_n) < +\infty$. Then $\det Du(x) > 0$ a.e. in Ω and $(\det Du)^{-1} \in L^\beta(\Omega)$.*

PROOF. It follows directly from assumption (c) and the sequential weak lower semicontinuity of the functional

$$G(u) = \int_\Omega g(\det Du(x))\, dx \quad \text{where} \quad g(t) = \begin{cases} t^{-\beta}, & \text{if } t > 0 \\ +\infty, & \text{if } t \leq 0\,. \end{cases}$$ ■

THEOREM 4.5. *Let f as above, $p > (\beta n(n-1))/(\beta+1-n)$ and suppose that $u_0(\Omega)$ is strongly Lipschitz. Then the functional F is sequentially weakly lower semicontinuous on $W^{1,p}_{u_0}(\Omega, \mathbb{R}^n)$.*

PROOF. First we observe that the functional F is well defined; as a matter of fact, since f is $\mathcal{L}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R})$-measurable, there exists a Borel function $\tilde{f}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that $\tilde{f}(s, t) = f(s, t)$ for every

$(s, t) \in (\mathbb{R}^n \setminus N) \times \mathbb{R}$, where $N \in \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{L}(N) = 0$. Finally, if we define

$$h(s, t) = \begin{cases} \tilde{f}(s, t) & \text{if } s \notin N \\ +\infty & \text{if } s \in N, \end{cases}$$

we have that h is a Borel function and by Lemma 7 in ([RR], § V.2.4.) we get

$$h(u(x), \det Du(x)) = f(u(x), \det Du(x)) \qquad a.e. \ in \ \Omega$$

for every $u \in W^{1,p}(\Omega, \mathbb{R}^n)$. Consequently $f(u(\cdot), \det Du(\cdot))$ is $\mathcal{L}(\Omega)$-measurable; besides f is positive by (c) and therefore F is well defined.

Now we consider $u \in A_p$ such that $(\det Du)^{-1} \in L^{\beta}(\Omega)$. By Corollary 4.3, u is a homeomorphism of $\overline{\Omega}$ onto $u_0(\overline{\Omega})$ and the inverse function $w \in W^{1,q}(u_0(\Omega), \mathbb{R}^n)$ for a suitable $q \in (n, \beta + 1)$. If we apply the change of variables formula (4.1) to the function $h(x) = f(u(x), \det Du(x))$, we obtain

$$F(u) = \int_{\Omega} h(x)\, dx = \int_{u_0(\Omega)} h(w(s)) \det Dw(s)\, ds =$$

$$= \int_{u_0(\Omega)} f(s, \det Du(w(s))) \det Dw(s)\, ds\,.$$

Now we define a new functional. Let

$$g(s, t) = \begin{cases} f\left(s, \dfrac{1}{t}\right) t & \text{if } t > 0 \\ +\infty & \text{if } t \leqslant 0 \end{cases}$$

and let consider g^{**}. It is easy to check that g^{**} is a convex integrand and $g^{**}(s, t) = g(s, t)$ if $t \neq 0$. Then we define

$$G(v) = \int_{u_0(\Omega)} g^{**}(s, \det Dv(s))\, ds$$

for every $v \in W^{1,q}(u_0(\Omega); \mathbb{R}^n)$. We observe that

$$F(u) = G(w) \tag{4.4}$$

and G is sequentially weakly lower semicontinuous in $W^{1,q}(u_0(\Omega); \mathbb{R}^n)$ since $q > n$. Finally we prove the semicontinuity of F. Let $u \in$

$\in W^{1,p}_{u_0}(\Omega, \mathbb{R}^n)$ and $\{u_n\}_n$ be a sequence in $W^{1,p}_{u_0}(\Omega, \mathbb{R}^n)$ such that $u_n \rightharpoonup u$ in $W^{1,p}_{u_0}(\Omega, \mathbb{R}^n)$ and $\liminf_n F(u_n) < +\infty$. We can suppose that $\sup_{n \in \mathbb{N}} F(u_n) < +\infty$ and, by (c), that

$$\sup_{n \in \mathbb{N}} \int_\Omega (\det Du_n(x))^{-\beta}\, dx < +\infty. \tag{4.5}$$

Hence, by Lemma 4.4, we have that $u, u_n \in A_p$ and $(\det Du)^{-1}$, $(\det Du_n)^{-1} \in L^\beta(\Omega)$ for every $n \in \mathbb{N}$. Therefore, for every $n \in \mathbb{N}$, we have that u, u_n are homeomorphisms of $\overline{\Omega}$ onto $u_0(\overline{\Omega})$ (Corollary 4.3) and we denote by w, w_n their inverse functions respectively. We recall that $w, w_n \in W^{1,q}(u_0(\Omega), \mathbb{R}^n)$ for a suitable $q \in (n, \beta + 1)$. Hence, by (4.4) and the lower semicontinuity of G, to prove the weakly lower semicontinuity of F, it is enough to prove that $w_n \rightharpoonup w$ in $W^{1,q}(u_0(\Omega), \mathbb{R}^n)$). First we check that the sequence $\{w_n\}_n$ is bounded in $W^{1,q}(u_0(\Omega), \mathbb{R}^n)$. Indeed,by (4.3) and Hölder inequality, there exist positive costants $c_2 = c_2(n)$ and $c_3 = c_3(\Omega, n, q, s, \beta)$ such that

$$\int_{u_0(\Omega)} \|Dw_n(s)\|^q\, ds \le c_2 \int_\Omega \|Adj\, Du_n(x)\|^q (\det Du_n(x))^{1-q}\, dx \le$$

$$\le c_3 \left(\int_\Omega \|Adj\, Du_n(x)\|^{p/(n-1)}\, dx \right)^{(q(n-1))/p} \times \left(\int_\Omega (\det Du_n(x))^{-\beta}\, dx \right)^{(q-1)/\beta}.$$

Besides, by (4.1), we obtain

$$\int_{u_0(\Omega)} \|w_n(s)\|^q\, ds \le \int_\Omega \|x\|^q \det Du_n(x)\, dx.$$

Since the sequence $\{Du_n\}_n$ is bounded in $L^p(\Omega, \mathbb{R}^{n \times n})$ and by (4.5), we obtain that the sequence $\{w_n\}_n$ is bounded in $W^{1,q}(u_0(\Omega), \mathbb{R}^n)$. Then there exist a subsequence of $\{w_n\}_n$ (we will denote it as the original sequence) and $\widetilde{w} \in W^{1,q}(u_0(\Omega), \mathbb{R}^n)$ such that $w_n \rightharpoonup \widetilde{w}$. It remains to prove that $w = \widetilde{w}$ on $u_0(\Omega)$. But, since $q > n$, $\{w_n\}_n$ converges uniformly to $\widetilde{w}$ in $u_0(\Omega)$ and then we get

$$\|x - \widetilde{w}(u(x))\| = \lim_n \|w_n(u_n(x)) - \widetilde{w}(u_n(x))\| = 0$$

for every $x \in \Omega$. In the same way one can verify that $u(\widetilde{w}(y)) = y$ for every $y \in u_0(\Omega)$. Therefore $w = \widetilde{w}$ on $u_0(\Omega)$ and this concludes the proof. ■